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PHYSICS OF MAGNETIC PHENOMENA

METHOD FOR SOLVING PHYSICAL PROBLEMS DESCRIBED BY LINEAR DIFFERENTIAL EQUATIONS

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A method for solving physical problems is suggested in which the general solution of a differential equation in partial derivatives is written in the form of decomposition in spherical harmonics with indefinite coefficients. Values of these coefficients are determined from a comparison of the decomposition with a solution obtained for any simplest particular case of the examined problem. The efficiency of the method is demonstrated on an example of calculation of electromagnetic fields generated by a current-carrying circular wire. The formulas obtained can be used to analyze paths in the near-field magnetic (magnetically inductive) communication systems working in moderately conductive media, for example, in sea water.

Keywords: current-carrying coil, electromagnetic field, linear equations in partial derivatives, spherical harmonics, near-field magnetic communication.

INTRODUCTION

It is well known that to solve complex equations of mathematical physics describing various physical processes a number of methods are used that allow one to succeed in each specific case. In particular, the method of variable separation or the Fourier method is one of the most widespread methods of solving problems described by differential equations in partial derivatives [1–4]. According to this method, the sought-after function of many arguments is represented in the form of a product of several functions with smaller number of arguments. Imposing boundary conditions on the functions with smaller number of arguments leads to the problem on eigenvalues (for separation constants) often referred to as the Storm–Liouville problem. As a result, the problem solution is generally expressed by an infinite sum of partial solutions each with its own separation constant.

The method of partial areas or the matching method is often used to solve various problems of electrodynamics, in particular, to study the propagation of electromagnetic waves in complex waveguide structures. Such structures are subdivided into simpler areas, and the electric and magnetic fields in each area are determined by solving the Helmholtz equation by the method of variable separation [5, 6]. To obtain the general problem solution, conditions of field continuity on both boundaries of these partial areas are used that lead to an infinite system of linear algebraic equations for the unknown eigenwave amplitudes. It is generally impossible to obtain exact solutions of infinite system of equations; therefore, the researchers often limit themselves to approximate solutions obtained by the method of reduction or successive approximation.

The Wiener–Hopf method is suitable for solving boundary problems in which a solution is searched in areas infinite in two opposite directions [6]. According to this method, an integral equation relating the required solution with

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a preset external action (characteristics of an incident wave) is first written in one of the semi-infinite areas taking advantage of the Green's functions. This equation is transformed into the Wiener–Hopf algebraic equation taking advantage of the Fourier transform and introducing additional indefinite function in the rest of semi-infinite area. Then the Fourier transform of the Green's function is factorized into functions one of which is regular in the upper part of the complex plane and the other is regular in the lower part of the complex plane. Simultaneously the function describing the external action normalized by the factor of the Fourier transform of the Green's function regular in the lower half-plane is represented as a sum of functions one of which is regular in the upper part of the complex plane and the other is regular in its lower part. As a result, two algebraic equations are derived: the first equation comprising functions regular in the upper part of the complex plane, and the second equation comprising functions regular in the lower part.

The method of solving the Helmholtz equation described in [7] can be called self-similar. According to this method, an auxiliary function is constructed from all arguments so that it can be chosen as a new single argument for which the differential equations in partial derivatives are reduced to an ordinary differential equation.

Various examples of solving problems of electrodynamics using the methods listed above can be found in [8–10]. However, for some problems these methods are either too complex, or do not provide sufficient accuracy. Therefore, creation of new highly effective methods for solving problems, including those described by the wave equations, is an important and urgent problem.

In the present paper, a new method is suggested for solving problems described by linear differential equations, according to which a solution of physical problem is represented in the form of decomposition in spherical harmonics with indefinite coefficients. Values of the indefinite coefficients are determined from a comparison of the written decomposition with the well-known solution of this problem for a simplest special case. Therefore, this method is applicable only when a solution of significantly simplified problem can be obtained, for example, static solution or solution on the symmetry axis of the examined structure.

Details of application of the suggested method are disclosed on an example of calculating magnetic fields generated by a circular wire coil carrying direct or alternating current. Practical interest to this problem has arisen in connection with the development and research of various systems of near-field magnetic (magnetically inductive) communication that can operate in low conductive media, for example, in sea water [11, 12].

