



# Associative memory on qutrits by means of quantum annealing

Vladimir Zobov<sup>1</sup> · Ivan Pichkovskiy<sup>1</sup>

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## Abstract

When associative memory is implemented on the well-studied Hopfield network, patterns are recorded in the interaction constants between binary neurons. These constants are chosen so that each pattern should have its own minimum energy of the system described by the Ising model. In the quantum version of the Hopfield network, it was proposed to recall such states by the adiabatic change of the Hamiltonian in time. Qubits, quantum elements with two states, for example, spins with  $S = 1/2$  were considered as neurons. In this paper, for the first time, we study the function of associative memory using three-level quantum elements—qutrits, represented by spins with  $S = 1$ . We record patterns with the help of projection operators. This choice is due to the need to operate with a state with a zero spin projection, whose interaction with the magnetic field vanishes. We recall the state corresponding to one of the patterns recorded in the memory, or superposition of such states by means of quantum annealing. To equalize the probabilities of finding the system in different states of superposition, an auxiliary Hamiltonian is proposed, which is turned off at the end of evolution. Simulations were performed on two and three qutrits and an increase in the memory capacity after replacing qubits with qutrits was shown.

**Keywords** Quantum adiabatic algorithm · Quantum annealing · Qubit · Qutrit · Associative memory · Memory capacity

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✉ Ivan Pichkovskiy  
pichkovskiy@bk.ru  
Vladimir Zobov  
rsa@iph.krasn.ru

<sup>1</sup> Kirensky Institute of Physics, Federal Research Center KSC SB RAS, Akademgorodok 50, bld. 38, Krasnoyarsk, Russia

## 1 Introduction

At the present stage of development of information technologies, quantum properties of many-body systems open up new possibilities of artificial neural networks [1, 2], including associative memory [3–7]. When implementing associative memory, several patterns are recorded in a neural network [3]. Then, we give the network a hint, which is a distorted version of the desired pattern. The network should be able to return the true instance. An example of associative memory is the Hopfield network, in which each binary neuron is represented as a spin  $S = 1/2$  and interacts with all other spins [3]. The interactions between the spins are described by the classical Ising model. In such a classical Hopfield network, the patterns are recorded in the interaction constants between the spins, which are chosen so that each pattern should have its own minimum system energy. The recognition process is carried out by iteratively changing the states of the spins until the system is in the state corresponding to one of the energy minima. When considering a quantum artificial neural network in [4–8], both the memorized patterns and the recognized pattern are represented by means of the problem (target) Hamiltonian. The recognition is carried out by an adiabatic change of the Hamiltonian with time or by quantum annealing. In this way, in the quantum case, it is possible to prepare the system not only in the state corresponding to an individual pattern [4–6], but also in the state of their superposition of the given form [7, 8].

A qubit, a quantum system with two states, for example, a spin with  $S = 1/2$  is usually considered as a neuron. However, in this role a system with three states—qutrit [9–13] can be used. Among the advantages of qutrits over qubits one expects a faster growth of the Hilbert space, and hence the size of the computational basis with the addition of new elements. This circumstance may be important for increasing the memory capacity. As qutrits, one suggests using, for example, objects with the spin  $S = 1$  in magnetic and crystal fields. These include quadrupole nuclei [9, 12, 13] of deuterium, nitrogen or lithium, as well as NV centers in a diamond (paramagnetic color centers formed by electrons on the vacancies near the nitrogen atoms) [14, 15]. The latter variant is preferable because of the presence of a strong dipole–dipole interaction between NV centers, which is necessary for the implementation of conditional operations in quantum algorithms. Moreover, in this case, there is a large difference in the frequency of transitions between different energy levels, which makes it possible to control the state of the system with the help of transition-selective high-frequency field pulses.

