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Multi-channel bound states in the continuum in coaxial cylindrical waveguide

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Abstract

Bound states in the continuum (BICs) or embedded trapped modes are widely studied in different physical systems. The studies are restricted to a single open scattering channel. In the present paper we consider BICs embedded into several continua in a cylindrical resonator opened by two coaxially attached cylindrical waveguides with different radii. We demonstrate that engineering the BICs requires a degeneracy of three eigenmodeds of the closed resonator. That is achieved by variation of both the length and the radius of the resonator.

Keywords: bound states in the continuum, embedded trapped modes, cylindrical resonator, effective Hamiltonian

(Some figures may appear in colour only in the online journal)

1. Introduction

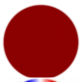
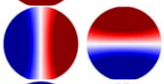
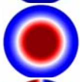
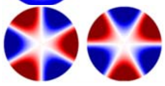
Bound states in the continuum (BICs), also known in the field of acoustics as embedded trapped modes, are solutions localized despite of the fact that they coexist with the continuous spectrum of propagating waves. This concept was originally proposed by von Neumann and Wigner [1] in 1929, in the context of quantum mechanics, and has recently become a topic of interest in optics and acoustics [2] because of the possibility to significantly increase the quality factor of the resonators due to complete [3, 4] or partial [5] destructive interference of modes leaking from the resonator.

One of the first researchers who pointed out the possibility of the existence of localized solutions with discrete frequencies embedded into the continuum (BICs) was Parker. He showcased BICs in the air stream holding a row of parallel plates and demonstrated that they may cause destruction of periodic mechanical structures [6, 7]. Evans and Porter first provided convincing numerical evidence for BICs of both Neumann and Dirichlet types in the case of a rigid circular cylinder placed on the center-plane between parallel walls [8, 9]. Linton and McIver [10] proved the existence of an infinite number of trapped modes for the case of cylindrical waveguide containing an axisymmetric obstacle, in particular a thin circular sleeve.

Specific geometric configurations of the systems were chosen in the majority of studies on this subject which made it possible to reduce their effective dimensionality usually to two [11, 12]. A different class is the fully three-dimensional systems. For example, in the case of non-axisymmetric obstacle inside the cylindrical waveguide, Hein and Coch [13] numerically computed acoustic resonances and BICs having solved the eigenvalue problem. BICs with orbital angular momentum were shown to occur in the cylindrical resonator opened by coaxial [14] and non-coaxial [15] attachment of cylindrical waveguides of a lesser radius.

In the cylindrical waveguides the propagating modes are classified by two indices p (azimuthal, OAM) and q (radial) (see table 1) while eigenmodes of the closed cylindrical resonator are specified by three indices mnl, azimutal (OAM), radial and longitudial, respectively. In the wake of Friedrich–Wintgen approach [3] the majority of papers consider the BICs embedded into a single continuum. However, in the waveguides as soon as the frequency exceeds the cut-off of the propagating modes with $p = \pm 1$, $q = 1$ the number of continua increases. In order to obtain a BIC it is necessary for the solution of Helmholtz equation to have zero coupling with all propagating channels. For the first time the problem of the BIC residing in a finite number of continua was considered by Pavlov-Verevkin *et al* [16] in the framework of the phenomenological Weisskopf–Wigner model. The rigorous statement on the BICs was formulated as follows. The interference among N degenerate states which decay into K non-interacting continua generally leads to the formation of $N - K$ BICs. The

Table 1. Cut-off frequencies and corresponding indices and profiles of propagating modes in the cylindrical waveguide.

