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The nontrivial ground state topology in the coexistence phase of chiral *d*-wave superconductivity and 120-degree magnetic order on a triangular lattice

V.V. Val'kov *, A.O. Zlotnikov, M.S. Shustin

Kirensky Institute of Physics, Federal Research Center KSC SB RAS, Krasnoyarsk 660036, Russia

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ABSTRACT

The Z_2 topological invariant is defined in the chiral *d*-wave superconductor having a triangular lattice in the presence of the 120-degree magnetic ordering. Analyzing the Z_2 invariant, we determine the conditions of implementing topological phases in the model with regard to superconducting pairings between the nearest and next nearest neighbors. It is often supposed in such a system that the pairing parameter between the nearest neighbors should be equal to zero due to the intersite Coulomb interaction. We show that taking into account even weak pairings in the first coordination sphere leads to the disappearance of the gapless excitations of the bulk spectrum in the wide region of the parameter space. Thus, topological invariants can be defined in this region. In solving the problem of open edges it is shown that the zero energy modes are realized basically in the topologically nontrivial phases. Such zero modes are topologically protected Majorana modes. A connection between the Z_2 invariant and the integer topological invariant of the ground state of the 2D lattice is established in the presence of the electron–hole symmetry and noncollinear magnetic ordering.

1. Introduction

Recently, much attention has been paid to topological superconductors supporting Majorana zero modes. In pioneering works [1,2], such quasiparticles were predicted in the *p*-wave and effective *p*-wave superconductors. However, this type of superconductivity is still rather exotic for real materials. For the systems with *s*-wave pairing several mechanisms have been proposed for the formation of the Majorana zero modes. One of the mechanisms is characterized by the proximity-induced triplet $p_x + ip_y$ pairings on the surface layer of a topological insulator in the *s*-wave superconductor/topological insulator hybrid structures [3]. Another mechanism is connected with the combined influence of strong spin–orbit interaction, proximity-induced superconductivity, and magnetic field [4–6]. In this case the Majorana zero modes arise when the external (or exchange) magnetic field is greater than some critical field.

At present, a new mechanism of the formation of the Majorana edge states in topological spin-singlet superconductors due to the presence of the long-range magnetic order is often considered [7–10]. The symmetry of the superconducting state is considered to be chiral $d_{x^2-y^2} + id_{xy}$ supporting the non-trivial topology and edge states [11]. It should be noted that the time-reversal symmetry is broken in such a state. It is widely believed that the chiral *d*-wave superconductivity may be realized in materials with a triangular lattice (for example, Na_xCoO_2 [8]) and hexagonal lattice (graphene [10]).

For the topological classification of the systems with many degrees of freedom as well as in the systems with strong electron correlations the topological invariant N_3 expressed in terms of the Green functions was derived [12,13]. Using this invariant the quantum topological phase transitions were studied in liquid helium ^{3}He-B [14], semiconducting nanowires [15], and quantum Hall systems. It should be noticed that the N_3 topological invariant is introduced for the gapped ground state in the systems with $2+1$-dimensions [13].

The non-zero values of N_3 indicate the non-trivial topology and the possibility of the edge state formation. For 1D systems with the particle-hole symmetry the well-known Z_2 invariant (Majorana number) was proposed [2]. This invariant expressed in terms of the Pfaffian of the Bogoliubov Hamiltonian in the Majorana representation allows one to study the conditions supporting the Majorana zero modes in the systems with the gapped bulk excitation spectrum. Later, the connection between N_3 and Z_2 numbers was established for the noncentrosymmetric superconductors with

* Corresponding author.
E-mail address: vvv@iph.krasn.ru (V.V. Val'kov).

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the broken time-reversal symmetry [15]. The main result is that the Majorana zero modes is expected to appear in the states with the odd N_3 invariant.

On a triangular lattice the appearance of the Majorana zero modes was demonstrated in Ref. [8] for the coexistence phase of $d_{x^2-y^2}+id_{xy}$-wave superconductivity and non-collinear stripe magnetic ordering. The superconducting pairings between the nearest neighbors were assumed to be suppressed by the intersite Coulomb interaction. Therefore, the pairing interaction between the next nearest neighbors was considered.

Recently, by solving self-consistent integral equations for the coexistence phase in the framework of the $t-J-V$ model it has been shown that in the presence of the stripe magnetic ordering the superconducting order parameter does not have the chiral structure. Thus, the conditions for the realization of the Majorana zero modes on the triangular lattice were analyzed for the coexistence phase of chiral superconductivity and 120° magnetic ordering [17]. In Ref. [17], as well as in Ref. [8], the superconducting pairings in the second coordination sphere were only considered. It turns out that the analysis of the topological phases in such a model is complicated due to the fact that there is a continual range of the parameters for which the bulk excitation spectrum is gapless. This is rather rare since topological indices are usually introduced for a set of parameters in which the bulk excitation spectrum is gapped. Therefore, the edge states with the zero excitation energy have been found in the region with the gapped bulk spectrum.