In this paper, we consider the associative memory on the qutrits represented by the spins with  $S = 1$ . Qutrit control is qualitatively different from qubit control due to the presence of a third state with a zero spin projection. The fact is that when controlling the associative memory on the qubits represented by the spins  $S = 1/2$  by means of quantum annealing [4–8], a magnetic field is applied, whose action on the state of qutrit with the zero spin projection is absent. Due to such states of qutrits, the well-known Hebbian learning rule [4–8] for recording patterns through the interaction constants also ceases to work. Therefore, in the paper, we will develop methods for controlling associative memory in qutrits using the projection operator method. Of course, this method can also be applied to qubits. The difference is that the memorized patterns are either encoded in the ternary number system in the states of qutrits or in the binary number system in the states of qubits. The recording and recall of the patterns will be

carried out by quantum annealing. We have proposed a new method for equalizing the amplitudes of states in a quantum superposition, which does not require the addition of auxiliary spins, in contrast to the method in [7, 8]. A numerical simulation of the work of memory on two and three qutrits will be performed with a different number of the stored patterns.

## 2 Quantum adiabatic algorithm for the memory recall

To implement associative memory, memorized patterns should be encoded in the states of a quantum system. Take a system of  $n$  spins with  $S = 1$ , and as the computational basis, we will use the basis  $|m_1, m_2, \dots, m_n\rangle$  of the eigenfunctions of the operators  $S_i^z$  of the projections of the spins on the  $Z$ -axis. Each of the projections  $m_i$  can take one of three values: 1, 0,  $-1$ . Let it be necessary to keep in memory  $p$  elements represented by  $p$  quantum vectors  $|\psi_\mu\rangle = |m_1^\mu, m_2^\mu, \dots, m_n^\mu\rangle$ , where  $\mu = 1, 2, \dots, p$ . Choose the projection method of memorization [4, 5], in which the recording is performed through the memory Hamiltonian

$$H_{\text{mem}} = - \sum_{\mu=1}^p |\psi_\mu\rangle\langle\psi_\mu|, \tag{1}$$

where  $|\psi_\mu\rangle\langle\psi_\mu|$  is the projector on the vector  $|\psi_\mu\rangle$ . With the help of Hamiltonian (1) we prepare a superposition of the memorized states as follows:

$$|\Psi\rangle = \sum_{\mu=1}^p a_\mu |\psi_\mu\rangle, \tag{2}$$

where  $a_\mu = a(m_1^\mu, m_2^\mu, \dots, m_n^\mu)$  are the amplitudes of the states. For this purpose, let us implement the evolution of the system over time  $T$  with a sufficiently slow (adiabatic) change in time of the Hamiltonian

$$H(t) = (1 - t/T)H_0 + (t/T)H_p, \quad 0 \leq t \leq T. \tag{3}$$

where  $H_p = H_{\text{mem}}$ , and  $H_0$  is the initial Hamiltonian, the ground state of which is easy to prepare. As the last one, we take an interaction with the magnetic field directed along the  $x$ -axis:

$$H_0 = -h \sum_{i=1}^n S_i^x, \tag{4}$$

the ground state of which is a direct product of the eigenvectors  $|1\rangle_x = (|1\rangle + \sqrt{2}|0\rangle + |-1\rangle)/2$  of the spin operators  $S^x$  of individual spins with positive eigenvalues 1:

$$|\Psi_0\rangle = |1\rangle_x^{\otimes n} \tag{5}$$

As a result of the evolution of the system from the initial state (5) under the action of the Hamiltonian (3), we obtain

$$|\Psi(T)\rangle = \hat{Q} \exp\left(-i \int_0^T H(t)dt\right) |\Psi_0\rangle, \tag{6}$$

where  $\hat{Q}$  is the ordering operator in time. If the evolution is adiabatic, then the system enters the ground state (2) of the Hamiltonian (1).

