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Self-consistent approximation: Development and application to the problem of waves in inhomogeneous media

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ABSTRACT

A new self-consistent approximation proposed earlier, is compared with various existing approximations, as well as with a numerical simulation of solutions of the wave equation for a medium with one-dimensional inhomogeneities. The Green's function, found using the new approach, is the closest to the result obtained by the numerical simulation. The results of the work show that the new approach has undoubted advantages in the study of stochastic problems in media with longwave inhomogeneities. The new self-consistent approximation in some cases has advantages over a numerical method: a more rapid process of calculation and the possibility of consideration of three-dimensional problems.

1. Introduction

A self-consistent approximation (SCA) is widely used in different fields of physics for the approximate calculation of Green's functions. It was proposed by Migdal in the study of electron-phonon interaction [1]. In those same years, a similar version of the SCA was independently proposed by Kraichnan [2] to investigate the effect of inhomogeneities on the dynamic susceptibility of waves in disordered media. A similar version was proposed to study the scattering of electrons in disordered media, as a generalization of the well-known non-self-consistent Born approximation, and has became known as the self-consistent Born approximation (see, e.g., [3]). We will use for all these versions the name of the standard self-consistent approximation. The standard SCA corresponds to taking into account of only the first term of the expansion of the vertex function in a series. In this approximation, there are no diagrams with intersecting lines of correlations (and those of the majority). Lack of diagrams with intersecting correlation lines imposes restrictions on both the range of applicability of the standard SCA and the accuracy of the results obtained with its help. Therefore, intensive studies of amendments to the self-energy by taking into account the next term in the expansion of the vertex function (vertex corrections) are carried out [4–13]. In these works, a significant progress in the study of the vertex corrections has been achieved. However, the discrepancy between the results of different approaches still remains significant. In [14], the self-consistent approximation of a higher level relative to the standard SCA, which taken into account both the first and second term of the expansion of the vertex function, was derived and compared with the standard SCA.

The aim of this work is to compare both the new and the standard SCA with the ladder approximation [15] and with the numerical simulation of the problem.

2. New self-consistent approximation and its properties

The derivation of the new SCA was carried out in [14]. In contrast to the standard SCA, the new SCA is described by a system of two coupled nonlinear integral equations: either for the self-energy Σ and the vertex function Γ (we omit the frequency ω in all expressions, where this does not lead to misunderstandings)

$$\Sigma_{\mathbf{k}} = \gamma^2 (2\pi)^{-d} \int \frac{S_{\mathbf{k}-\mathbf{k}_1} \Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1} d\mathbf{k}_1}{g_{\mathbf{k}_1}^{-1} - \Sigma_{\mathbf{k}_1}}, \tag{1}$$

$$\Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1} \approx \frac{1}{1 - \gamma^2 (2\pi)^{-d} \int \frac{S_{\mathbf{k}_1-\mathbf{k}_2} \Gamma_{\mathbf{k}_2,\mathbf{k}_1-\mathbf{k}_2} d\mathbf{k}_2}{\left[g_{\mathbf{k}_2}^{-1} - \Sigma_{\mathbf{k}_2}\right]\left[g_{\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2}^{-1} - \Sigma_{\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2}\right]}}, \tag{2}$$

or for the Green's function G and the vertex function Γ

$$G_{\mathbf{k}} = \frac{1}{\mathbf{g}_{\mathbf{k}}^{-1} - \gamma^2 (2\pi)^{-d} \int S_{\mathbf{k}-\mathbf{k}_1} G_{\mathbf{k}_1} \Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1} d\mathbf{k}_1}, \tag{3}$$

$$\Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1} \approx \frac{1}{1 - \gamma^2 (2\pi)^{-d} \int S_{\mathbf{k}_1-\mathbf{k}_2} G_{\mathbf{k}_2} G_{\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2} \Gamma_{\mathbf{k}_2,\mathbf{k}_1-\mathbf{k}_2} d\mathbf{k}_2}. \tag{4}$$

Here $S_{\mathbf{k}}$ is the Fourier transform of the normalized correlation function $K(\mathbf{x}', \mathbf{x}'')$ of inhomogeneities, of the normalized function of the electron-phonon interaction $D(\mathbf{x}', \mathbf{x}'')$ or of the average potential of the interaction between electrons and impurities and γ is a rms

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fluctuation of the correspondent physical value.

The new SCA contains, as limiting cases at small γ, all lower-level approaches: the standard SCA and the non-self-consistent Bourret (Born) approximation [16].