In this paper we study the conditions supporting the Majorana zero modes on the triangular lattice in the coexistence phase of chiral d-wave superconductivity and 120° spin ordering with regard to the superconducting pairing in the second and first coordination spheres. It is shown that taking into account the pairing between the nearest neighbors with an arbitrarily small amplitude Δ_{21} leads to the disappearance of the continuous parameter region with the gapless bulk excitations. As a result, the Majorana number and N_3 topological invariant for the 2D lattice are calculated. A series of the topological phase transitions upon changing the chemical potential and exchange field is demonstrated. The connection between the N_3 and Z_2 invariants is determined in the presence of noncollinear magnetism. Topologically, the non-trivial phases with the Majorana number equal to -1 and odd N_3 invariant coincide with each other as well as with the parameter regions supporting the Majorana zero modes which are found by solving the problem with open boundary conditions.

2. Model and method

Let us consider the model describing the coexistence phase of chiral superconductivity and noncollinear magnetic order in the mean-field approximation on the triangular lattice. It is assumed that superconductivity is proximity-induced with the $d_{x^2-y^2}+id_{xy}$ symmetry of the order parameter. We consider the superconducting pairings between the nearest and next-nearest neighbors. This assumption allows us to study the topological phases of the system in a relatively simple manner. However, it should be noted that the coexistence phase is caused not only by the proximity effect but also as a consequence of the internal electron interactions [16].

The long-range magnetic ordering is considered in the mean-field approximation assuming that an average magnetic moment $\langle \mathbf{S}_f \rangle = M(\cos(\mathbf{QR}_f), -\sin(\mathbf{QR}_f), 0)$ is formed at the lattice site f. Here, $\mathbf{Q}$ is the magnetic structure vector, M is the average on-site magnetization. Hereinafter, we consider the 120° spin ordering with $\mathbf{Q}=(Q,Q), Q=2\pi/3$ and define the coordinates in the real and quasi-momentum space as $\mathbf{R}_f = n\mathbf{a}_1 + m\mathbf{a}_2, \mathbf{k} = k_1\mathbf{b}_1 + k_2\mathbf{b}_2$, where $\mathbf{a}_i$ and $\mathbf{b}_i$ are the basic and reciprocal vectors of the triangular lattice, respectively. The Hamiltonian has the form:

$$H = -\mu\sum_{f\sigma} c^{\dagger}_{f\sigma}c_{f\sigma} + \sum_{fm\sigma} t_{fm} c^{\dagger}_{f\sigma}c_{m\sigma} + h(\mathbf{Q})\sum_{\mathbf{f}}\Big(\exp(\mathbf{iQR_f})\mathbf{c}^{\dagger}_{\mathbf{f}\uparrow}\mathbf{c}_{\mathbf{f}\downarrow} + \exp(-\mathbf{iQR_f})\mathbf{c}^{\dagger}_{\mathbf{f}\downarrow}\mathbf{c}_{\mathbf{f}\uparrow}\Big) + \sum_{\mathbf{fm}}\Big(\Delta_{\mathbf{fm}}\mathbf{c}_{\mathbf{f}\uparrow}\mathbf{c}_{\mathbf{m}\downarrow} + \Delta^{*}_{\mathbf{fm}}\mathbf{c}^{\dagger}_{\mathbf{m}\downarrow}\mathbf{c}^{\dagger}_{\mathbf{f}\uparrow}\Big), \quad (1)$$

where μ is the chemical potential, t_{fm} and Δ_{fm} are the electron hopping and superconducting pairing amplitudes, respectively. The exchange field parameter is defined as follows:

$$h(\mathbf{Q}) = \mathbf{M/2}\sum_{\mathbf{m}} \mathbf{I_{fm}}\exp(-\mathbf{iQ}(\mathbf{R_f}-\mathbf{R_m})). \quad (2)$$

I_{fm} is the parameter of the exchange interaction, being considered within the two coordination spheres. An important difference between the system (1) and the model studied in [17] is the consideration of the superconducting pairings between the nearest neighbors. Hereinafter, the model is considered both in the case of the periodic boundary conditions along the direction $\mathbf{a}_2$ (the cylinder topology), and in the case of the periodic boundary conditions in two spatial directions (the torus topology). In both cases, the operator part of the Hamiltonian (1) has the form:

$$H = \frac{1}{2}\sum_k \mathbf{C}(k)^{+}\cdot H(k)\cdot \mathbf{C}(k),$$

$$H(k) = \begin{pmatrix} \xi_k & h & 0 & \Delta_k \\ h^{+} & \xi_{k-Q} & -\Delta^{T}_{-k+Q} & 0 \\ 0 & -\Delta^{*}_{-k+Q} & -\xi^{*}_{-k+Q} & -h^{+} \\ \Delta^{+}_{k} & 0 & -h & -\xi^{*}_{-k} \end{pmatrix}, \quad (3)$$

where $\mathbf{C}(k) = \left(\mathbf{c}_{k\uparrow}, \mathbf{c}_{k-Q\downarrow}, \mathbf{c}^{+}_{-k+Q\uparrow}, \mathbf{c}^{+}_{-k\downarrow}\right)^{T}$.

