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Spin-wave oscillations in gradient ferromagnets: Exactly solvable models

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ABSTRACT

The method of searching for the profiles of the gradient dependence of the material parameters of matter on the coordinates that allow the exact solution of wave equations, developed previously for electromagnetic and elastic waves, was generalized to spin waves in gradient ferromagnets. Such profiles were found and exact solutions of the wave equations for a ferromagnet with uniaxial magnetic anisotropy $\beta(z)$ or exchange $\alpha(z)$ varying in space were obtained. The obtained solutions were used to develop the theory of spin-wave resonance in gradient thin magnetic films. The dependences of the eigenfunctions $m_n(z)$, the frequencies of the discrete spectrum ω_n, and the high-frequency susceptibility χ_n on the number of spectral levels n were found. The cardinal differences between the spin-wave spectra of films with gradients $\beta(z)$ and $\alpha(z)$ are shown. The variable anisotropy $\beta(z)$ changes the shape of the energy potential of the magnetic film and leads to a change in the discrete spectrum for frequencies $\omega_n(n)$ lower than the frequency of the gradient potential well or potential barrier ω_c. The variable exchange $\alpha(z)$ does not change the shape of the energy potential. Spin-wave oscillations occur in a rectangular potential well created by the surfaces of the film, regardless of profile $\alpha(z)$. The discrete frequency spectrum $\omega_n(n)$ is quadratic on n, or has negligible deviations from the quadratic, for all n. An analytical expression for the effective exchange parameter is obtained. Exact solutions of the Schrödinger equation with spatially dependent effective mass $\mathrm{m}(z)$ were found for the profile of $\mathrm{m}(z)$ inverse to the function of $\alpha(z)$.

1. Introduction

Inhomogeneous materials with a smoothly aperiodically varying along the coordinate axes value of some material parameter (or several parameters) make up the class of gradient metamaterials. Gradient heterogeneity can be due to both natural causes and specially formed in the process of creating the material. Electromagnetic and elastic waves in substances with artificially created gradient inhomogeneities of the refractive index and elastic constants, respectively, are currently being intensively studied experimentally and theoretically [1]. The one-dimensional wave equations in this case contain a coordinate-dependent coefficient, the profile of which is described by some function $U(z)$. The development of the theory of waves in gradient material involves the sequential solution of two problems: (i) finding the exact solution of a differential equation with a coordinate-dependent coefficient and (ii) analyzing this exact solution for a specific physical model using approximate analytical or numerical methods. The first, purely mathematical problem, has different solutions for cases of different locations of the gradient coefficient $U(z)$ in the equation: before the spatial derivative(s), the temporal derivative, or the desired function of the wave equation. Exact solutions of wave equations are known for many $U(z)$ profiles [2]. The case when the structure of the wave equation corresponds to the Schrödinger equation for electrons in the external potential $U(z)$ (a coordinate-dependent coefficient in front of the desired function) has been studied especially well [3,4].

Various technologies for producing artificial gradient optical and elastic media with a predetermined law $U(z)$ of a change in a material parameter have been developing since the 1970s [1]. For example, co-evaporation of two low and high refractive index materials is used. A predetermined change in the relationship between evaporation intensities over time leads to a predetermined law $U(z)$ of a change in the refractive index along the thickness of the formed film. The development of methods for producing films with a given law of variation of the material parameter $U(z)$ stimulated the search for exact solutions for new profiles of potential $U(z)$ [5–9].

In recent years, a new approach to this problem has been formed, which is summarized and consistently presented in the book [1]. In the usual approach, the form of the potential $U(z)$ is given, and a search for changes of variables and functions of the equation of oscillation is made which would lead to an exact solution of this equation. In the new method, the function $U(z)$ is assumed to be unknown and the standard replacements of variables and functions found in [1] are carried out, with the help of which an additional nonlinear differential equation for the desired function $U(z)$ is derived from the equation of oscillations.

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Solutions of this additional equation, which can be several, give a concrete form of coefficients $U(z)$ for which the oscillation equation has an exact solution. Finding each solution to the additional equation is a separate, not simple mathematical problem. Thus, the form of the potential $U(z)$ in this method is determined by purely mathematical and not physical considerations. However, it is well known that almost every potential for which an exact solution of the oscillation equation is obtained, sooner or later finds its application in any physical problem. Equations from different areas of physics (optics, acoustics, magnetism) for some gradient parameters are similar to each other and the exact solutions obtained for them are used in all these areas. The developed approach was applied by the authors [1] to finding exact solutions for a number of problems of propagation of electromagnetic and elastic waves in gradient metamaterials. It gives the theory of waves in gradient materials a particular orderliness and generality.

