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Accidental bound states in the continuum in an open Sinai billiard

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ABSTRACT

The fundamental mechanism of the bound states in the continuum is the full destructive interference of two resonances when two eigenlevels of the closed system are crossing. There is, however, a wide class of quantum chaotic systems which display only avoided crossings of eigenlevels. As an example of such a system we consider the Sinai billiard coupled with two semi-infinite waveguides. We show that notwithstanding the absence of degeneracy bound states in the continuum occur due to accidental decoupling of the eigenstates of the billiard from the waveguides.

1. Introduction

If waveguides are attached to a resonator, a billiard, or a quantum dot, *etc.* the bound states of the resonator residing in the propagation band of waveguides become resonant states with finite resonant widths. There could be, however, an exception to this widely accepted rule. According to Friedrich and Wintgen [1] if two eigenlevels pass each other as a function of continuous parameter one of the states can acquire zero resonance width. Thus, this resonance state becomes a bound state in the continuum (BSC). BSCs are localized solutions which correspond to discrete eigenvalues coexisting with extended modes of continuous spectrum in resonator-waveguide configurations. The existence of such modes was first reported in Ref. [2] at the dawn of quantum mechanics. To the best of our knowledge, the term *bound state (embedded) in the continuum* was introduced in [3] in the context of resonance reactions in the presence continuous channels. Since then *bound state in the continuum* has been universally used to designate an BSC in quantum mechanics [4].

An equivalent explication of the BSCs is that a degeneracy of the bound states of the same symmetry occurs under variation of some parameter of the resonator, for example at discrete points of the length of integrable rectangular resonator [5,6]. Then the state superposed from two degenerate eigenstates $a_1\psi_1 + a_2\psi_2$ can be decoupled from the waveguides by variation of the superposition coefficients a_1 and a_2 [5]. Importantly, the full destructive interference of two degenerate eigenmodes of resonator represents a generic mechanism of BSC formation [1,7] whose implementations go far beyond the above two resonance arguments [5,8–14]. Recently such BSCs were experimentally observed in microwave setups [15,16] and around a vertical surface-piercing circular cylinder placed symmetrically between the parallel walls of a water waveguide [17]. The generic model of two interfering resonances was also exploited for electron bound states in the ionized continuum in atomic systems [18–21].

The degeneracy is, however, inherent only to integrable systems. In the present paper we consider the non-integrable Sinai billiard which, as a chaotic system, has only avoided crossings of the eigenlevels [22]. Then the Friedrich–Wintgen mechanism of the BSC due to degeneracy of eigenstates of closed billiard is not applicable. There is another way to realize a BSC by decoupling an individual eigenstate from the waveguides [9,13]. We assume that a circular potential

$$V(x, y) = V_g \exp\left[-\frac{(x-x_0)^2 + (y-y_0)^2}{R^2}\right] \quad (1)$$

is imposed onto the rectangular billiard as shown in Fig. 1. Then irrespective to the parameters of the potential the separable rectangular billiard becomes a nonseparable analog of the chaotic Sinai billiard which we will refer to as the soft Sinai billiard.

We demonstrate in this paper that under the effect of the finger gate potential V_g the eigenfunctions of the Sinai billiard undergo transformations so that one of them can acquire zero overlapping with the propagating functions of the waveguides. As a result the resonance width proportional to the squared coupling constant becomes equal to zero [23].

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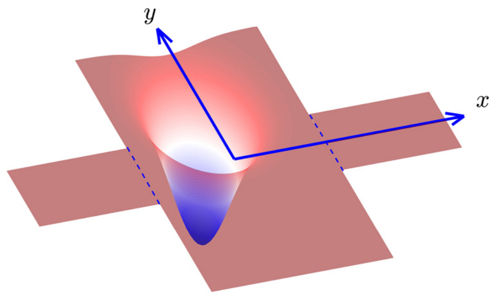

Fig. 1. (Color online.) Potential of circular symmetry (1) with negative V_g is applied onto the open rectangular billiard transforming it into the soft Sinai billiard. The geometrical parameters of the system are $L_x = 3$, $L_y = 4$, $R = 1.5$, $x_0 = 0$, $y_0 = 1$ in terms of the waveguide width.