Channel	Frequency	Indices	Mode profile
1	0	$p = 0,\ q = 1$	
2, 3	1.8412	$p = \pm 1,\ q = 1$	
4, 5	3.0542	$p = \pm 2,\ q = 1$	
6	3.8317	$p = 0,\ q = 2$	
7, 8	4.2012	$p = \pm 3,\ q = 1$	

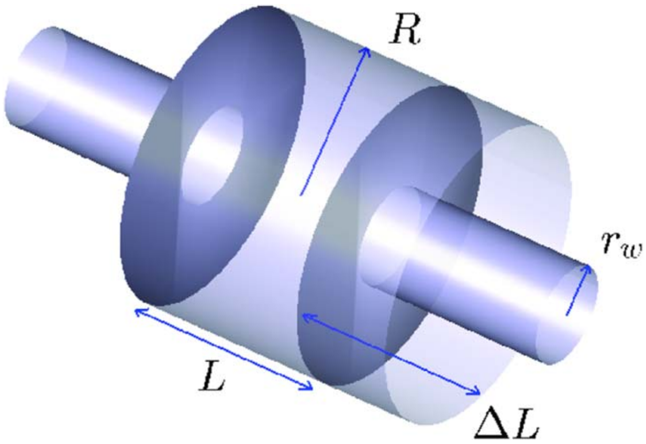


Figure 1. Cylindrical resonator of radius R and length L with two coaxially attached waveguides of the unit radius.

equivalent point of view [4, 17] is that the linear superposition of the N degenerate eigenstates $\sum_{n=1}^{N} a_n \psi_n$ can be adjusted to have zero coupling with K different continua in $N - K$ ways by variation of the N superposition coefficients a_n. Respectively, these coefficients a_n define an expansion of the BIC over the eigenstates of the closed resonator. The aim of present paper is to demonstrate the existence of BICs embedded into several continua of propagation bands of cylindrical waveguides in accordance to table 1. The considered system is schematically depicted in figure 1.

2. Acoustic coupled mode theory

For readers' convenience in this section we review the acoustic coupled mode theory [18] which allows to easily find BICs as eigenmodes of the effective non-hermitian Hamiltonian with real eigenvalues, i.e. with zero line width. It is important that all quantities are dimensionless. The resonator parameters are measured in terms of waveguides radius r_w, and the dimensionless frequency is expressed in terms of the dimensional one as follows: $\omega = \tilde{\omega} r_w / s$, where r_w is the radius of waveguides and s—the sound velocity. The propagating modes in the waveguides have the following form:

$$\psi_{pq}^{(C)}(\rho, \alpha, z) = \phi_{pq}^{(C)} \frac{1}{\sqrt{2\pi k_{pq}}} \exp(\mathrm{i}p\alpha + \mathrm{i}k_{pq} z),$$
$$\phi_{pq}^{(C)}(\rho) = \begin{cases} \frac{\sqrt{2}}{J_0(\mu_{0q})} J_0(\mu_{0q}\rho),\ p = 0 \\ \sqrt{\frac{2}{\mu_{pq}^2 - p^2}} \frac{\mu_{pq}}{J_p(\mu_{pq})} J_p(\mu_{pq}\rho),\ p = 1, 2, 3, \ldots, \end{cases} \quad (1)$$

where ρ and α are the polar coordinates in the reference frame of the waveguides and μ_{pq} are the roots of equation $J_p'(\mu_{pq}\rho)|_{\rho=0}$ implied by the Neumann boundary conditions on the waveguides hard walls.

The solutions inside the closed resonator are:

$$\psi_{mnl}(r, \phi, z) = \psi_{mn} \frac{1}{\sqrt{2\pi}} \exp(\mathrm{i}m\phi) \psi_l(z),$$
$$\psi_{mn}(r) = \begin{cases} \frac{\sqrt{2}}{R J_0(\mu_{0n})} J_0(\mu_{0n} r/R),\ m = 0 \\ \sqrt{\frac{2}{\mu_{mn}^2 - m^2}} \frac{\mu_{mn}}{R J_m(\mu_{mn})} J_m(\mu_{mn} r/R), \end{cases}$$
$$m = 1, 2, 3, \ldots$$
$$\psi_l(z) = \sqrt{\frac{2 - \delta_{l,1}}{L}} \cos\left[\frac{\pi(l-1)z}{L}\right], \quad (2)$$

where r and ϕ are the polar coordinates in the reference frame of the resonator and μ_{mn} are the roots of equation $J_m{}'(\mu_{mn} r/R) = 0$. The corresponding eigenfrequencies are:

$$\omega_{mnl}^2 = \left[\frac{\mu_{mn}^2}{R^2} + \frac{\pi^2(l-1)^2}{L^2}\right]. \quad (3)$$