In the case of the cylinder topology the operator $\mathbf{C}$ has $4N_1$ components, where N_1 is the number of sites along the $\mathbf{a_1}$ direction. Then, the N_1 by N_1 matrices $\hat{\xi}_k, \hat{\Delta}_k$ and $\hat{h}$ in the Hamiltonian (3) have the form ($k\equiv k_2$):

$$\hat{\xi}_{k_2} = \begin{pmatrix} t_{k_2}-\mu & T_{k_2} & \Gamma_{k_2} & 0 & 0 \\ T_{-k_2} & \ddots & \ddots & \ddots & 0 \\ \Gamma_{-k_2} & \ddots & \ddots & \ddots & \Gamma_{k_2} \\ 0 & \ddots & \ddots & \ddots & T_{k_2} \\ 0 & 0 & \Gamma_{-k_2} & T_{-k_2} & t_{k_2}-\mu \end{pmatrix},$$

$$\hat{h} = h\cdot \mathrm{diag}(e^{iQ}, e^{2iQ}, \ldots, e^{N_1 iQ}),$$

$$\hat{\Delta}_{k_2} = -\begin{pmatrix} \tilde{\Delta}^{*}_{k_2} & \psi^{*}_{-k_2} & \Delta^{*}_{22}e^{ik_2} & 0 & 0 \\ \psi^{*}_{k_2} & \ddots & \ddots & \ddots & 0 \\ \Delta^{*}_{22}e^{-ik_2} & \ddots & \ddots & \ddots & \Delta^{*}_{22}e^{ik_2} \\ 0 & \ddots & \ddots & \ddots & \psi^{*}_{-k_2} \\ 0 & 0 & \Delta^{*}_{22}e^{-ik_2} & \psi^{*}_{k_2} & \tilde{\Delta}^{*}_{k_2} \end{pmatrix}. \quad (4)$$

Here

$$t_{k_2} = 2t_1\cos(k_2) + 2t_3\cos(2k_2), \tilde{\Delta}_{k_2} = 2\Delta_{21}\cos(k_2),$$
$$T_{k_2} = t_1(1+\exp(ik_2)) + t_2(\exp(-ik_2) + \exp(2ik_2)),$$
$$\Gamma_{k_2} = t_2\exp(ik_2) + t_3(1+\exp(i2k_2)),$$
$$\Psi_{k_2} = \Delta_{22}\exp(i2\pi/3)(\exp(i2k_2) + \exp(i2\pi/3 - ik_2)) + \Delta_{21}\exp(i2\pi/3)(1+\exp(i2\pi/3+ik_2)),$$

and t_1, t_2, t_3 are the hopping parameters for the first, second, and third coordination spheres. The parameters Δ_{21} and Δ_{22} denote the amplitudes of the superconducting pairings of the d-wave symmetry (angular momentum $l = 2$) which are implemented between the nearest and next-nearest neighbors, respectively.

The eigenvalues and eigenstates of the Hamiltonian (3) determine the spectrum of elementary excitations as well as the amplitudes of the Bogoliubov quasiparticles:

$$\alpha_{k_2 j} = \sum_{n=1}^{N_1} \left(A_{jn,k_2} c_{nk_2\uparrow} + B_{jn,k_2} c_{n,k_2-Q_2\downarrow} + C_{jn,k_2} c^{\dagger}_{n,-k_2\downarrow} + D_{jn,k_2} c^{\dagger}_{n,-k_2+Q_2\uparrow} \right). \tag{5}$$

Considering the lattice with the torus topology the value of h is determined by the expression (2), $\xi_k \equiv t_{\mathbf{k}} - \mu, \Delta_k \equiv -\Delta^*_{\mathbf{k}}$, and functions $t_{\mathbf{k}}$ and $\Delta_{\mathbf{k}}$ are the Fourier transforms of the hopping integral and the superconducting order parameter with the $d_{x^2-y^2} + id_{xy}$ symmetry type, respectively. It should be noted that the pairing interaction in the first coordination sphere is considered to be sufficiently suppressed by the inter-site Coulomb interaction, so $\Delta_{21} \ll \Delta_{22}$ [8,18].

3. Hamiltonian symmetry and Z_2 topological invariant

Regardless of the consideration of the system with the cylinder or torus topology the Hamiltonian (3) has the symmetry:

$$\Lambda H(k) \Lambda = -H^*(-k+Q), \quad \Lambda = \begin{pmatrix} \hat{0} & \hat{I} \\ \hat{I} & \hat{0} \end{pmatrix}, \tag{6}$$

where $\hat{0}$ and $\hat{I}$ are the zeros and identity matrices of the corresponding size (2×2 for the torus topology and $2N_1 \times 2N_1$ for the cylinder topology). Due to this symmetry the eigenvalues of the Hamiltonian $H(k)$ are grouped in pairs $\varepsilon_n(k)$ and $-\varepsilon_n(-k+Q)$. Following the paper [15], let us consider the particle-hole invariant momenta K (PHIM points) of the Brillouin zone when $K = -K + Q + G$ holds, where G is a reciprocal-lattice vector. At these points the Hamiltonian has the particle-hole symmetry. In the case of the cylinder topology $K_2 = -2\pi/3; \pi/3$, while in the case of the torus topology we have four PHIM points $\mathbf{K} = (-2\pi/3, -2\pi/3)$; $(-2\pi/3, \pi/3)$; $(\pi/3, -2\pi/3)$; $(\pi/3, \pi/3)$. Then, we can define the matrices