The extension of this approach to the problem of describing spin waves in gradient ferromagnets is an important task. The application of the found exact solutions to the problems of each of the physical areas has its own specifics. In the theory of electromagnetic and elastic waves, the main sought quantities are transmission, reflection, refraction, and other characteristics of traveling waves. In some cases, the calculation of these quantities is also necessary for spin waves. But the primary problem of the theory of spin waves is the calculation of the spectrum and amplitudes of standing spin-wave oscillations in thin magnetic films. These characteristics are both the subject of intensive theoretical [10–16] and experimental [17–29] studies. Therefore, we use the exact solutions for spin waves obtained in this work to develop the theory of spin-wave resonance in gradient thin magnetic films.

The situation with the development of both experimental and theoretical studies of spin waves in gradient magnetic metamaterials is fundamentally different from the situation with the study of electromagnetic and elastic waves. Studies of spin waves in gradient ferromagnets have until recently dealt only with natural gradient inhomogeneity, which occurs uncontrollably when producing thin magnetic films [10–29]. A brief description of this stage of research development is given below in the next section of work.

Targeted experimental studies of artificially created gradient magnetic materials are just beginning. The authors of [30] developed a technology for creating layered films in which the magnetic parameters of neighboring layers differ slightly from each other along the film thickness, simulating a predetermined law of magnetization $M(z)$ or exchange $\alpha(z)$. By appropriate selection of the composition of the alloy for each layer, the authors obtained samples in which only one of the magnetic parameters changed according to a given law, while the others remained approximately constant. Thus, a change in the ratio between the components of the Co_xNi_y alloy in a certain concentration range leads to a strong change in the magnetization M of the alloy with a small change in the exchange parameter α. In the Co_xP_y alloy, small changes in the addition of phosphorus (from 7 to 9 atomic %) lead to strong changes in the exchange parameter α with an almost unchanged magnetization M. The dependence of the material parameter on z with this technology is obtained in steps, but with small thicknesses of layers it can be approximated by a smooth function. The spin-wave resonance on the obtained gradient samples was also investigated in [30]. Further development of the technology proposed in [30] would make it possible to create gradient magnets with the most exotic laws of the dependence of magnetic parameters on z, which are necessary for practical applications. This makes it relevant to expand the class of existing gradient profiles of magnetic parameters that allow exact solutions of spin-wave equations.

The objectives of our work are (i) to search for new gradient dependencies of the parameters of magnetic anisotropy (Section 2) and exchange (Section 3) in addition to the existing ones, which allow one to obtain exact solutions of the equations of spin-wave oscillations, and (ii) to develop the theory of spin-wave resonance for these new accurate solutions. Existing exact solutions for these parameters are briefly discussed at the beginning of each section.

2. Coordinate-dependent magnetic anisotropy

The potential energy density in a magnetouniaxial crystal is

$$\mathscr{H} = \frac{1}{2}\alpha(\nabla\mathbf{M})^2 - \frac{1}{2}\beta(\mathbf{x})(\mathbf{M}\mathbf{l})^2 - \mathbf{H}\mathbf{M} - \frac{1}{2}\mathbf{H}^{\mathrm{m}}\mathbf{M}, \tag{1}$$

where $\mathbf{M}$ is the magnetization vector, α is the exchange parameter, β is the parameter of uniaxial magnetic anisotropy with the axis directed along ort $\mathbf{l}$, $\mathbf{H}$ is the external dc magnetic field, $\mathbf{H}_m$ is the demagnetizing field depending on $\mathbf{M}$, $x_i = \{x, y, z\}$. Oscillations of the magnetization vector are described by the Landau-Lifshitz equation

$$\frac{d\mathbf{M}}{dt} + g[\mathbf{M}\times\mathbf{H}_e] = 0, \tag{2}$$

where the effective magnetic field $\mathbf{H}_e$ is determined by the expression

$$\mathbf{H}_e = -\frac{\partial\mathscr{H}}{\partial\mathbf{M}} + \frac{\partial}{\partial x_i}\frac{\partial\mathscr{H}}{\partial\left(\frac{\partial\mathbf{M}}{\partial x_i}\right)}. \tag{3}$$

We consider a ferromagnetic layer (thin film), where the axis of anisotropy $\mathbf{l}$ and the magnetic field $\mathbf{H}$ are perpendicular to the film plane, along the coordinate axis z. Eq. (1) describes a film with a light anisotropy axis, if $\beta > 0$, and with a light plane, if $\beta < 0$. We consider a model where only the anisotropy parameter β depends on the coordinate z. We suppose that the magnitude of the magnetic field H is sufficient to magnetize the film perpendicular to its plane for any value of the coordinate z, that is, for any z, the inequality holds

$$H - [4\pi - \beta(z)]M > 0. \tag{4}$$

Performing the usual linearization of Eq. (2) ($M_x, M_y \ll M$, $M_z \approx M$), we obtain the equation for the resonance circular projection of the magnetization $m = M_x + iM_y$

$$\frac{d^2m}{dz^2} + \frac{1}{\alpha}\left[\frac{\omega-\omega_0}{\omega_M} - \beta(z)\right]m = 0, \tag{5}$$