2. Effective non-hermitian Hamiltonian

The unambiguous tool for detection of the BSCs is the effective non-Hermitian Hamiltonian obtained by the Feshbach projection [25] of the total Hamiltonian onto the inner states of the closed billiard [7,5]. Then, neglecting the dispersion properties of the waveguides the effective non-Hermitian Hamiltonian takes the following Weisskopf–Wigner form [25,28,27,26]

$$\widehat{H}_{eff} = \widehat{H}_B - i\widehat{W}\widehat{W}^+, \tag{2}$$

where $\widehat{H}_B$ is the Hamiltonian of closed billiard and $\widehat{W}$ is the coupling matrix [29–31]

$$W_{b,pC} = \sqrt{\frac{1}{k_p}} \int_0^d dy \phi_p(y) \frac{\partial \psi_b(x,y)}{\partial x}\Bigg|_{x=x_b}, \tag{3}$$

where $C = L, R$ enumerates the interfaces between the left and right waveguides shown in Fig. 1 by dashed lines, ψ_b are the eigenfunctions of the closed billiard. The scattering state within the billiard is given by the Lippmann–Schwinger equation [30,31]

$$(\widehat{H}_{eff} - E)\psi_c = \widehat{W}\psi_{in}, \tag{4}$$

where ψ_{in} is the wave injected via, say, the left waveguide. The solution in the waveguides is given by the reflection and transmission amplitudes of the scattering matrix. The last is given by

$$\widehat{S} = \widehat{W}^+ \frac{1}{\widehat{H}_{eff} - E + i0} \widehat{W}. \tag{5}$$

The resonant properties of open billiards are studied numerically as a rule by the use of the finite difference method. That leads to the tight-binding formulation of the effective Hamiltonian [32,30,31]

$$\widehat{H}_{eff} = \widehat{H}_B - v^2 \sum_{C=L,R} \sum_p \sum_{j_y=1}^{N_W} \exp(ik_p a_0)\phi_p(j_y)\phi_p(j_y)^+ \delta_{j_x, j_C}. \tag{6}$$

Here $\widehat{H}_B$ is the Hamiltonian of the soft Sinai billiard, the vector $\mathbf{j} = (j_x, j_y)$ runs over discretized space $x = a_0 j_x$, $y = a_0 j_y$ where $j_x = 1, 2, \ldots N_x$, $j_y = 1, 2, \ldots N_y$, N_x, N_y are numerical sizes of the rectangular billiard and $N_W = 1/a_0$ is the numerical width of the waveguide, $j_L = 1$, $j_R = N_x$. Wave functions

$$\phi_p(j) = \sqrt{\frac{2}{N_W + 1}} \sin\left(\frac{k_p j}{N_W + 1}\right)$$

are the transverse waveguide solutions with corresponding propagating subbands

$$E = [4 - 2\cos k_p a_0 - 2\cos(\pi p/(N_W + 1))]/a_0^2. \tag{7}$$

The effective Hamiltonian (6) coincides with the Hamiltonian of the billiard everywhere except the interfaces with the waveguides.

3. The Sinai billiard symmetrical relative to $x \to -x$

The eigenfunctions are classified as even and odd $\psi(x, y) = \pm\psi(-x, y)$. Respectively, the eigenvalues in each irreducible representation undergo avoided crossings with variation of V_g as illustrated in Fig. 2. For clarity we show some patterns of the eigenfunctions at $V_g = 50$. A variation of different parameter of the potential (1), for example, the radius or position shows a similar result. Thus, we have no degeneracy of the eigenfunctions of the same irreducible representation in the soft Sinai billiard.

Fig. 3 shows the transmission probability calculated via Eq. (5). For the peaks of the transmittance to follow the eigenvalues of the closed billiard we reduce the coupling between the waveguides and the billiard by choosing $v = 0.5$. Then the resonance widths scale as v^2 [30,33] as clearly seen from Eq. (6). In microwave billiards that is achieved by implementation of diaphragms between the waveguides and the billiard [34].

The BSC occurs if the resonance width turns to zero [1,7]. The resonant width is defined by the imaginary part of the complex eigenvalue z_r of the effective non-Hermitian Hamiltonian (2)

$$\widehat{H}_{eff}|\lambda) = z_\lambda|\lambda) \tag{8}$$

The corresponding eigenmode of the effective non-Hermitian Hamiltonian is a BSC. Fig. 4 shows multiple events of the resonant widths turning to zero, i.e., BSCs in the soft Sinai billiard. The even BSCs sorted by their energies are shown in Fig. 2 (a) by open circles and listed in Table 1. Respectively the odd BSCs are shown in Fig. 2 (b) and listed in Table 2. Besides these BSCs one can see