$$W(k) = H(k)\Lambda, \quad \widetilde{W}(k) = R^T W(k) R,$$
$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{I} & -i\hat{I} \\ \hat{I} & i\hat{I} \end{pmatrix},$$

which satisfy the relations $W(k) = -W^T(-k+Q), \widetilde{W}(k) = -\widetilde{W}^T(-k+Q)$. These matrices are antisymmetric at the PHIM points. It can be shown that the matrix $\widetilde{W}$ coincides with the Hamiltonian (3) at the PHIM points if the Majorana representation is used in the expression (3):

$$c_{k\sigma} = \gamma_{Ak\sigma} + i\gamma_{Bk\sigma}; \; c^{+}_{k\sigma} = \gamma_{A-k\sigma} - i\gamma_{B-k\sigma}. \tag{7}$$

Thus, following Kitaev [2], one can introduce the Z_2 Pfaffian invariant $M(K_2)$ in the cylinder topology only for the PHIM points $K_2 = -2\pi/3, \pi/3$:

$$M(K_2) = P(K_2, K_1 = -2\pi/3) P(K_2, K_1 = \pi/3), \tag{8}$$

where $P(\mathbf{K})$ is the fermionic parity of the ground state of the system with the torus topology:

$$P(\mathbf{K}) = \text{sign}\left(\text{Pf}\left(-i\widetilde{W}(\mathbf{K})\right)\right). \tag{9}$$

If $M(K_2) = -1$ the system is in the topologically nontrivial phase supporting the Majorana zero modes. Otherwise, if $M(K_2) = 1$ the ground state is topologically trivial and there are no topologically protected edge states with the zero excitation energy. Note that the bulk spectrum should be gapped to define the Majorana number. In general, $P(\mathbf{K}) = \text{sign}\left(h^2 - \xi^2_{\mathbf{K}} - |\Delta_{\mathbf{K}}|^2\right)$ and it can be shown by direct calculations that $P(K_2 = \pi/3, K_1 = -2\pi/3) = P(K_2 = \pi/3, K_1 = \pi/3)$. It means that at $K_2 = \pi/3$ there are no Majorana zero modes in the system considering the cylinder topology.

At $K_2 = -2\pi/3$ the Majorana number is defined by the relation

$$M = \text{sign}\left(\left(h^2 - (3t_1 - 6t_2 + 3t_3 + \mu)^2\right) \cdot \left(h^2 - (t_1 - 2t_2 - 3t_3 - \mu)^2 - 4(2\Delta_{22} - \Delta_{21})^2\right)\right). \tag{10}$$

In Fig. 1 the parameters for which the Majorana number (10) changes the sign are depicted by the bold lines. As is shown in the last section the bulk excitation spectrum becomes gapless for these parameters. The conditions for which the gapless elementary excitations occur on the triangular lattice with the cylinder topology and $N_1 = 48$ are shown by thin lines (curves $h(\mu)$). It can be seen that the majority of the zero modes lies in the topologically nontrivial phase. As N_1 increases, the distribution of the zero energy curves becomes dense and all of them are found in the topologically nontrivial phases with $M = -1$. The more N_1 decreases, the more zero modes appear in the topologically trivial phase. These modes are not topologically protected. This indicates that the correspondence between the bulk and boundary is well established when considering a sufficiently large number of sites. The set of the parameters is chosen as $\Delta_{21} = 0.05|t_1|, \Delta_{22} = 0.3|t_1|, t_2 = t_3 = 0$.

It should be noted that changing the parameters in the topologically nontrivial phase leads to the oscillations of the minimal excitation energy ε_0 and its dropping to zero on the lines of the zero modes (thin lines in Fig. 1). At the points where $\varepsilon_0 = 0$ the quantum phase transition is realized: the ground state containing a

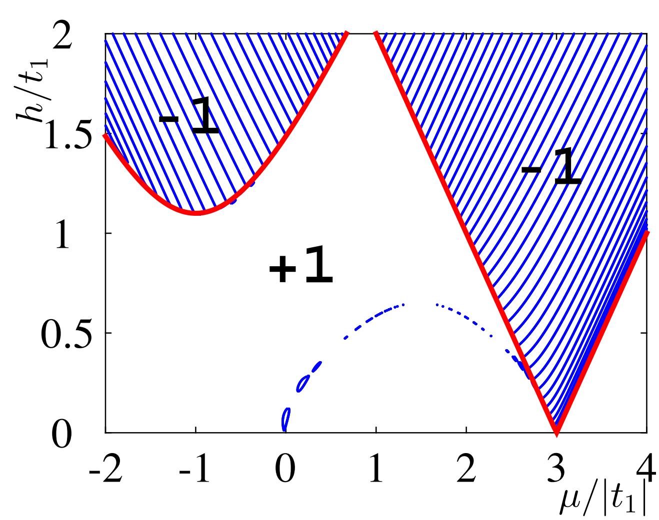

Fig. 1. The conditions for the realization of the zero energy excitations obtained with the consideration of the periodic boundary conditions along $\mathbf{a}_2$ (thin blue lines), h is the exchange field, μ is the chemical potential, t_1 is the hopping parameter between the nearest neighbors. The parameters are $t_1 < 0, t_2 = t_3 = 0, \Delta_{22} = 0.3|t_1|, \Delta_{21} = 0.05|t_1|, N_1 = 48$. The red bold lines show a boundary between the phases with different values of the topological Z_2 invariant $M = \pm 1$.