The effective non-Hermitian Hamiltonian has the following form [15, 18]

$$H_{\text{eff}} = H_{\text{B}} - \mathrm{i} \sum_{C=L,R} \sum_{pq} k_{pq} W_{pq}^{(C)} W_{pq}^{(C)\dagger}, \quad (4)$$

where H_{B} is the Hamiltonian of the closed resonator and the coupling matrices of the resonator eigenmodes with the propagating modes are determined by the overlapping integrals:

$$W_{mnl;pq}^{(C)} = \int_0^1 \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\alpha \phi_{pq}^{(C)}(\rho, \alpha) \psi_{mnl}^*(r, \phi, z = z_C)$$
$$= \psi_l(z_C) \int_0^{2\pi} \mathrm{d}\alpha \int_0^1 \rho \mathrm{d}\rho \phi_{pq}^{(C)}(\rho, \alpha) \psi_{mn}^*(r(\rho, \alpha), \phi(\rho, \alpha)), \quad (5)$$

where $C = L, R$ enumerates the waveguides.

Complex eigenvalues z of H_{eff} have simple physical meaning: their real parts define positions of resonances with a width defined by corresponding imagine part [19]

$$H_{\text{eff}} \psi_r(r, \phi, z) = z_r \psi_r(r, \phi, z). \quad (6)$$

In order to find a BIC one has to find those eigenvalues of H_{eff} which have zero imaginary part, i.e. resonances with zero width. The eigenvector of $H_{\rm eff}$ corresponding to such eigenvalue is the BIC.

Therefore one has to solve the fix-point equations [15, 20]

$$\omega_c = \mathrm{Re}(z(\omega_c, L_c, R_c)), \; \mathrm{Im}(z(\omega_c, L_c, R_c)) = 0, \qquad (7)$$

for two varying parameters of the resonator, its length L and radius R. After the fix-point equations are solved we can determine the eigenmodes of the effective Hamiltonian with real eigenvalues [4, 15]. Numerically these fix-point equations were solved by method of successive approximations.

3. BICs embedded into the continua $p = 0$, $q = 1$ and $p = \pm 1$, $q = 1$

BICs embedded into the propagation band of the first channel $p = 0$, $q = 1$ of the coaxial waveguide for $\omega < \mu_{11}$ were considered in previous papers [14, 15, 21]. Now assume $\mu_{11} < \omega < \mu_{21}$. Then the waveguides support three propagating modes with indices $p = 0$, $q = 1$ and $p = \pm 1$, $q = 1$ depicted in table 1. For the BIC to occur it is necessary for its overlapping with these propagating modes to vanish. Firstly, it is obvious that any eigenmode with $|m| > 1$ is orthogonal to these propagating modes so it is a symmetry protected BIC. More interesting are the BICs occurring due to full destructive interference by Friedrich–Wintgen mechanism [3] which are embedded into the continua $p = 0$, $q = 1$ and $p = \pm 1$, $q = 1$.

Here we use more evident explanation of BICs realization based on linear superposition of degenerate eigenmodes [4]. We start with the BICs superposed of the eigenmodes with $m = 0$ but with different radial indices n. Then by the symmetry arguments all these eigenmodes have zero coupling with the continua $p = \pm 1$, $q = 1$. Therefore one can form a BIC superposed of