In the adiabatic method, the recall of any of the memorized states  $|\psi_{\text{prob}}\rangle$  is realized by adding a probe Hamiltonian of the hint  $H_{\text{prob}}$  to the target Hamiltonian:

$$H_p = H_{\text{mem}} + \Gamma H_{\text{prob}}, \tag{7}$$

where  $\Gamma$  is a small coefficient that specifies the value of the hint. In [4–6], the probe Hamiltonian for spins  $S = 1/2$  was set using the interaction with the magnetic field. For spins  $S = 1$ , this option is inconvenient, because it does not allow one to take into account the state with the zero spin projection. Therefore, we will take

$$H_{\text{prob}} = -|\psi_{\text{prob}}\rangle\langle\psi_{\text{prob}}|. \tag{8}$$

When the probe Hamiltonian is added in the form of a projector with one of the recorded states (patterns), the energy of this state in the Hamiltonian (7) decreases by  $\Gamma$ . The energies of other states will not change. This removes the degeneracy, the recalled state becomes the ground state of the Hamiltonian (7), and adiabatic evolution to the time  $T$  prepares this state.

### 3 Quantum associative memory on two qutrits

As an example, consider an associative memory on a two-spin system with the stored states  $|\psi_1\rangle = |0, 1\rangle$ ,  $|\psi_2\rangle = |1, 0\rangle$  and  $|\psi_3\rangle = |-1, -1\rangle$  that corresponds to the Hamiltonian:

$$H_{\text{mem}} = -|0, 1\rangle\langle 0, 1| - |1, 0\rangle\langle 1, 0| - |-1, -1\rangle\langle -1, -1|. \tag{9}$$

Express the projection operators through the spin operators of an individual spin  $i$  ( $i = 1, 2$ ):

$$\begin{aligned} |-1\rangle\langle -1|_i &= -S_i^z \frac{1 - S_i^z}{2}, \quad |0\rangle\langle 0|_i = 1 - (S_i^z)^2, \quad |1\rangle\langle 1|_i = S_i^z \frac{1 + S_i^z}{2}, \\ |m_1, m_2\rangle\langle m_1, m_2| &= |m_1\rangle\langle m_1| \otimes |m_2\rangle\langle m_2| = |m_1\rangle\langle m_1|_1 |m_2\rangle\langle m_2|_2. \end{aligned} \tag{10}$$

To calculate the evolution of system (6) with the time-dependent Hamiltonian, we divide the total time  $T$  into  $N$  small intervals  $\Delta t = 0.1$ , in each of them we will neglect

the change in Hamiltonian (3) [5, 16]. The solution to our problem  $|\Psi\rangle$  will be sought as a product:

$$|\psi(t = T)\rangle \cong \prod_{l=0}^N \exp\left\{-i \Delta t \left(\frac{l}{N} H_p + \left(1 - \frac{l}{N}\right) H_0\right)\right\} |\psi(t = 0)\rangle. \tag{11}$$

The calculation results are shown in Fig. 1. First, the figure shows that when  $\Gamma = 0$ , we get a superposition (2) with different amplitudes  $a_\mu$ . This feature of the adiabatic preparation of the superposition state was discussed in theoretical works [7, 8] and was observed experimentally [17]. The reason is that each of the three states approaches the final state at time  $T$  along different paths, as shown in Fig. 1a for instant energy levels. When  $t$  approaches  $T$ , the energy difference tends to zero and the transition from adiabatic to diabatic evolution occurs [7, 8]. A description of the dynamics of the system at  $t \rightarrow T$  is given in Supplementary Material. For the probability of finding the system when  $t = T$  in the state  $|-1, -1\rangle$  using the perturbation theory [4, 7, 8, 18], we obtained the expression

$$|a_3|^2 \approx A(h/T)^{2/3} \tag{12}$$

which at  $A = 1.22$ , is consistent with the results of the numerical simulation and qualitatively correctly conveys dependencies on the parameters  $h$  from 0.5 to 4 and  $T$  from 100 to 400. For equal probabilities of the system being in two other stored states,  $(1 - |a_3|^2)/2$  is obtained.