superposition of states with an even number of fermions is replaced by a state with an odd number of fermions and vice versa. Such switching of the fermionic parity was obtained for the Kitaev model [19] and it may be a general property of the finite quasi-one-dimensional systems in the topologically nontrivial phase.

It is sufficient that all the zero modes shown in Fig. 1 are the edge ones. This is an important difference from the case $\Delta_{21} = 0$ considered in [17] where the continual region with the bulk gapless excitations appears in the space of the parameters h and μ. As a result, the zero energy modes of the system with the cylinder topology which are found in this region are not the edge ones and represent the bulk excitations modified due to the boundary effects. With regard to the weak nearest neighbor superconducting pairing $\Delta_{21} \ll \Delta_{22}$ all the zero modes in the topologically nontrivial phases become the edge ones.

In Figs. 2, 3 the realization of the edge states is demonstrated for $\Delta_{21} \neq 0$ in the topological phase. We consider the dependence of the site-dependent parameter

$$p_n(h) = |A_{0n,K_2}|^2 + |B_{0n,K_2}|^2 + |C_{0n,K_2}|^2 + |D_{0n,K_2}|^2 \quad (11)$$

on the exchange field h and number of sites n. The Bogoliubov coefficients appearing in (11) correspond to the elementary excitation with a minimal energy ε_0 and $K_2 = -2\pi/3$. Such dependencies are shown for $\mu = 2|t_1|$ and $\mu = 4|t_1|$. Other parameters are the same as in Fig. 1. In these cases the transition between the topologically trivial and nontrivial phases corresponds to $h = t_1$. It is seen in Fig. 3 that the edge states including the Majorana zero modes are realized in the phase with $M = -1$ of the Z_2 topological invariant (8). In Fig. 2 the edge states and zero modes are found even in the phase with $M = 1$. As it is shown in the next section the edge states can exist in this region but the zero modes are not topologically protected. Upon increasing N_1 the Majorana zero modes become more localized at the edges.

The following two features deserve mentioning. First, the topologically protected edge states with the non-zero excitation energy can be realized even if the value of the Z_2 invariant corresponds to the topologically trivial phase ($M = 1$, see Fig. 2). This result is in agreement with the calculation of the N_3 invariant for the 2D system considered below. Second, in the considered system the edge states with the zero energy can be realized with the quasi-momenta $k_2 \neq -2\pi/3$ but such states are not topologically protected ones.

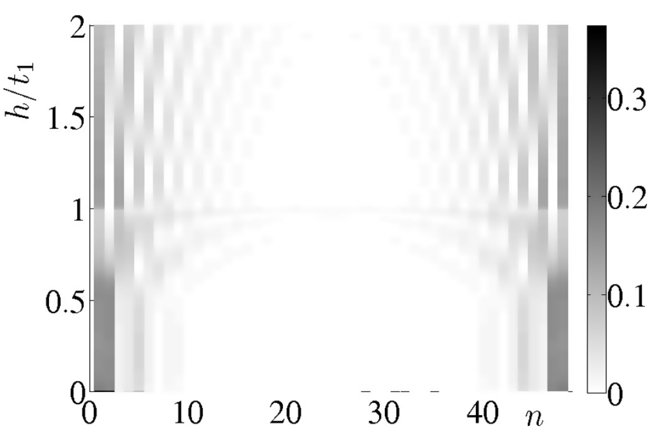

Fig. 2. Spatial distribution of the sum of the Bogoliubov coefficients $p_n(h)$ (color bar) vs exchange field h at $\mu/|t_1| = 2$. Other parameters are the same as in Fig. 1. The darkest and lightest areas correspond to the largest and smallest values of p_n, respectively.

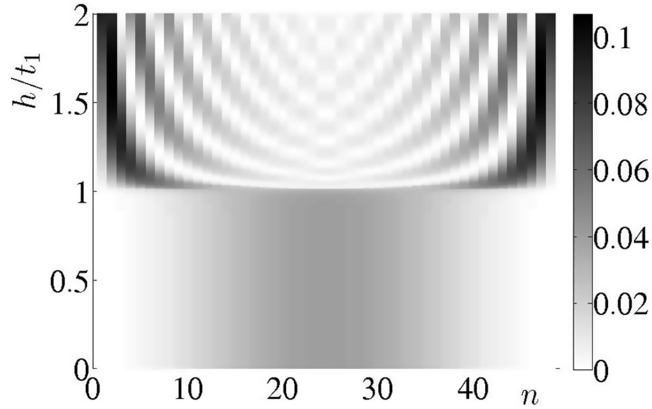

Fig. 3. Spatial distribution of the sum of the Bogoliubov coefficients $p_n(h)$ (color bar) vs exchange field h at $\mu/|t_1| = 4$. Other parameters are the same as in Fig. 1. The darkest and lightest areas correspond to the largest and smallest values of p_n, respectively.