In the limiting case k_c=0 (where k_c is the correlation wave number, $r_c = k_c^{-1}$ is the correlation radius of inhomogeneities), $S_{\mathbf{k}-\mathbf{k}_1} \to (2\pi)^d \delta(\mathbf{k} - \mathbf{k}_1)$. In this case a quantity of diagrams in the expansion of a Green's function can be found analytically. In [14] a number of diagrams have been found for the standard and new SCA. By using the rules of a function expansion in the binomial series [17], we obtain the following representation for a Green's function

$$G = g \sum_{n=0}^{\infty} N_n z^n, \quad z = \gamma^2 g^2. \tag{5}$$

The coefficient N_n is a number of diagrams in each n-th order of the Green's function expansion.

$$G; \; N_n = (2n - 1)!!, \tag{6}$$

$$G_{Sta}; \; N_n = \frac{(2n)!}{n!(n + 1)!} \; - \text{ Catalan numbers}, \tag{7}$$

$$G_{New}; \; N_n = \sum_{m=0}^{\infty} \frac{2^{m+1}(2m)!}{3^{2m+1}(m + 1)!(m - n)!n!}. \tag{8}$$

The general formula for the exact Green's function coefficients N_n, Eq. (6), is well known. General formulas for the coefficients N_n of the standard and the new SCA, Eqs. (7) and (8), respectively, were derived in [14]. To find the coefficients of the ladder approximation [15], we perform the following operations here. The system of Eqs. (3) and (4) in the limiting case of k_c=0 is simplified:

$$G_{\mathbf{k}} = \frac{1}{g_{\mathbf{k}}^{-1} - \gamma^2 G_{\mathbf{k}} \Gamma_{\mathbf{k},0}}, \tag{9}$$

$$\Gamma_{\mathbf{k},0} = \frac{1}{1 - \gamma^2 G_{\mathbf{k}}^2 \Gamma_{\mathbf{k},0}}. \tag{10}$$

This system led to a quadratic equation for the Green's function, and this function was obtained in an explicit form. Substituting it in Eq. (5), we received the coefficients N_n of the new SCA, Eq. (8). The ladder approximation [15] for k_c=0 leads to a system of equations in which the first equation is identical with Eq. (9) and the second can be obtained from Eq. (10) when we put $\Gamma_{\mathbf{k},0} = 1$ in the denominator of the latter. This system leads to a cubic equation for $G_{\mathbf{k}}$ and the Green's function can not be represented in the form of Eq. (5). Therefore, to find the coefficients N_n of the ladder approximation, we substitute the simplified ladder analogue of Eq. (10) into Eq. (9), expand the latest in a series of γ^2, and carry out the process of iteration of the resulting expression. As a result, we get

$$G_{Lad}; \; N_1 = 1, \; N_2 = 3, \; N_3 = 12, \; N_4 = 55, \; \dots. \tag{11}$$

In Table 1 the values N_n up to the 6-th order are shown. One can see that for $n > 1$ the new approach takes into account in every n-th order more diagrams than standard SCA and then non-self-consistent ladder approximation.

Fig. 1 shows relative portions (in percentage) of the amount of the exact diagrams taken into account in each approximation. The standard SCA accurately takes into account only the first order, the new accurately takes into account both the first and second order. In the following orders of the expansion the new SCA is also much closer to the exact value than the standard SCA. The ladder approximation is also worse than ours, and, besides, it is difficult in the application because it is non-self-consistent.

Table 1
The number of diagrams in each n-th order of the expansion of the Green's function for the first six orders of n, taken into account in the standard (G_{Sta}), the ladder (G_{Lad}), and the new (G_{New}) approximations and in the exact expression for the Green's function (G).

n	1	2	3	4	5	6
G_{Sta}	1	2	5	14	42	132
G_{Lad}	1	3	12	55	273	1428
G_{New}	1	3	13	67	381	2307
G	1	3	15	105	945	10395

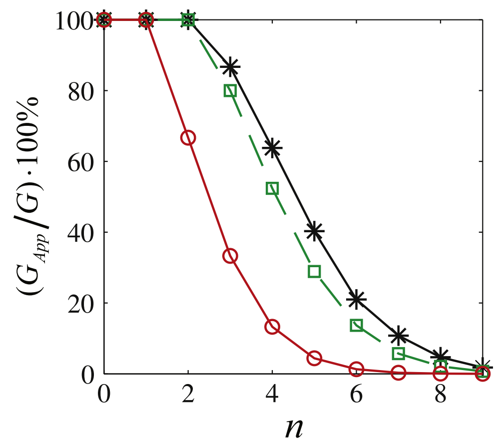

Fig. 1. (Color online) The relative proportions of the exact number of diagrams (in percent) accounted for the standard (circles, red curve), ladder (squares, green curve), and the new (asterisks, black curve) approximations in each n-th term of the expansion of the Green's function.

In the general case of an arbitrary correlation radius ($k_c \neq 0$) the number of diagrams in the Green's functions is determined by the same formulas, Table 1, and Fig. 1, that for k_c=0.