With the inclusion of a probe Hamiltonian in (7), as shown in Fig. 1b–d, as  $\Gamma$  increases, the probability of finding the selected state increases rapidly, while the other two come down. When  $\Gamma \ll 1$  in the Supplementary Material, the corresponding dependencies are calculated and obtained as follows:

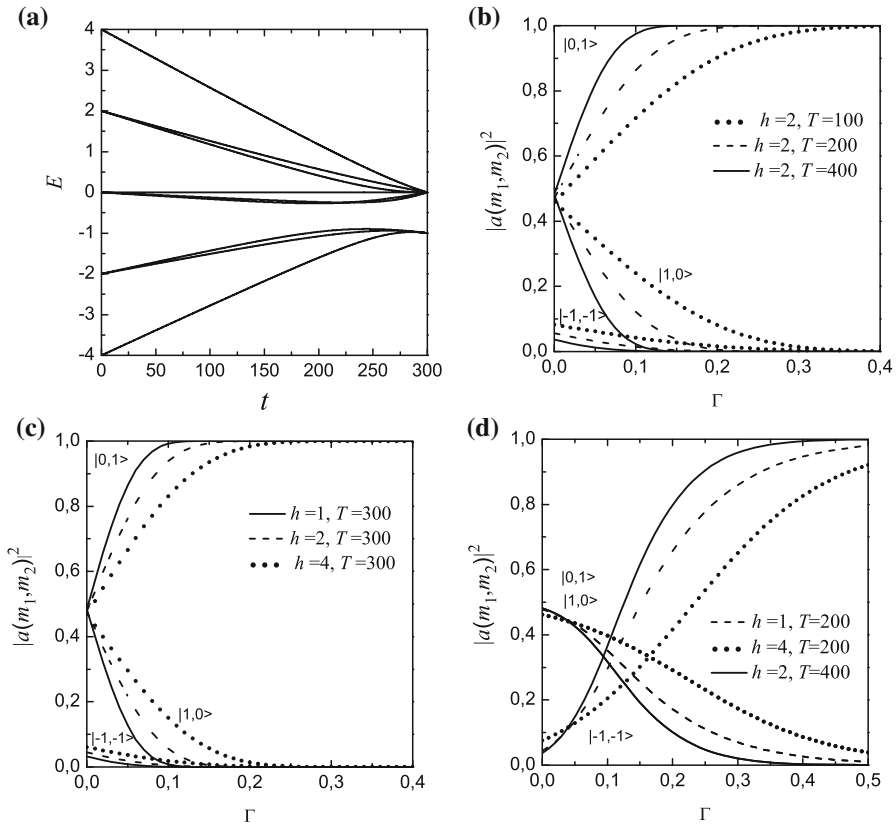
$$\text{at } \Gamma H_{\text{prob}} = -\Gamma|0, 1\rangle\langle 0, 1|$$

$$|a_1(\Gamma)|^2 \approx \left(1 + 2\Gamma K_1(T/h)^{2/3}\right)/2, \tag{13}$$

$$\text{at } \Gamma H_{\text{prob}} = -\Gamma|-1, -1\rangle\langle -1, -1|$$

$$|a_3(\Gamma)|^2 \approx |a_3(\Gamma = 0)|^2 \left(1 + 2\Gamma K_3(T/h)^{2/3}\right), \tag{14}$$

where  $|a_3(\Gamma = 0)|^2$  defined in (12),  $K_1 = 0.203$ ,  $K_3 = 0.53$ . With a further increase in  $\Gamma$ , the growth of probabilities is accelerated. This is due to the removal of the degeneration between the three states and the transition from the diabatic to the adiabatic evolution as  $t$  approaches  $T$ , accompanied by an increase in the gap  $\Gamma$  and a decrease in the probabilities of transitions from the ground state highlighted by the hint. The steepness of dependence on  $\Gamma$  increases with increasing  $T$  and decreasing  $h$ .