4. The topological invariant N_3 of the 2D lattice and its connection with the Z_2 invariant. The analysis of the bulk spectrum

It is known that topological transitions changing the topological index occur when the gap closes in the bulk spectrum. For the system under consideration, the bulk spectrum has the form:

$$E^{\pm}_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(\xi^2_{\mathbf{k}} + \xi^2_{\mathbf{k-Q}} + 2h^2 + |\Delta_{\mathbf{k}}|^2 + \left|\Delta_{\mathbf{-k+Q}}\right|^2\right) \pm v^2_{\mathbf{k}}}, \quad (12)$$

where

$$v^2_{\mathbf{k}} = \left\{\frac{1}{4}\left(\xi^2_{\mathbf{k}} - \xi^2_{\mathbf{k-Q}} + |\Delta_{\mathbf{k}}|^2 - |\Delta_{\mathbf{-k+Q}}|^2\right) + h^2\left[\left(\xi_{\mathbf{k}} + \xi_{\mathbf{k-Q}}\right)^2 + \left|\Delta_{\mathbf{k}} + \Delta_{\mathbf{-k+Q}}\right|^2\right]\right\}^{1/2}. \quad (13)$$

The conditions for the zero energy in the bulk spectrum are described by the equation:

$$|h^2 - \xi_{\mathbf{k}}\xi_{\mathbf{k-Q}} - \Delta_{\mathbf{k}}\Delta^*_{\mathbf{-k+Q}}|^2 + |\xi_{\mathbf{k}}\Delta_{\mathbf{-k+Q}} - \xi_{\mathbf{k-Q}}\Delta_{\mathbf{k}}|^2 = 0.$$

At the PHIM points $\mathbf{K} = -\mathbf{K} + \mathbf{Q} + \mathbf{G}$ the second term in the equation is also equal to zero, and the first term is the same as $\left(\mathrm{Pf}(\widetilde{W}(\mathbf{K}))\right)^{\mathbf{2}}$. Thus, at the symmetric points of the Brillouin zone the change in the sign of the Majorana number (8), as it should be, is accompanied by the existence of the zero energy in the bulk spectrum at these points.

In the case when $\mathbf{k} \neq -\mathbf{k} + \mathbf{Q}$, the equations determining the conditions for the realization of the gapless bulk excitations have the form:

$$\begin{array}{l} h^2 - \xi_{\mathbf{k}}\xi_{\mathbf{k-Q}} - \mathrm{Re}(\Delta_{\mathbf{k}}\Delta^*_{\mathbf{-k+Q}}) \\ |\xi_{\mathbf{k}}\Delta_{\mathbf{-k+Q}} - \xi_{\mathbf{k-Q}}\Delta_{\mathbf{k}}| \\ \mathrm{Im}(\Delta_{\mathbf{k}}\Delta^*_{\mathbf{-k+Q}}) \end{array} \quad (14)$$

The formation of the gapless bulk excitations at the non-PHIM points according to the solution of Eqs. (14) also leads to topological phase transitions. However, at this transition the Z_2 invariant (8) does not change. A characteristic which allows one to identify such transitions in two-dimensional systems (including the systems with the interaction) is the topological invariant of the ground state introduced in Ref. [12]:

$$N_3 = \frac{1}{24\pi^2}\varepsilon_{\mu\nu\lambda}\int_{-\infty}^{\infty}d\omega\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}dk_1dk_2\mathrm{Tr}\left(G\partial_{\mu}G^{-1}\cdot G\partial_{\nu}G^{-1}G\partial_{\lambda}G^{-1}\right), \quad (15)$$

where $\mu, \nu, \lambda = 1, 2, 3, \varepsilon_{\mu\nu\lambda}$ is antisymmetric Levi-Civita tensor, $\partial_1 = \partial/\partial_{k_1}, \partial_2 = \partial/\partial_{k_2}, \partial_3 = \partial/\partial_\omega$. By the repeated indices we mean the summation. In the system of non-interacting electrons the matrix Green function G is a matrix 4×4 for the torus topology and it has the form $G = [i\omega I - H(\mathbf{k})]^{-1}$.