1. CALCULATION OF THE MAGNETIC FIELD OF A WIRE COIL WITH DIRECT CURRENT

Let us calculate the magnetic field created by a uniform constant electric current in a circular wire of radius R (Fig. 1). The static magnetic field $\boldsymbol{H}$ in the region of space containing no currents is irrotational. Therefore, it is possible to describe it by the formula

$$\boldsymbol{H} = -\operatorname{grad}\psi \,, \tag{1}$$

where ψ is the scalar potential of the magnetic field satisfying the Laplace equation, that is, a harmonic function. In the spherical system of coordinates the general solution of the Laplace equation is given by the formula [1]

$$\psi(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{A_{nm}}{r^{n+1}} + B_{nm}r^{n}\right)P_{nm}(\cos\theta)\cos(m\varphi+\varphi_m)\,, \tag{2}$$

where $P_{nm}(x)$ are the associated Legendre functions of the first kind expressed in terms of the Legendre polynomials $P_n(x)$ by the formula [13]

$$P_{nm}(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)\,. \tag{3}$$

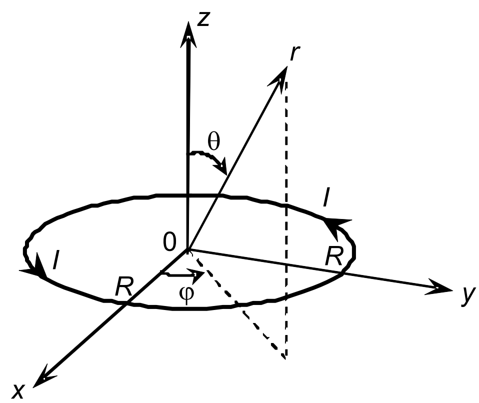

Fig. 1. Current-carrying circular wire.

Let us consider that the axis and the origin of the system of spherical coordinates coincide with the axis and the center of the circular coil. We seek a solution to the problem for the region of space outside of the sphere of radius R. In this case, the coefficients B_{nm} in formula (2) must vanish, and the formula itself is simplified

$$\psi(r,\theta) = \sum_{n=1}^{\infty} \frac{A_n}{r^{n+1}} P_n(\cos\theta). \tag{4}$$

In the above formula we have taken into account the axial symmetry of the problem, that is, the absence of dependence on the angle φ. From here, using formula (1), we derive the magnetic field components

$$H_r = \sum_{n=1}^{\infty} A_n \frac{n+1}{r^{n+2}} P_n(\cos\theta), \quad H_\theta = \sum_{n=1}^{\infty} \frac{A_n}{r^{n+2}} P_{n1}(\cos\theta)\,. \tag{5}$$

In particular, on the coil axis these components are expressed by formulas

$$H_r(r,\theta)\big|_{\theta=0} = \sum_{n=1}^{\infty} A_n \frac{n+1}{r^{n+2}}, \quad H_\theta(r,\theta)\big|_{\theta=0} = 0\,, \tag{6}$$

since $P_n(1) = 1$ and $P_{n1}(1) = 0$.

Let us compare formula (6) for the component H_r with the well-known formula [14]

$$H_r(r) = \frac{IR^2}{2(R^2+r^2)^{3/2}}\,, \tag{7}$$

which is easily obtained when calculating the field on the coil axis by integrating the expression for the Biot–Savart–Laplace law. Here I is the current carried by the wire coil. For convenience of comparison, we expand the formula in reciprocal powers of $1/r$:

$$H_r(r) = \frac{I}{2R} \sum_{n=1}^{\infty} C_{-3/2}^{n-1} \left(\frac{R}{r}\right)^{2n+1}, \tag{8}$$

where the binomial coefficients are determined by the formula [15]

$$C_\alpha^m = \begin{cases} 1 & \text{for } m = 0, \\ \dfrac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+1)}{m!} & \text{for } m \geq 1. \end{cases} \tag{9}$$

We note that expansion (8) contains only odd degrees of the ratio R/r. Therefore, because formulas (6) and (8) describe the same quantity, expansion (6) must not contain even degrees of R/r. Then formulas (6) and (5) must be re-written in the form

$$H_r = \sum_{n=1}^{\infty} A_n \frac{2n}{r^{2n+1}} P_{2n-1}(\cos\theta), \quad H_\theta = \sum_{n=1}^{\infty} \frac{A_n}{r^{2n+1}} P_{2n-1,\,1}(\cos\theta). \tag{10}$$