**Fig. 1** Associative memory on two qutrits with three stored states:  $|\psi_1\rangle = |0, 1\rangle$ ,  $|\psi_2\rangle = |1, 0\rangle$  and  $|\psi_3\rangle = |-1, -1\rangle$ . **a** Instantaneous energy levels of the Hamiltonian  $H(t)$  (3) with  $H_p = H_{\text{mem}}$  (9) versus time as a parameter, with  $h = 2, T = 300$ . **b–d** Dependencies on  $\Gamma$  of the probabilities of finding the system in the states indicated in the figure with  $\Gamma H_{\text{prob}} = -\Gamma|0, 1\rangle\langle 0, 1|$  (**b, c**) and with  $\Gamma H_{\text{prob}} = -\Gamma|-1, -1\rangle\langle -1, -1|$  (**d**), and for different values of the field  $h$  and time  $T = \Delta t N$  ( $\Delta t = 0.1, N = T/\Delta t$ )

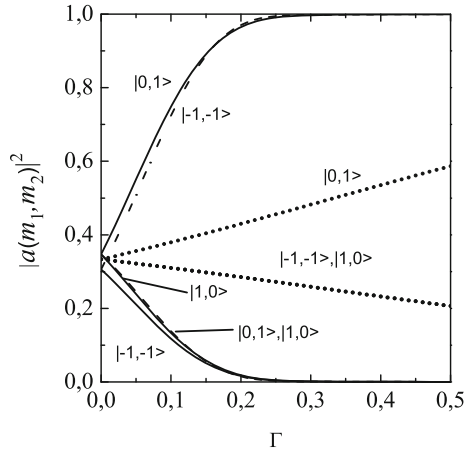
### 4 Amplitude aligning in superposition

Comparison of Fig. 1b, c shows that if hint with a small coefficient  $\Gamma < 0.1$  is already enough to recall state  $|0, 1\rangle$  then to confidently recall state  $|-1, -1\rangle$ , a larger coefficient  $\Gamma > 0.1$  is required. This occurs due to the difference in amplitudes  $a_\mu$  in superposition (2), prepared at  $\Gamma = 0$ . To align the amplitudes in superposition (2) in the case of qubits, it was proposed in [7, 8] to introduce auxiliary spins, with the help of which a multi-spin interaction with the coefficients needed for alignment is created in the effective Hamiltonian.

Another way to obtain an equal superposition state follows from the memory Hamiltonian proposed in [4]

$$H'_{\text{mem}} = -|\Psi\rangle\langle\Psi|, \tag{15}$$

**Fig. 2** The probabilities of finding the system in three stored states:  $|\psi_1\rangle = |0, 1\rangle$ ,  $|\psi_2\rangle = |1, 0\rangle$  and  $|\psi_3\rangle = |-1, -1\rangle$ , versus  $\Gamma$  with the permanent auxiliary Hamiltonian (16) (dotted lines) or with switched off one (20) (solid line  $\Gamma H_{\text{prob}} = -\Gamma|0, 1\rangle\langle 0, 1|$  and dashed lines  $\Gamma H_{\text{prob}} = -\Gamma|-1, -1\rangle\langle -1, -1|$ ). The probability of a state of the same name as a hint deviates upward, and the probabilities of the other two states deviate downward. Values of other parameters are:  $h = 2, T = 300$



in which  $|\Psi\rangle$  is defined in (2) with  $a_\mu = 1/\sqrt{p}$ . For the above example of two qutrits, we have

$$\begin{aligned}
 H'_{\text{mem}} &= -(1/3) \{|0, 1\rangle + |1, 0\rangle + |-1, -1\rangle\} \{|0, 1\rangle + |1, 0\rangle + |-1, -1\rangle\} \\
 &= (1/3)H_{\text{mem}} + (1/3)H_{\text{help}},
 \end{aligned}
 \tag{16}$$

where

$$\begin{aligned}
 H_{\text{help}} &= -\{|0, 1\rangle\langle 1, 0| + |1, 0\rangle\langle 0, 1| + |-1, -1\rangle\langle 1, 0| + |-1, -1\rangle\langle 0, 1| \\
 &\quad + |0, 1\rangle\langle -1, -1| + |1, 0\rangle\langle -1, -1|\},
 \end{aligned}
 \tag{17}$$