The non-zero integer values of the invariant N_3 (15) determine the topologically nontrivial phases in which the edge states can form. In Ref. [15] a connection between the Z_2 invariant (8) and topological invariant N_3 (15) was established in the case of noncentrosymmetric systems with the broken time-reversal symmetry though preserving the electron–hole symmetry. It is shown that the product of the Majorana numbers (8) at the points $\mathbf{k} = -\mathbf{k} + \mathbf{G}$ coincides with the parity of the topological index N_3. In the system with magnetic ordering this relation is generalized:

$$(-1)^{N_3} = \mathrm{sign}\left(\prod_{\mathbf{K}=-\mathbf{K}+\mathbf{Q}+\mathbf{G}} \mathrm{Pf}\left(\widetilde{W}(\mathbf{K})\right) \right). \tag{16}$$

The phase diagram with different topological phases in the space of the chemical potential μ and exchange field h is shown in Fig. 4. In each phase the values of the topological invariant N_3 are marked. The solid lines defining the boundaries between different topological phases are obtained as the solutions of the system of Eqs. (14) which are determined in the presence of the bulk gapless excitations. The parameters are the same as in Fig. 1. It should be noted that this invariant is ill-defined at the topological transition point. In the vicinity of the transition the calculation of the invariant requires increased accuracy. As can be seen from Fig. 4 the increase of the chemical potential leads to a series of topological transitions. The topologically trivial phase with $N_3 = 0$ at $\mu < -2|t_1|$ is implemented when the chemical potential intersects the bottom of the bare electron band (not shown in the Figure).

As is well known, the difference between the values of the N_3 invariant in the neighboring phases determines the values of the topological invariants of the Fermi points in which the bulk spectrum has the zero energy at the transition between the phases. In the model under consideration the invariants of the Fermi points are equal to ± 1. There is only one exception at the transition between the phase with $N_3 = 0$ to the phase with $N_3 = 4$ at negative μ (not shown in Fig. 4) where the invariant of each of the two Fermi points is 2. Thus, in other cases, the difference corresponds to the number of the nodal points of the bulk spectrum at the topological transition.

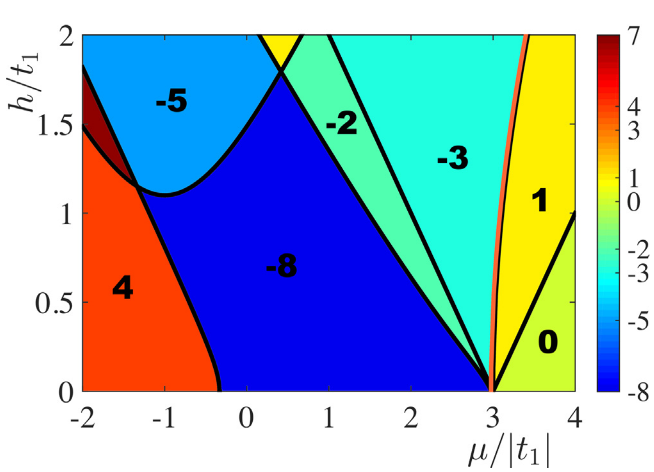

Fig. 4. The diagram of the topological phases with different N_3 (15) in the variables h, μ, where h is the exchange field, μ is the chemical potential. The parameters are the same as in Fig. 1. The phases with the odd N_3 value correspond to the phases with the Majorana zero modes in Fig. 1 having $M = -1$ (8).

The excitation spectrum in the coexistence phase of superconductivity and noncollinear magnetic order differs from the spectrum in the superconducting phase. Moreover, the spectrum in the coexistence phase is determined by two superconducting order parameters $\Delta_{\mathbf{k}}, \Delta_{-\mathbf{k}+\mathbf{Q}}$, which have different systems of the nodal points. This leads to several significant differences in the analysis of the zeros of the bulk spectrum. The spectrum in superconductors has zero energy at the boundaries and in the middle of the Brillouin zone only when the chemical potential intersects the bottom or the top of the bare electron band. In the coexistence phase due to the exchange field the spectrum has zero energy at these points, when the chemical potential lies inside the band. Such a picture can be seen in Fig. 4 at the transition from the phase with $N_3 = -2$ to the phase with $N_3 = -3$ when the gap closes at the point $(-2\pi/3, -2\pi/3)$ under the condition $h = |\mu + 3t_1 - 6t_2 + 3t_3|$. This point corresponds to the one of the nonequivalent points lying at the intersection of the edges of the hexagonal Brillouin zone. In this case $\Delta_{\mathbf{k}} = \Delta_{-\mathbf{k}+\mathbf{Q}} = 0$. The second analogous transition is realized between the phases with $N_3 = 3$ and $N_3 = 1$ when the spectrum becomes gapless at the points $(0, 0)$ and $(2\pi/3, 2\pi/3)$. At small values of Δ_{21} the phase with $N_3 = 3$ is rather narrow and lies between the phases with $N_3 = -3$ and $N_3 = 1$. This narrow phase is schematically shown in Fig. 4. For a superconductor without magnetic ordering the intersection of the nodal points of the superconducting order parameter by the Fermi contour leads to the gapless excitations. When noncollinear magnetism is taken into account this condition is not satisfied due to the parameter $\Delta_{-\mathbf{k}+\mathbf{Q}}$. However, there are conditions when the energy spectrum is equal to zero at the points in which $\Delta_{\mathbf{k}}, \Delta_{-\mathbf{k}+\mathbf{Q}} \neq 0$. This picture corresponds to the remaining transitions in Fig. 4. It should be noted that disregarding Δ_{21}, the relation $\Delta_{\mathbf{k}} = \Delta_{-\mathbf{k}+\mathbf{Q}}$ is satisfied, and, as a result, the energy spectrum is considerably simplified. When Δ_{21} is taken into account the condition $\Delta_{\mathbf{k}} = \Delta_{-\mathbf{k}+\mathbf{Q}} \neq 0$ is valid only at the points $(-2\pi/3, \pi/3), (\pi/3, -2\pi/3), (\pi/3, \pi/3)$. The zeros of the spectrum at these points are realized, for example, at the transition from the phase with $N_3 = -8$ to the phase with $N_3 = -5$ upon increasing h.