$$\psi_{\rm BIC} = a_{0nl}\psi_{0nl} + a_{0n'l'}\psi_{0n'l'}$$

to give rise to the following condition for the BIC

$$a_{0nl}W_{0nl;01} + a_{0n'l'}W_{0n'l';01} = 0, \qquad (8)$$

where the coupling matrix elements are given by (5). Theoretically it is easy to reach a degeneracy of the eigenmodes ψ_{0nl} and $\psi_{0n'l'}$ by variation of the resonator's radius R. However it is not easy to vary the radius experimentally. It is much easier to vary the length of the resonator by use of piston-like hollow-stem waveguides tightly fitted to the interior boundaries of a cylindric resonator [14]. Figure 2 shows typical wave function of such BIC superposed of the eigenmodes ψ_{032} and ψ_{014} with coefficients shown in figure 3. Open circles indicate the areas of waveguides connection. It is clear that the BIC is orthogonal to the waveguide modes with the indices $p = \pm 1$, $q = 1$ propagating in the second and third channels. Vanishing of the coupling coefficient with the first scattering channel with indices $p = 0$, $q = 1$ is ensured by the complete destructive interference of the resonator eigenstates with the indices $m = 0$, $n = 3$, $l = 2$ and $m = 0$, $n = 1$, $l = 4$ as depicted in figure 3.

Another way is to superpose the eigenmodes with the indices $m = \pm 1$, $n = 3$, $l = 2$ and $m = \pm 1$, $n = 2$, $l = 4$. At the

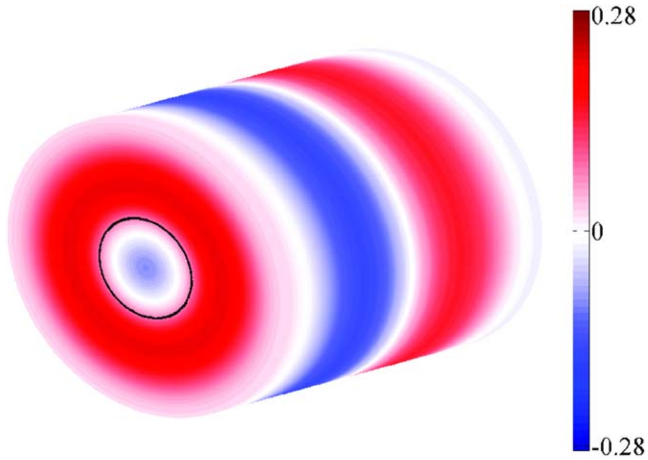


Figure 2. Wave function of BIC withdrawn on the surface of the cylindrical resonator at $\omega^2 = 6.3878$, $L = 3.729$, $R = 3$.

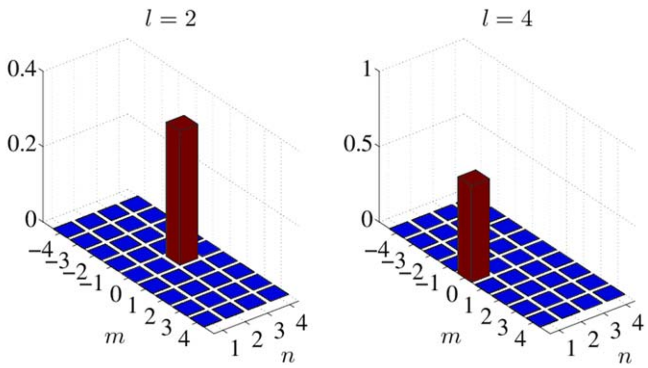


Figure 3. Expansion coefficients $|a_{mnl}|$ formed the BIC depicted in figure 2.

point of degeneracy of the eigenfrequencies $\omega_{132}(L)$ and $\omega_{124}(L)$

$$\psi_{\rm BIC} = a_{132}\psi_{132} + a_{124}\psi_{124}.$$

This BIC has zero coupling with the first channels $p = 0$, $q = 1$ and $p = -1$, $q = 1$ by the symmetry arguments while zero coupling with the channel $p = 1$, $q = 1$ can be achieved by variation of the resonator's length to have

$$a_{132}W_{132;11} + a_{124}W_{124;11} = 0. \qquad (9)$$

The BIC projection onto the boundary of the closed resonator is shown in figure 4 with coefficients shown in figure 5.

As in the case of a single open channel, BICs can be detected in the transmittance spectrum as collapsing Fano resonances [4, 22], i.e. points in which the zero and unit transmittance coincide. However, the picture essentially depends on which wave is injected as shown on figures 6 and 7. If a wave is injected in the first channel with indices $p = 0$, $q = 1$ only those BICs are accompanied by collapses of the Fano resonance which are results of the Friedrich–Wintgen destructive interference [3]. For example the BIC shown in figure 2 is depicted in figure 6 by the white circle. On the contrary, if the wave with indices $p = 1$, $q = 1$ is injected the aforementioned BICs can not be seen in the transmittance.