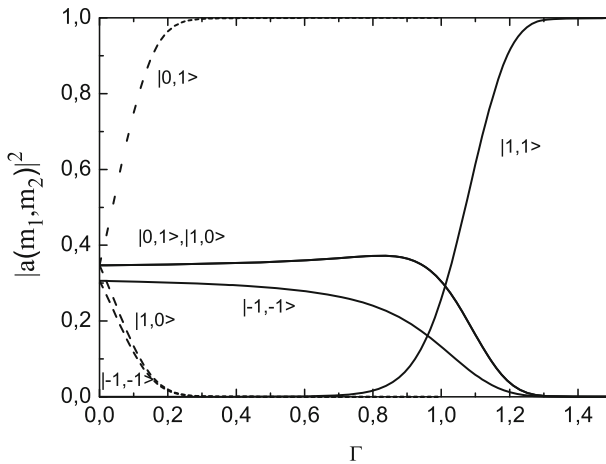
and  $H_{\text{mem}}$  is defined in (9). The off-diagonal projectors in (17) will be expressed through the spin projection operators of the individual spin:

$$\begin{aligned}
 |0\rangle\langle 1| &= S^- S^z, \quad |1\rangle\langle 0| = S^z S^+, \quad |-1\rangle\langle 0| = -S^z S^-, \\
 |0\rangle\langle -1| &= -S^+ S^z, \quad |-1\rangle\langle 1| = S^- S^-, \quad |1\rangle\langle -1| = S^+ S^+,
 \end{aligned}
 \tag{18}$$

where we entered the operators:

$$S^\pm = (S^x \pm iS^y)/\sqrt{2}.
 \tag{19}$$

The calculations performed with the Hamiltonian (16) showed (Fig. 2) that at the end of evolution the equal-superposition state is actually realized. However, after adding the probe Hamiltonian of the hint (8), the call to the desired state occurs slowly, i.e., for substantially larger values of  $\Gamma$  than in the previous case. This is due to the changed view of the Hamiltonian, the ground state of which we are preparing. In the previous case, there was a choice between three degenerate states, whereas in the latter case, a rearrangement of the only nondegenerate ground state occurs.



**Fig. 3** Dependencies of the probabilities of finding the system in the states shown in the figure on the magnitude of the probe Hamiltonian  $\Gamma H_{\text{prob}} = -\Gamma|1, 1\rangle\langle 1, 1|$  with the state that is absent in the memory (solid lines) and on the magnitude of the incomplete probe Hamiltonian  $\Gamma H_{\text{prob}} = -\Gamma|1\rangle\langle 1|_2$  (dashed lines). Values of other parameters are:  $h = 2, T = 300$

To combine the merits of the two approaches, we apply the technique proposed in [19]. We will turn off the interaction  $H_{\text{help}}$  at the end of evolution, i.e., instead of (3), we take the time-dependent Hamiltonian in the form

$$H(t) = (1 - t/T)H_0 + (t/T)(1 - t/T)H_{\text{help}} + (t/T)H_p. \tag{20}$$

The calculation results are presented in Fig. 2. From comparison with Fig. 1b–d it can be seen that the amplitudes of the three states in the superposition (2) (although not complete) and the type of their dependences on the magnitude of  $\Gamma$  are really equalized. At the same time, as expected, there remains good sensitivity to the hint.

In the above examples, the small coefficient  $\Gamma$  in the probe Hamiltonian of the hint determines the degree of its imperfection. Additionally, the incompleteness of the hint can be specified in the probe Hamiltonian using an incomplete projector, in which only a part of the spin variables are represented:  $m$  from  $n$ . The corresponding energy shift  $-\Gamma$  will be acquired by all  $3^{n-m}$  states with a given set of these  $m$  variables and all possible states of the remaining  $n-m$  spins. With this kind of incomplete hint, several patterns can be recalled as a result of evolution, for which the states included in the hint coincide, but the states that are not included differ. On the contrary, if the patterns have different states included in the hint, then degeneracy is removed. As an illustration, Fig. 3 shows the calculation for an incomplete hint with the Hamiltonian  $\Gamma H_{\text{prob}} = -\Gamma|1\rangle\langle 1|_2$ . The obtained dependence on  $\Gamma$  almost coincided with the dependence shown in Fig. 2 for a full projector  $\Gamma H_{\text{prob}} = -\Gamma|0, 1\rangle\langle 0, 1|$ . The reason is that the projector gives only two possible answers: Whether or not the state coincides the one presented in the hint. There is no other more detailed description, i.e., there is no dependence on the Hamming distance obtained in [4–6] for qubits for the probe Hamiltonian of a hint in the form of interaction with a magnetic field.