From Eq. (16) we conclude that the Majorana modes exist in the phases with the odd N_3. The transition to such phases is accompanied by the gap closing in the bulk spectrum in the odd number of points in the Brillouin zone. In the phases with the even N_3 the edge states can arise but the topologically protected zero modes are not found. This agrees with the calculation results shown in Figs. 1–3. These conclusions indicate that the definition of the topological invariant (15) allows one to search for possible conditions of the realization of the Majorana modes in electron systems with interaction and magnetic order.

5. Conclusions

The topological properties of the coexistence phase of the $d_{x^2-y^2} + id_{xy}$-wave superconductivity and noncollinear 120° magnetic ordering on a triangular lattice are studied. When the superconducting pairings are taken into account only in the second coordination sphere, the gap in the bulk excitation spectrum is closed in the continuous region of the parameter space. This feature means that the topological invariants cannot be introduced in a standard way, in spite of the fact that the edge zero modes are found in the system. In the present work it is shown that taking into account the arbitrarily small superconducting amplitude, induced by the pairing interaction in the first coordination sphere, leads to a gap opening in the bulk spectrum. The bulk spectrum becomes gapless only on the boundaries between topologically different phases. This allows us to introduce the Z_2 topological invariant M (the Majorana number) and to analytically determine the conditions of implementing the topologically nontrivial phases with $M = -1$.

Considering the triangular lattice with the periodic boundary conditions along the basic vector $\mathbf{a}_2$, the zero modes are found to exist on the specific curves in the parameter space of the chemical potential and exchange field. It is shown that for the lattice with a finite number of sites N_1 along the direction $\mathbf{a}_1$ the zero modes can arise in the topologically trivial phase. Such zero modes are not topologically protected. However, the majority of the zero modes (the Majorana modes) at large N_1 is found in the topologically nontrivial phase with $M = -1$.

The topological invariant N_3 of the 2D lattice expressed in terms of the Green functions is calculated for the coexistence phase. We find a series of topological transitions in the coexistence phase upon increasing the chemical potential. The relationship between two topological invariants M and N_3 is determined with regard to the noncollinear magnetism. It is shown that the topologically nontrivial phases with the Majorana number equal to -1 correspond to the phases with the odd N_3. In the topologically nontrivial phases with the even N_3 the edge states can exist but they cannot be the topologically protected Majorana edge states with the zero excitation energy.

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References

[1] N. Read, D. Green, Phys. Rev. B 61 (2000) 10267.
[2] A.Yu. Kitaev, Physics-Uspekhi 44 (Suppl.) (2001) 131.
[3] L. Fu, C.L. Kane, Phys. Rev. Lett. 100 (2008) 096407.
[4] M. Sato, Y. Takahashi, S. Fujimoto, Phys. Rev. Lett. 103 (2009) 020401.
[5] J.D. Sau, R.M. Lutchyn, S. Tewari, S. Das Sarma, Phys. Rev. Lett. 104 (2010) 040502.
[6] J. Li, H. Chen, I.K. Drozdov, A. Yazdani, B.A. Bernevig, A.H. MacDonald, Phys. Rev. B. 90 (2014) 235433.
[7] I. Martin, A.F. Morpurgo, Phys. Rev. B 85 (2012) 144505.
[8] Y.-M. Lu, Z. Wang, Phys. Rev. Lett. 110 (2013) 096403.
[9] A. Gupta, D. Sa, Solid State Commun. 203 (2015) 41.
[10] A.M. Black-Schaffer, K. Le Hur, Phys. Rev. B 92 (2015) 140503.
[11] G.E. Volovik, JETP Lett. 66 (1997) 522.
[12] G.E. Volovik, V.M. Yakovenko, J. Phys. Condens. Matter 1 (1989) 5263.
[13] G.E. Volovik, The Universe in a Helium Droplet, Oxford Press, 2003.
[14] G.E. Volovik, JETP Lett. 90 (2009) 398.
[15] P. Ghosh, J.D. Sau, S. Tewari, S. Das Sarma, Phys. Rev. B 82 (2010) 184525.
[16] V.V. Val'kov, A.O. Zlotnikov, JETP Lett. 104 (2016) 483.
[17] V.V. Val'kov, A.O. Zlotnikov, A.D. Fedoseev, M.S. Shustin, J. of Magn. Magn. Mater. 440 (2017) 47.
[18] V.V. Val'kov, T.A. Val'kova, V.A. Mitskan, JETP Lett. 102 (2015) 361.
[19] S. Hegde, S. Vishveshwara, Phys. Rev. B 94 (2016) 115166.