At $k_c \neq 0$, the fourth and higher orders of expansion of self-energy Σ of the new SCA, along with correct diagrams contain small quantities of defective diagrams [14]. It is difficult to evaluate the effect of such diagrams in general form. Therefore, the comparison of results obtained in the framework of the new SCA with the results of a numerical simulation is of special importance to assess the accuracy of the proposed method. The next section of the paper is devoted to this comparison.

3. Applications of the new SCA. Waves in inhomogeneous media

We carried out in [14] the comparison of the results of the new and standard SCA for the simplest model of the wave equation in a randomly inhomogeneous medium. In this section, we carry out a numerical simulation of the same wave equation. We then compare the results of the new and standard SCA with the numerical simulation, that is almost with the exact solution of this problem. The considering wave equation is

$$\nabla^2 \varphi + [\nu + \gamma \rho(\mathbf{x})]\varphi = 0, \tag{12}$$

where $\rho(\mathbf{x})$ is a centered ($\langle \rho \rangle = 0$) and normalized ($\langle \rho^2 \rangle$=1) random function. For scalar models of electromagnetic or elastic waves $\nu = (\omega/s)^2$, where ω is the frequency, s is the velocity of corresponding waves in the medium; for spin waves $\nu = (\omega - \omega_0)/g\alpha M$, ω_0 is the frequency of the uniform ferromagnetic resonance, g is the gyromagnetic ratio, α is the exchange parameter, M is the magnetization. In all cases, γ is the rms fluctuation of the correspondent inhomogeneities. We consider the case of one-dimensional inhomogeneities and model the stochastic properties of the random function $\rho(x)$ by exponential correlations.

We found in [14] dynamic susceptibilities (Green's functions) of the waves in both the new and the standard SCA. Here we additionally find a Green's function in the non-self-consistent Bourret/Born approximation. We find also a solution of this problem using a numerical simulation method.

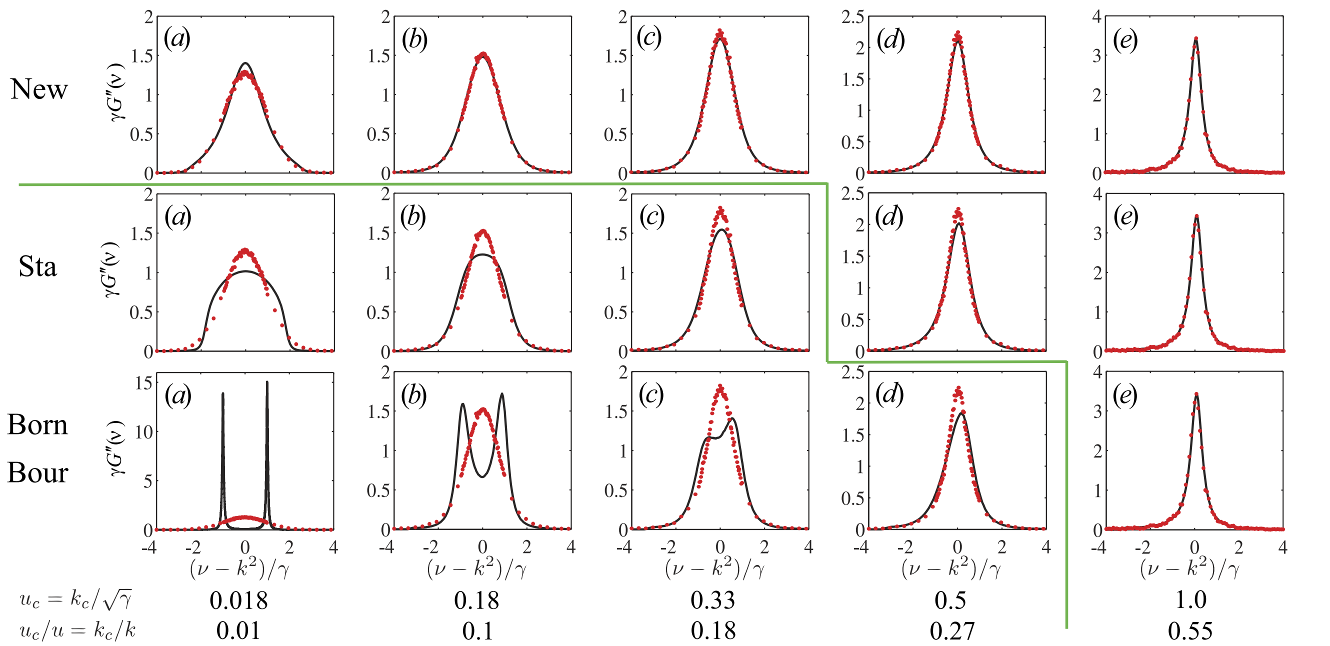

Fig. 2. (Color online) The imaginary part of the Green's function calculated in different approximations (black curves) and its numerical simulation (red dots). The scales of the axes of G are different.