Finally, to demonstrate the selectivity of the system response to the recall of the memorized and not-memorized states, Fig. 3 shows the calculated dependences on  $\Gamma$  of the probabilities of the system being in different states when a probe Hamiltonian is presented on a state that is not in memory:  $\Gamma H_{\text{prob}} = -\Gamma|1, 1\rangle\langle 1, 1|$ . The energy of the Hamiltonian  $H_p$  (7) for this state will change the value from zero to  $-\Gamma$ . While  $\Gamma < 1$ , a gap is maintained between it and the energies of the recorded states. For large values of  $\Gamma$  exceeding the energy gap (for  $\Gamma > 1$ ), the state becomes the ground state and the system passes into it at the end of evolution [6]. As shown in Figs. 2 and 3, the probability value  $|a(\Gamma)|^2 = 0.7$  for the not-memorized state is achieved at  $\Gamma = 1.13$ , while for the memorized state it is achieved at  $\Gamma = 0.09$ .

The calculations described above were performed by us for an associative memory with the other three states. Similar results were obtained. The reason is that with the projective method of recording states, all states are sorted according to only one criterion—coincidence with the state in the projector: if it coincides, then the projector gives 1, if it does not coincide, then the projector gives 0 (it doesn't matter how many spin projections did not coincide what is the Hamming distance). Therefore, the memory Hamiltonian in the form (9) in the computational basis is represented by a diagonal matrix in which three elements corresponding to any recorded states are equal to  $-1$  and the rest to zero. Nevertheless, within the interval  $0 < t < T$ , there may not be a complete coincidence of the instantaneous energy levels of the total Hamiltonian  $H(t)$  and the “populations” of the levels due to the difference in the matrix elements of the Hamiltonian  $H_0$  (4) between the recorded states. In our chosen example, these elements are equal to zero; therefore, transitions between the corresponding states occur in the second order of the perturbation theory and dependencies (12)–(14) are observed. If with a different choice of states there will be transitions in the first order of the perturbation theory between them, then such dependencies will change.

Finally, on two qutrits, calculations with four and five recorded states were performed. Four-state associative memory is successful, but five-state one is not. This shows that the memory capacity on two qutrits is higher than on two qubits, since no more than two states can be memorized on two qubits [4].

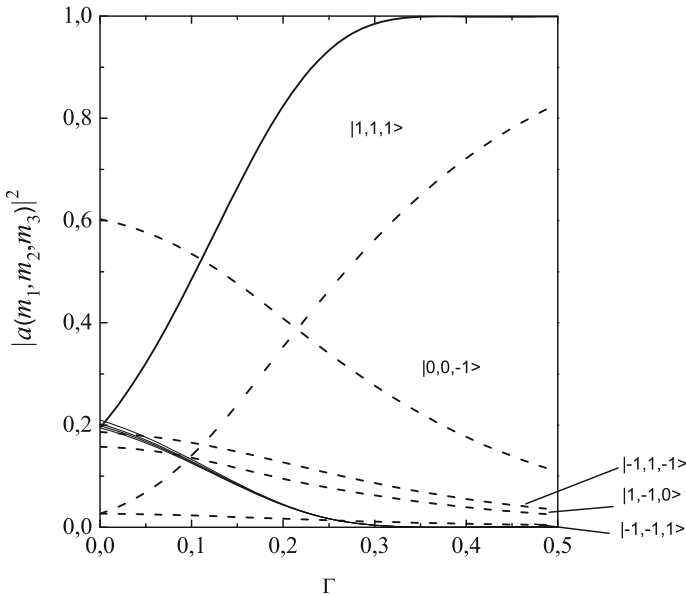
### 5 Quantum associative memory on three qutrits