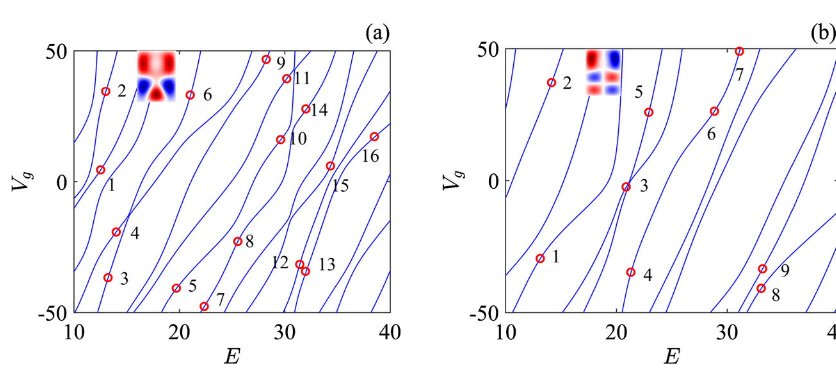

Fig. 2. (Color online.) Eigenvalues of the soft Sinai billiard vs height of the potential (1). The corresponding eigenfunctions are even (a) and odd (b) relative to $x \to -x$. Open circles mark the BSC points listed in Tables 1 and 2.

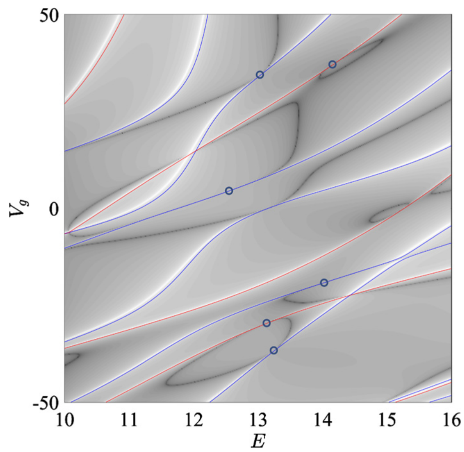

Fig. 3. (Color online.) Transmission probability of the Sinai billiard in Log scale.

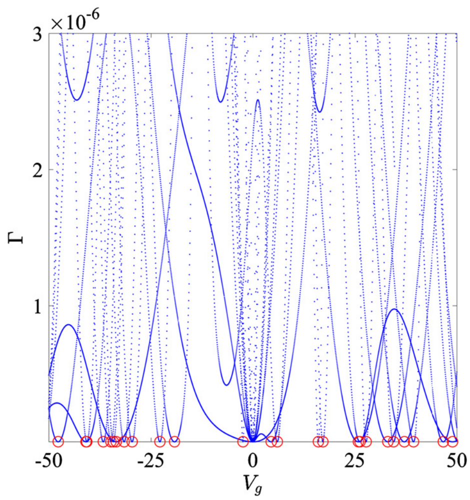

Fig. 4. (Color online.) Evolution of the resonant widths for variation of the potential. Red open circles mark numerous BSCs.

in Fig. 4 numerous symmetry protected BSCs at the point $V_g = 0$ which are the odd eigenfunctions of the rectangular billiard.

Fig. 3 clearly demonstrates that the BSC points are positioned at those points in the parametric space of E and V_g where the transmission zero coalesces with the transmission unit similar to the BSCs due to the Friedrich–Wintgen mechanism [5], i.e., the collapse of Fano resonance occurs [24]. The phenomenon of BSCs stimulated by potential (1) is a result of deformation of the eigenfunctions of the closed billiard [13]. With variation of the potential some of the coupling matrix elements (3) can turn to zero as illustrated in Fig. 5. Let us choose for example the eigenfunction of the closed billiard $b = 3$ with the eigenvalue $E_3 < 4\pi^2$, i.e., below the second propagating band. We have for the coupling matrix (3)

$$W_{b,C} = (W_1\ W_2\ 0\ W_4\ \ldots). \tag{9}$$

Then there is a vector

$$\psi_3^+ = (0\,0\,1\,0\ \ldots) \tag{10}$$

which is the eigen null vector of the matrix $WW^+\psi_{BSC} = 0$. On the other hand, the vector (10) is the eigenvector of the closed billiard with the Hamiltonian

$$\widehat{H}_B = \begin{pmatrix} E_1 & 0 & 0 & 0 & \cdots \\ 0 & E_2 & 0 & 0 & \cdots \\ 0 & 0 & E_3 & 0 & \cdots \\ 0 & 0 & 0 & E_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}, \tag{11}$$

Table 1
BSCs even relative to $x \to -x$ marked by open circles in Fig. 2 (a).