The field component H_θ vanishes on the coil axis, and the component H_r is simplified:

$$H_r(r,\theta)\big|_{\theta=0} = \sum_{n=1}^{\infty} A_n \frac{2n}{r^{2n+1}}. \tag{11}$$

Requiring equality of terms in expressions (8) and (11), we find the expansion coefficients

$$A_n = \frac{I}{4R} C_{-3/2}^{n-1} \frac{R^{2n+1}}{n}. \tag{12}$$

After substitution of coefficients (12) into formulas (10), we obtain the sought-after components of the magnetic field:

$$\begin{aligned} H_r &= \frac{I}{2R} \sum_{n=1}^{\infty} C_{-3/2}^{n-1} \left(\frac{R}{r}\right)^{2n+1} P_{2n-1}(\cos\theta), \\ H_\theta &= \frac{I}{4R} \sum_{n=1}^{\infty} C_{-3/2}^{n-1} \left(\frac{R}{r}\right)^{2n+1} \frac{P_{2n-1,\,1}(\cos\theta)}{n}. \end{aligned} \tag{13}$$

Formulas (13) coincide with the well-known formulas [14]

$$H_r = \frac{I}{2R}\left(\frac{R}{r}\right)^3 \cos\theta, \quad H_\theta = \frac{I}{4R}\left(\frac{R}{r}\right)^3 \sin\theta, \tag{14}$$

obtained in the magnetic dipole approximation for $r \gg R$ if in formulas (13) we retain only one term ($n = 1$) that decreases with increasing distance r more slowly than all other terms.

2. CALCULATION OF THE ELECTROMAGNETIC FIELD OF A WIRE COIL WITH ALTERNATING CURRENT

Let us calculate the components of fields created by a coil with alternating current when the coil sizes are much less than the wavelength. We now write down the general expression for an arbitrary high-frequency electromagnetic field in the spherical system of coordinates in the region of space $r > R$. As is well known [1, 2], the components of the electromagnetic field can be expressed in terms of the two Debye scalar potentials $U(\boldsymbol{r}, k)$ and $V(\boldsymbol{r}, k)$. The potential $U(\boldsymbol{r}, k)$ is used to describe waves of electric type that can have a longitudinal electric field component E_r, but have no longitudinal magnetic field component H_r. The potential $V(\boldsymbol{r}, k)$, on the contrary, is used for a description of waves of magnetic type that can have a longitudinal magnetic field component H_r, but have no longitudinal electric field components E_r. These potentials are solutions of the Helmholtz equation

$$\Delta U + k^2 U = 0, \quad \Delta V + k^2 V = 0, \tag{15}$$

where the wave number

$$k = \frac{\omega}{c}\sqrt{\varepsilon_r + \frac{i\sigma}{\varepsilon_0 \omega}}, \tag{16}$$

ω is the circular frequency in the time dependence exp $(-i\omega t)$, ε_r is the relative dielectric permittivity of the medium, and σ is the conductivity of the medium. The field components are expressed through the Debye potentials by formulas [1, 2]

$$E_r = \left(\frac{\partial^2}{\partial r^2} + k^2\right)(rU), \; H_r = \left(\frac{\partial^2}{\partial r^2} + k^2\right)(rV),$$

$$E_\theta = \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}(rU) + \frac{iZ_c k}{\sin\theta}\frac{\partial}{\partial\varphi}V, \; H_\theta = \frac{-ik}{\sin\theta Z_c}\frac{\partial}{\partial\varphi}U + \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}(rV), \tag{17}$$

$$E_\varphi = \frac{1}{r\sin\theta}\frac{\partial^2}{\partial\varphi\partial r}(rU) - iZ_c k\frac{\partial}{\partial\theta}V, \; H_\varphi = \frac{ik}{Z_c}\frac{\partial}{\partial\theta}U + \frac{1}{r\sin\theta}\frac{\partial^2}{\partial\varphi\partial r}(rV),$$

where the characteristic impedance of the medium

$$Z_c = \sqrt{\frac{\mu_0}{\varepsilon_0\left(\varepsilon_r + \dfrac{i\sigma}{\varepsilon_0\omega}\right)}}. \tag{18}$$