Let us turn to a system of three qutrits (of three spins  $S = 1$ ). The effective Hamiltonians were constructed according to the rules described above for the five stored states:  $|-1, -1, 1\rangle, |0, 0, -1\rangle, |1, -1, 0\rangle, |1, 1, 1\rangle$  and  $|-1, 1, -1\rangle$ . This time a probe Hamiltonian was chosen:

$$\Gamma H_{\text{prob}} = -\Gamma|1, 1, 1\rangle\langle 1, 1, 1|. \tag{21}$$

The calculation results are shown in Fig. 4. We are convinced of the successful realization of the memorization of the five states on three qutrits.

To conclude this section, we discuss the issue of associative memory capacity on a system of  $n$  spins. The projection method of recording states can be applied to both qutrits and qubits. In the computational basis, the Hamiltonian of memory (1) is



**Fig. 4** Depending on  $\Gamma$ , the probabilities of finding the system of three qutrits in five stored states:  $|-1, -1, 1\rangle, |0, 0, -1\rangle, |1, -1, 0\rangle, |1, 1, 1\rangle$  and  $|-1, 1, -1\rangle$ , with the auxiliary switchable (20) Hamiltonian (17) (solid lines) and without one (3) (dashed lines). The probe Hamiltonian of the hint is  $\Gamma H_{\text{prob}} = -\Gamma |1, 1, 1\rangle\langle 1, 1, 1|$ . The probability of a state of the same name as the hint deviates upward, the probabilities of other states deviate downward. The values of other parameters are:  $h = 2, T = 300$

represented by a diagonal matrix with the number of elements on the diagonal  $2^n$  for qubits and  $3^n$  for qutrits. The diagonal matrix elements corresponding to memorized states (patterns) are  $-1$ , and the rest are zero. In the projection method, patterns are recorded in the orthogonal states of a system of  $n$  spins and lead to maximum values of memory capacities:  $2^n/2$  for qubits and  $3^n/2$  for qutrits. It follows that the memory capacity of  $n$  qutrits is greater than  $n$  qubits. This is confirmed by the calculations performed above.

### 6 Conclusion

Thus, the modeling performed demonstrated a possibility of realizing associative memory on qutrits and pattern recall by means of quantum annealing. An increase in the memory capacity after replacing qubits with qutrits was shown. The main advantage of the qutrit system compared to the qubit system is the ability to provide the same size of the computational basis  $2^n \sim N \sim 3^m$  with a smaller number of elements:  $m < n$ . Control fewer elements is technically simpler. For example, in the projection method under consideration for storing patterns ([7]), it will be necessary to create a multi-spin interaction:  $n$ -spin for qubits and  $m$ -spin for qutrits, i.e., with a smaller number  $m < n$ . It is clear that the use of  $d$  level elements—qudits ( $d > 3$ ) can give even greater gain. Unfortunately, control of quantum systems with a large number of levels is technically

difficult. Therefore, today the main focus of researchers is still on qubits. Nevertheless, work is underway on the use of qutrits and qudits in quantum computing (see, for example, new preprints [20, 21]) in the hope that their advantages will be realized.

As qutrits in the experiment, various three-level quantum systems can be used, for example, objects with the spin  $S = 1$  in the magnetic and crystal fields listed in the introduction. The Hamiltonian in the form suitable for experimental implementation is obtained after substitution in  $H_{\text{mam}}$ ,  $H_{\text{help}}$  and  $H_{\text{prob}}$  the products of three projectors expressed in terms of spin operators (10) and (18). Note that along with the two-spin interaction, which has the form of the Ising interaction or the dipole–dipole interaction, three-spin interactions and interactions containing squares of spin operators are required. The methods for obtaining such interactions with the help of rotation operators selective in transitions between the levels of each of the three spins are described in [14, 15, 22, 23].

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