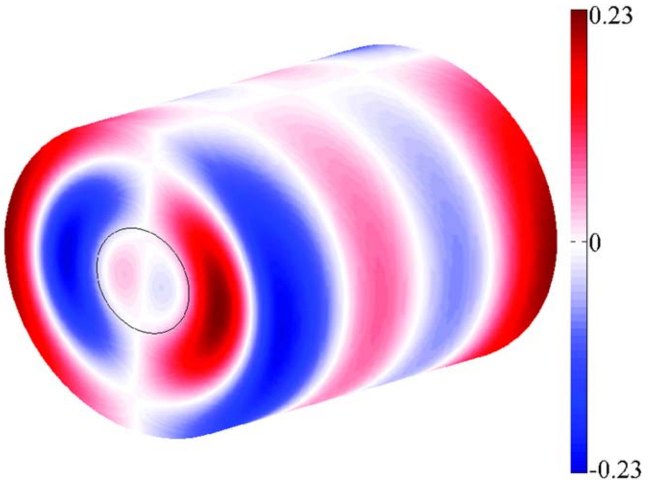


Figure 4. Wave functions of BIC at $\omega^2 = 8.7777$, $L = 3.9763$, $R = 3$.

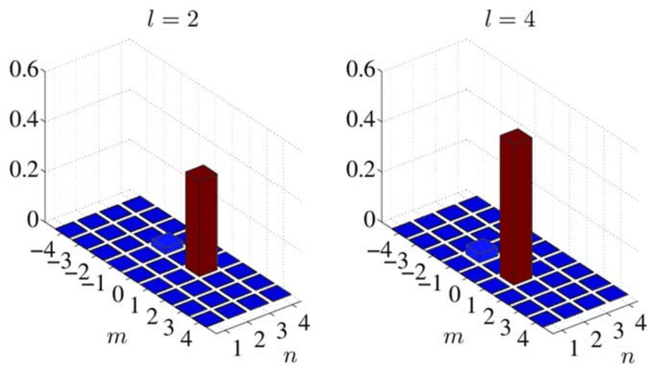


Figure 5. Expansion coefficients $|a_{mnl}|$ forming the BIC depicted in figure 4.

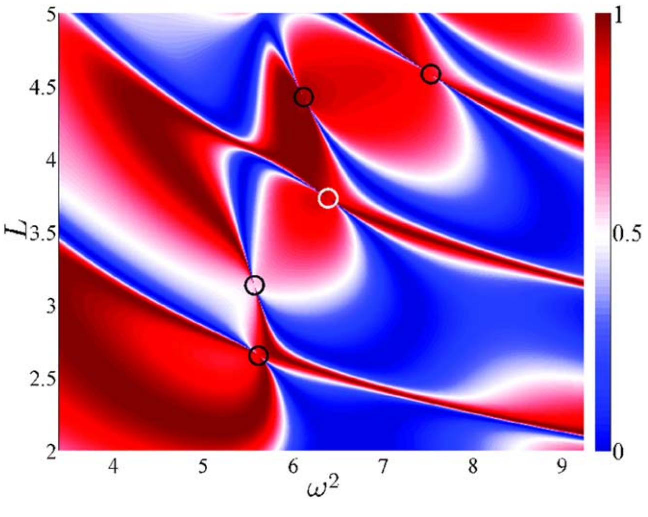


Figure 6. The transmittance versus frequency and resonator length in case of the wave with indices $p = 0$, $q = 1$ is supplied by the generator. White circle represents the BIC depicted in figure 2.