Number of the even BSC	E	V_g
1	12.550	4.5
2	13.029	34.45
3	13.244	−36.65
4	14.026	−19.2
5	19.709	−40.7
6	21.025	33.05
7	22.355	−47.7
8	25.541	−22.8
9	28.236	46.7
10	29.608	16.05
11	30.181	39.35
12	31.418	−31.55
13	31.960	−34.2
14	32.002	27.75
15	34.333	6.00
16	38.495	17.15

Table 2
BSCs odd relative to $x \to -x$ marked by open circles in Fig. 2 (b).

Number of the odd BSC	E	V_g
1	13.133	−29.6
2	14.155	37.1
3	20.882	−2.4
4	21.307	−34.8
5	22.927	25.85
6	28.844	26.25
7	31.099	48.95
8	33.063	−40.9
9	33.189	−33.5

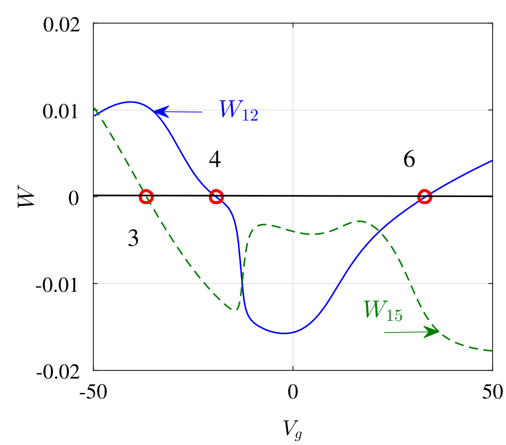

Fig. 5. (Color online.) Evolution of the coupling matrix element (3) with V_g.

with the eigenlevel E_3. Thus the null eigenvector (10) is the eigenstate of the effective non-Hermitian Hamiltonian (2) with real eigenenergy E_3, i.e., the BSC with energy E_3. We emphasize that this statement is true if the coupling matrices (3) coincide for both left and right waveguides, i.e., for the present case of the symmetry relative to $x \to -x$. That result does not depend on choice of the coupling strength v. Following Refs. [9,13,35] we term such eigenstates accidental BSCs. The above consideration is correct only until the evanescent modes $p > 1$ are neglected in Eq. (6). The evanescent modes give an additional Hermitian contribution into the effective Hamiltonian in the form

$$\widehat{\widehat{H}}_B = \widehat{H}_B - v^2 \sum_{p>1} \sum_{C=L,R} \sum_{j_y=1}^{N_W} \exp(-|k_p|a_0)\phi_p(j_y)\phi_p(j_y)^+ \delta_{j_x,j_C}. \tag{12}$$

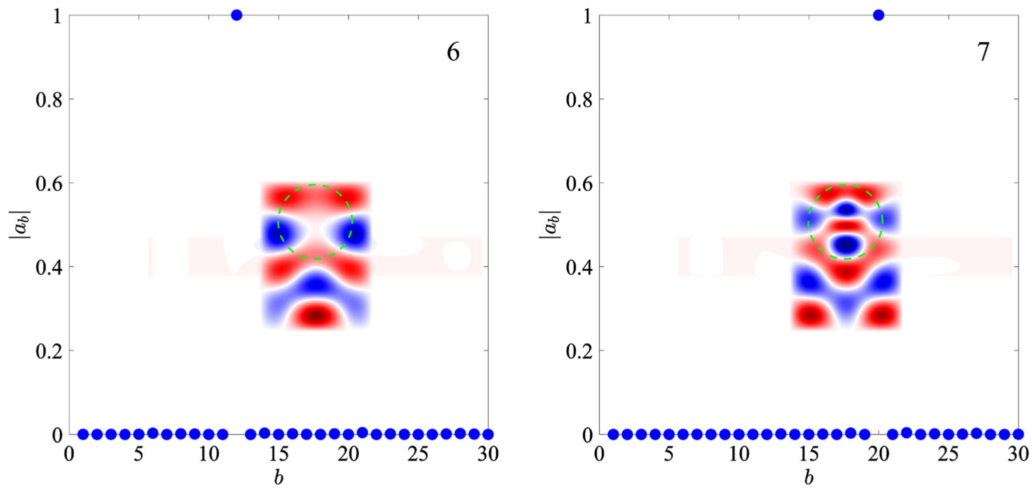

Fig. 6. (Color online.) Patterns of even BSCs 6 (left) and 7 (right) according to Table 1 with coefficients of the modal expansions. Position of potential (1) is shown by dash green circle.