The general solution of Helmholtz equation (15) in the region including a point at infinity and not containing the origin of coordinates has the form [1, 2]

$$U,V = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_{nm}(k)\, h_n^{(1)}(kr) P_{nm}(\cos\theta)\cos(m\varphi + \varphi_m), \tag{19}$$

where $h_n^{(1)}(x)$ is the spherical Bessel function of the third kind [13]. Since the circular wire coil possesses the axial symmetry and the electric current in it is uniform, the electromagnetic field generated by the current will have axial symmetry. In this case, formula (19) is simplified

$$U,V = \sum_{n=0}^{\infty} A_n(k)\, h_n^{(1)}(kr) P_n(\cos\theta). \tag{20}$$

Formulas (17) are also simplified:

$$E_r = \left(\frac{\partial^2}{\partial r^2} + k^2\right)(rU),\ H_r = \left(\frac{\partial^2}{\partial r^2} + k^2\right)(rV),$$

$$E_\theta = \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}(rU),\ H_\theta = \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}(rV), \tag{21}$$

$$E_\varphi = -iZ_c k\frac{\partial}{\partial\theta}V\ ,\ H_\varphi = \frac{ik}{Z_c}\frac{\partial}{\partial\theta}U\ .$$

Substituting expression (20) into formula (21) for the component H_r, we obtain

$$H_r = \sum_{n=0}^{\infty} kA_n(k)\left[\left(\frac{d^2}{dx^2}+1\right)\left(xh_n^{(1)}(x)\right)\right]_{x=kr} P_n(\cos\theta)\,. \tag{22}$$

To find indefinite coefficients $A_n(k)$, we consider a special case, namely, calculate fields on the coil axis, that is, at $\theta = 0$. In this case, formula (22) is simplified:

$$H_r(r,\theta,k)\big|_{\theta=0} = \sum_{n=0}^{\infty} kA_n(k)\left[\left(\frac{d^2}{dx^2}+1\right)\left(xh_n^{(1)}(x)\right)\right]_{x=kr}\ , \tag{23}$$

since for any arbitrary order n of the polynomial $P_n(x)$, $P_n(x)\big|_{x=1} = 1$.

To simplify further the examined special case, we calculate the static limit for formula (23). To this end, the principal term of the expression in square brackets of formula (23) must be separated as $x \to 0$. It is obvious that it is responsible for the principal term of the function $xh_n^{(1)}(x)$. Using formulas

$$x\,h_0^{(1)}(x) = -i\exp(ix),\ \ x\,h_1^{(1)}(x) = -\left(1+\frac{i}{x}\right)\exp(ix) \tag{24}$$

and the recurrent formula

$$x\,h_n^{(1)}(x) = (2n-1)\cdot h_{n-1}^{(1)}(x) - x\,h_{n-2}^{(1)}(x)\,, \tag{25}$$

presented in [13], for $n \geq 1$ we obtain

$$x\,h_n^{(1)}(x) = -i\cdot 1\cdot 3\cdot 5\ldots(2n-1)\,x^{-n}\left[1+O(x)\right], \tag{26}$$

where $O(x)$ denotes a small quantity of the order x. Substituting expression (26) into formula (23) and retaining in it only the principal terms, we obtain

$$H_r(r,\theta,k)\big|_{\theta=0} = -i\sum_{n=0}^{\infty} kA_n(k)\cdot 1\cdot 3\cdot 5\ldots(2n-1)\frac{n(n+1)}{(kr)^{n+2}}\,. \tag{27}$$

Since the expansion of the component H_r, according to formula (8), must not contain even powers of the ratio R/r, formulas (20) and (27) can be re-written in the form

$$V = \sum_{n=1}^{\infty} A_n(k)\, h_{2n-1}^{(1)}(kr) P_{2n-1}(\cos\theta), \tag{28}$$

$$H_r(r,\theta,k)\big|_{\theta=0} = -i \sum_{n=1}^{\infty} kA_n(k)\cdot 1\cdot 3\cdot 5\ldots(4n-3)\frac{(2n-1)2n}{(kr)^{2n+1}}. \tag{29}$$