Next, let us consider the frequency range $\mu_{02} < \omega < \mu_{31}$ where the waveguides support six open channels according to table 1. We consider the BICs superposed of the eigenmodes with $m = 0$ but with different radial quantum numbers n. Then for symmetry reasons all these eigenmodes have zero coupling with

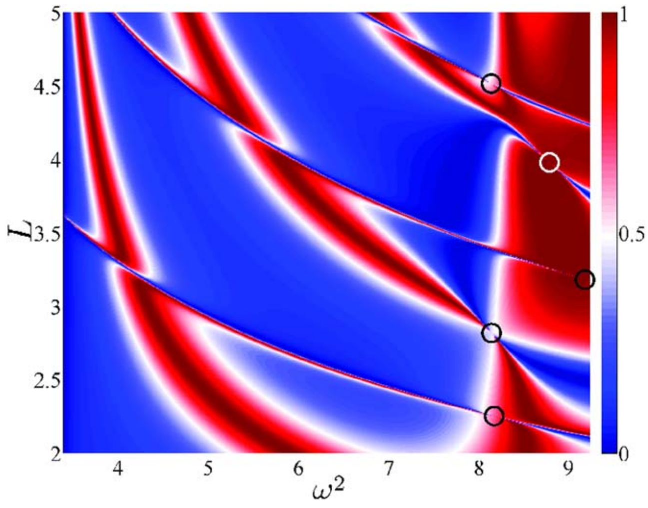


Figure 7. The transmittance versus frequency and resonator length in case of the wave with indices $p = 1$, $q = 1$ is supplied by the generator. White circle represents the BIC depicted in figure 4.

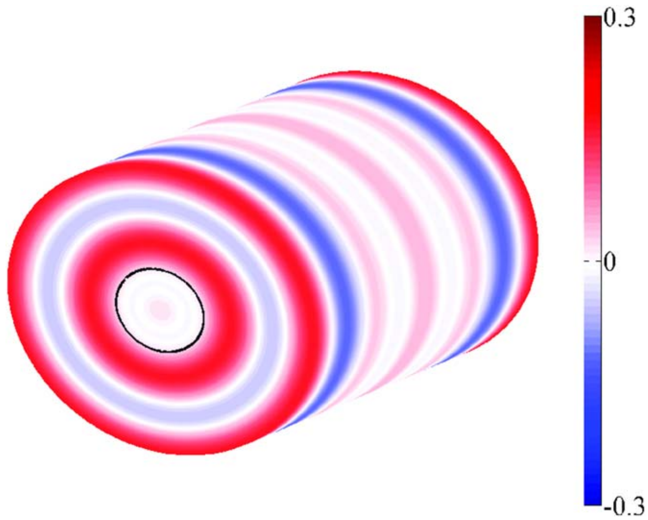


Figure 8. Wave function of BIC at $\omega^2 = 19.327$, $L = 5.7172$, $R = 3.5$.

four continua $p = \pm 1$, $q = 1$ and $p = \pm 2$, $q = 1$. In order the BIC to have zero coupling with the remaining open channels with $p = 0$, $q = 1$ and $p = 0$, $q = 2$ it is necessary to superpose three eigenmodes with the correspondence to the requirements of threefold degeneracy [16, 17]. As could be seen from (3), twofold degeneracy is easily achieved by variation of one parameter of the system, for example the resonator length [14]. Respectively, the threefold degeneracy requires variation of two parameters—bot the resonator length and radius. The point of threefold degeneracy is shown by the white circle on the figure 10.

Therefore the condition for the BIC embedded into two continua with $p = 0$, $q = 1$ and $p = 0$, $q = 2$ can be written as

$$a_{019} W_{019;01} + a_{047} W_{047;01} + a_{055} W_{055;01} = 0, \qquad (10)$$

$$b_{019} W_{019;02} + a_{047} W_{047;02} + a_{055} W_{055;02} = 0. \qquad (11)$$

Obviously this system of linear equations can be easily solved to provide zero coupling of this BIC with two continua.