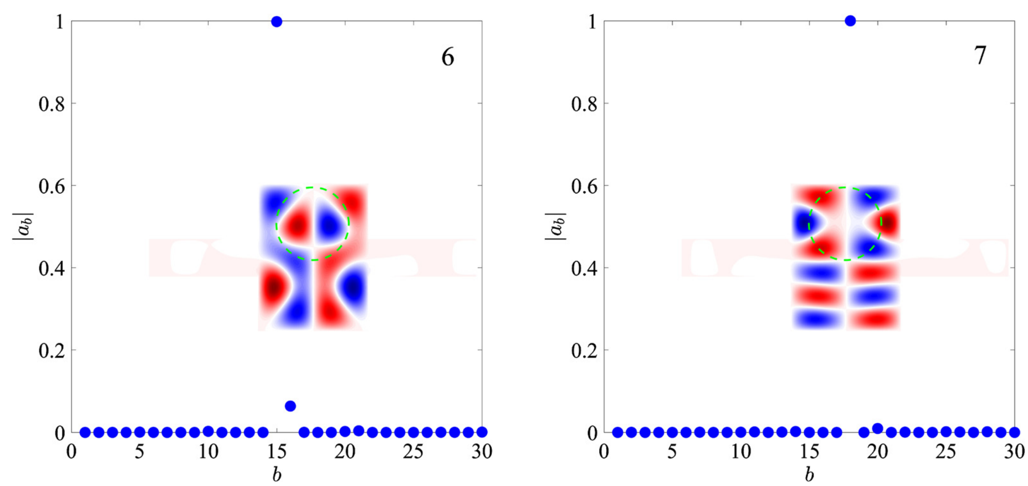

Fig. 7. (Color online.) Patterns of odd BSCs 6 (left) and 7 (right) according to Table 2 with coefficients of the modal expansions.

Then the above arguments for the BSCs in the non-integrable billiard are applicable except that we should identify which eigenstates of the modified Hamiltonian (12) acquire zero coupling with the first propagating mode ϕ_1 under variation of the potential. Respectively such an eigenstate of $\widehat{\widetilde{H}}_B$ becomes the BSC with energy equal to the eigenenergy $\widetilde{E}_b$. Obviously, this eigenenergy differs from former eigenenergy E_3 to give rise to the BSC points slightly different from the eigenenergies in Fig. 2. Some patterns of these BSCs are shown in Figs. 6 and 7 with the modal expansions which clearly show that the BSCs are given by a single dominant eigenfunction of the closed soft Sinai billiard.

4. Summary and conclusions

There are obvious measurement problems associated with BSCs in a quantum billiard. In the case of 2D quantum billiards there is, however, a beautiful way out. It turns out that single-particle states in a hard-wall quantum billiard obey the same stationary Helmholtz equation and same boundary condition as states in a flat microwave resonator [22]. That means a quantum billiard can be emulated by microwave analogs in which the perpendicular electric field plays the role of the wave function. The above said is correct until we include the potential shown in Fig. 1. Nevertheless there is a limited analogy between the quantum mechanical potential (1) and a dielectric disk with dielectric constant ϵ of radius R placed inside the rectangular resonator. Then the Helmholtz equation for the electric field $E_z = \psi(x, y)$ takes the following form

$$-\nabla \frac{1}{\epsilon(x, y)} \nabla \psi = \omega^2 \psi \tag{13}$$

where the light velocity is omitted. After projection of this equation onto the eigenbasis of the rectangular resonator the effective Hamiltonian (2) will take the following form

$$\widehat{H}_{eff} = \widehat{H}_B + \widehat{V} - i\widehat{W}\widehat{W}^+, \tag{14}$$

where the matrix elements of perturbation caused by the dielectric disk can be easily evaluated according to Eq. (13) as follows

$$\langle mn|V|m'n'\rangle = \left(1 - \frac{1}{\epsilon}\right) \int dxdy \nabla \psi_{mn}(x, y) \nabla \psi_{m'n'}(x, y), \tag{15}$$

where integration is performed over the disk. In quantum mechanical billiard with implied potential we would have

$$\langle mn|V|m'n'\rangle = \int dxdy \psi_{mn}(x, y) V(x, y) \psi_{m'n'}(x, y). \tag{16}$$

Although there is no exact equivalence between two types of the perturbation matrices it is clear effect of deformation of the eigenmodes of microwave resonator under the dielectric disk is similar to the effect of the potential (1). In both cases the overlapping of the eigenmodes with the propagating mode in waveguides can be canceled to give rise to the accidental BSC.

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