The requirement of term-wise equality of expressions (8) and (29) for the component H_r on the symmetry axis of the coil leads to the following expressions for the coefficients of expansion of the Debye potential V in formula (20):

$$A_n(k) = i\frac{I}{4n(2n-1)kR} C_{-3/2}^{n-1} \frac{(kR)^{2n+1}}{1\cdot 3\cdot 5\ldots(4n-3)}, \tag{30}$$

and the Debye potential is found to be

$$V = i\frac{I}{4kR}\sum_{n=1}^{\infty} C_{-3/2}^{n-1} \frac{(kR)^{2n+1}}{1\cdot 3\cdot 5\ldots(4n-3)} h_{2n-1}^{(1)}(kr)\frac{P_{2n-1}(\cos\theta)}{n(2n-1)}. \tag{31}$$

Recall that in the examined problem the sizes of the current-carrying coil are much less than the wavelength ($|k|R \ll 1$) and hence, it cannot radiate electromagnetic waves of electric type. Therefore, we consider that the Debye potential U is equal to zero.

Substituting potential (31) into formulas (17), we find nonzero components of the high-frequency electromagnetic field created by the circular wire coil with uniform alternating current:

$$H_r = i\frac{I}{4R}\sum_{n=1}^{\infty} C_{-3/2}^{n-1} \frac{(kR)^{2n+1}}{1\cdot 3\cdot 5\ldots(4n-3)}\left[\left(\frac{d^2}{dx^2}+1\right)\left(xh_{2n-1}^{(1)}(x)\right)\right]_{x=kr}\frac{P_{2n-1}(\cos\theta)}{n(2n-1)},$$

$$H_\theta = -i\frac{I}{4R}\sum_{n=1}^{\infty} C_{-3/2}^{n-1} \frac{(kR)^{2n+1}}{1\cdot 3\cdot 5\ldots(4n-3)}\left[\frac{1}{x}\frac{d}{dx}\left(xh_{2n-1}^{(1)}(x)\right)\right]_{x=kr}\frac{P_{2n-1,1}(\cos\theta)}{n(2n-1)},$$

$$E_\varphi = -Z_c\frac{I}{4R}\sum_{n=1}^{\infty} C_{-3/2}^{n-1} \frac{(kR)^{2n+1}}{1\cdot 3\cdot 5\ldots(4n-3)} h_{2n-1}^{(1)}(kr)\frac{P_{2n-1,1}(\cos\theta)}{n(2n-1)}. \tag{32}$$

If we retain only the first terms ($n = 1$) in formulas (32), we obtain the well-known formulas for the electromagnetic field of a magnetic dipole, that is, for the field of coil with alternating current under condition $kr \gg 1$ [16]:

$$H_r = \frac{IR^2}{2}\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\cos\theta,$$

$$H_\theta = \frac{IR^2}{4}\left(\frac{1}{r^3}-\frac{ik}{r^2}-\frac{k^2}{r}\right)e^{ikr}\sin\theta, \tag{33}$$

$$E_{\varphi} = \frac{IR^2}{4} Z_c \left(\frac{ik}{r^2} + \frac{k^2}{r} \right) e^{ikr} \sin\theta \, .$$

We note that formulas (33) for $kr \ll 1$ coincide with static formulas (14) to within small terms of the order of $(kr)^2$.

CONCLUSIONS

In this work, the new method for solving physical problems described by linear differential equations, in particular, various problems of electrodynamics has been suggested. According to this method, a solution of the differential equation in partial derivatives is written in the form of decomposition in spherical harmonics with indefinite coefficients. Values of the indefinite coefficients are retrieved from a comparison of the written decomposition with the solution obtained for any simplest special case of the examined problem, for example, for a static problem solution or a solution on the symmetry axis of the examined structure. Therefore, the suggested method can be used successfully to solve complex physical problems, but only when a solution of a simplified particular case can be obtained.

The efficiency of the method is demonstrated on the example of calculation of electromagnetic fields created by a circular wire carrying uniform constant or alternating current. The formulas obtained can be used to analyze paths in systems of near-field magnetic (magnetically inductive) communication that have been actively developed and investigated over the last few years. One of the advantages of such communication systems is their ability to work even in moderately conductive media, for example, in wet ground or sea water.

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