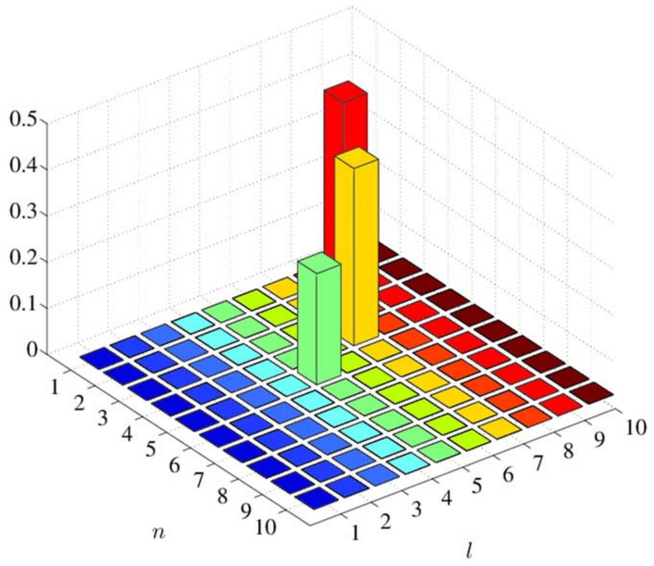


Figure 9. The modal decomposition of the BIC depicted in figure 8.

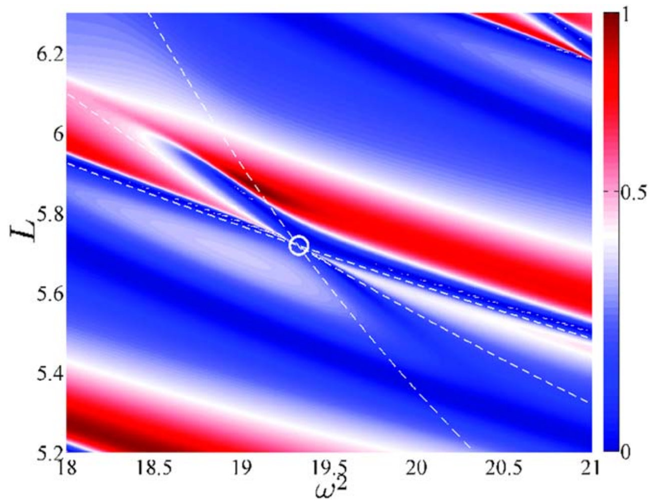


Figure 10. The transmittance versus frequency and resonator length for $R = 3.5$. White dashed lines represent eigenvalues of the closed resonator with indices $mnl = 0, 1, 9$, $mnl = 0, 4, 7$ and $mnl = 0, 5, 5$. The circle indicates the point of BIC3 depicted in figure 8.

The mode shape of BIC on the surface of the cylindrical resonator with the corresponding coefficients is shown in figure 8 with expansion coefficients shown in figure 9. Although many channels are open the BICs can still be diagnosed in the transmittance through collapses of Fano resonances as seen from figure 10.

4. Conclusion

We considered BICs embedded into several continua in the system composed of a cylindrical acoustic resonator with two coaxially attached waveguides of a lesser radius for variation of the resonator scales (length and radius). Due to the axial symmetry of the waveguide the BICs are classified by the OAM. In particular the BICs with zero OAM $m = 0$ are orthogonal to propagating modes of the attached waveguides with non-zero OAM $p \neq 0$. Then these BICs can be tuned to have zero coupling with the continuum with zero OAM $p = 0$ by variation of the resonator length via the Friedrich–Wintgen mechanism of full destructive interference of two resonances. However if two channels $p = 0$, $q = 1$ and $p = 0$, $q = 2$ with zero OAM are opened, the accidental BICs can occur due to full destructive interference of three resonances. These BICs are located at the point of threefold degeneracy of eigenmodes in correspondence with the papers [16, 17]. Such a degeneracy can be achieved by simultaneous variation of two parameters—the resonator length and radius.

Despite that there are fewer possible applications for the acoustical BICs than, for example, the optical ones, the physical mechanisms underlying this phenomenon are generic for all areas of physics, as was shown by Friedrich and Wintgen [3]. Therefore, the investigation of BICs embedded into several continua makes it possible to expand the scope of experimental study of this phenomenon, including a possibility to extend the frequency range of BICs.

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