For the numerical simulation, the medium is divided into layers of equal thickness Δx. Oscillations in each i-th layer is described by the homogeneous wave equation, Eq. (12), with the discrete values of the random function $\rho(x_i)$ in the middle of the i-th layer x_i. A solution in a separate layer is

$$\varphi_i = A_i e^{ik_i(x-x_i)} + B_i e^{-ik_i(x-x_i)}, \tag{13}$$

where $k_i = \sqrt{\nu + \gamma\rho(x_i)}$ is a wave number. At the interface between the i-th and $i+1$-th layer, the solution must meet the conditions of continuity of the function φ_i and its derivative. The Green's function must satisfy the radiation conditions and the conditions at the source x_0 [18]

$$G|_{x0-0} = G|_{x0+0}, \quad \left.\frac{dG}{dx}\right|_{x0-0} - \left.\frac{dG}{dx}\right|_{x0+0} = 1. \tag{14}$$

Solutions of equations in layers on the left and right of the source are connected by the products of transfer-matrices [19]. To simulate the random function $\rho(x_i)$ with exponential correlation function, the recursive algorithm [20] was used

$$\rho(x_i) = \sqrt{1-t^2}\,\xi_i + t\rho(x_{i-1}), \quad t = \exp(-k_c\Delta x), \tag{15}$$

where ξ_i is the discrete white Gaussian noise with zero mean and unit dispersion.

A number of realizations sufficient for the convergence of solutions depends on the selected parameters of the problem. With the increase of γ, a greater amount of realizations is required, while with the increase of k_c, we can confine ourselves to a smaller number of realizations. Thus, when the value $k_c/k = 0.01$, it was necessary to average 4.5×10^5 realizations, and for $k_c/k = 0.55$ only 2×10^4 realizations.

Fig. 2 shows the dynamic susceptibility $G''(\omega)$ found analytically in [14] for both the new (top row) and the standard (middle row) SCA and found analytically in this paper in the Bourret (Born) approximation (bottom row). Results of numerical simulation of the problem (red dots in all charts) are also shown in this figure. The calculation was performed for a fixed wave number $u = k/\sqrt{\gamma} = 1.8$. It is seen that a significant narrowing of the resonance curve $G''_k(\nu)$, increasing its height, and change its shape occur with the increase of the dimensionless correlation wave number $u_c = k_c/\sqrt{\gamma}$. Since the value of k is fixed, the increase in u_c also corresponds to an increase of the ratio k_c/k (lower row numbers). It is seen that on the left and below the green line in Fig. 2, the standard SCA unsatisfactorily reproduce both the form and the width of the resonance peak of the function $G''_k(\nu)$ in most part of the studied interval $0 \le u_c \le 1$. Most clearly the advantage of the new approach over the standard is seen at small u_c. The shape of the resonance peak, calculated in the standard SCA, has far from the reality, a domed appearance, and peak width exceeds the width of the exact (calculated by numerical simulation) resonance peak. In contrast, the resonance peak of the imaginary part of the Green function $G''_k(\nu)$, calculated using the new SCA, close to the exact peak as in the shape and width. In the Bourret (Born) approximation, most of the investigated interval corresponds to the physically meaningless solution: two resonance peaks of large amplitude; with an increase in u_c these two peaks converge and merge into a single peak. It is a well known fact, and the Bourret (Born) approximation is shown here to demonstrate that at $u_c \ge 1$ (Fig. 2*e*) the Green's functions obtained using a simple non-self-consistent approximation, the standard SCA, and the new SCA practically coincide with each other and with the results of a numerical experiment.

4. Conclusion

We investigate in the work the properties of the new SCA derived earlier [14] and the accuracy of the results obtained using this approach. A comparison of diagrams, taken into account in the standard, ladder, and new approach demonstrates the advantage of the latter method. A Green's function of waves in a medium with one-dimensional inhomogeneities is found by numerical simulation. It is shown that the Green's function calculated in the new approach, practically coincides with the results of the numerical experiment in the entire investigated interval of variations of the correlation wave number of inhomogeneities k_c. The results obtained with the standard SCA, differ significantly from the results found by numerical simulation method for small values of k_c. It is shown that the use of such sophisticated techniques as both the standard and the new SCA, the ladder approximation, or the numerical simulation gives no advantage at $k_c/\sqrt{\gamma} \ge 1$: virtually the same results can be obtained in a simple (non-self-consistent) Bourret (Born) approximation. At the same time, the new SCA has undoubted advantages in the study of stochastic problems of radio physics in media with longwave inhomogeneities (small k_c). The new SCA in some cases has advantages over the numerical method: the more rapid process of calculation and the possibility of consideration of three-dimensional problems.

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