where ω is the frequency, $\omega_0 = g(H - 4\pi M)$, $\omega_M = gM$, g is the gyromagnetic ratio. In the case when not the anisotropy, but the magnetization M depends on z, then the linearized equation of oscillations has a rather complicated form. Therefore, the authors of [10–16], considering this case, assumed that only variations of the magnetostatic field $H_m(z) = -4\pi M(z)$ has a significant effect on the properties of spin waves, and in the remaining terms M can be approximately considered constant. Then the equation takes the form

$$\frac{d^2m}{dz^2} + \frac{1}{\alpha}\left[\frac{\omega-\omega_0{}'}{\omega_M} + 4\pi M(z)\right]m = 0, \tag{6}$$

where $\omega_0{}' = g(H + \beta M)$. Both Eqs. (5) and (6) have a form similar to the Schrödinger equation for electrons in an external potential $U(z)$. Therefore, all exact solutions of the Schrödinger equation, known for different potential profiles $U(z)$ [3,4], are exact solutions of Eqs. (5) and (6) with the corresponding profiles $\beta(z)$ or $M(z)$.

This circumstance was used in the 60s-70s of the last century to develop the theory of spin-wave resonance in gradient films for some profiles of smooth variation of the magnitudes of magnetization $M(z)$ and uniaxial magnetic anisotropy $\beta(z)$. The development of the theory was stimulated by experimentally detected deviations of the dependence of the resonant frequencies ω_n (or resonant fields H_n) on the mode number n, from the quadratic law $\omega_n \propto n^2$ predicted by Kittel's theory [31] for homogeneous magnetic films. In the paper [10], it was supposed that these deviations are due to the smooth inhomogeneity of the magnetization M across the film thickness caused by various technological factors. The real dependences of the magnetization M on z were unknown, but there were grounds to assume that the function $M(z)$ decreases from the middle of the film to its surfaces. Therefore, for modeling, a decreasing parabolic function $M(z)$ was used, which, due to the minus sign in the expression for $H_m(z)$, leads to a parabolic potential well in the energy profile of the system. The exact solution of the Schrödinger equation, and, consequently, of Eqs. (5) and (6) in this case, can be written in terms of confluent hypergeometric function [2]. In some cases, technological factors led to another dependence of M on z: an almost linear decrease in this function from one surface of the film to another. For such cases, the solution to the Schrödinger equation and Eqs. (5) and (6) can be solved in terms of the Airy functions [2]. Regularities that take into account the features of the magnetic system and the boundary conditions on the surfaces of the film were obtained from these exact solutions by approximate analytical, graphical and numerical methods [11–16]. The main of these laws is the form of the function of the dependence of discrete frequencies ω_n (or resonance fields H_n) on the number of energy levels n. For frequencies $\omega_n < \omega_c$, where ω_c corresponds to the upper boundary of the gradient potential well, a model with a parabolic dependence of the magnetic parameters on z leads to the law $\omega_n \propto n$, and a model with a linear dependence of these parameters leads to the law $\omega_n \propto (n - 3/4)^{2/3}$, where $n = 1, 2, 3, \ldots$. For frequencies $\omega_n > \omega_c$, the dependence ω_n on n for both models with increasing n more and more approaches the law $\omega_n \propto n^2$ corresponding to a homogeneous film.

Theoretical works [11–16] allowed a qualitative explanation of the results of experimental studies of spin-wave resonance of those years and stimulated the improvement of the technology for producing films. Increasing technological requirements (high vacuum, temperature regimes, artificial formation of boundary conditions, etc.) led to the creation of more perfect permalloy films [17], for which Kittel's law $\omega_n \propto n^2$ was well implemented for all values n. However, deviations from this law for frequencies $\omega_n < \omega_c$ are still observed both on films of various alloys and on films of granulated materials consisting of small ferromagnetic particles in a nonmagnetic matrix. These deviations are qualitatively explained by the authors of experimental works in the framework of models of either a parabolic [18–24] or a linear [25–29] variations of magnetic parameters. It is possible that in order to describe such a naturally occurring gradient of magnetic parameters, it would be useful to apply the Pöschl-Teller potential [32], which makes it possible to simulate asymmetric potential wells [33]. Having finished the introductory part of this section, let us return to the goal of our work.

Using the method [1], we introduce the designation $V^2(z)$ for the normalized dependence of the gradient parameter on z

$$\beta(z) = \beta_0 V^2(z). \tag{7}$$

Equation (5) is coincides in its mathematical structure with Eq. (5.22) studied in Ref. [1] for electromagnetic waves in plasma with variable electron density. Therefore, we carry out transformations of Eq. (5), similar to the transformations of this equation. Representing $V^2(z)$ in the form

$$V^2(z) = \lambda + W^2(z), \tag{8}$$

where λ is an arbitrary dimensionless parameter, we do function replacement

$$m(z) = W^{-1/2}(z) f(z). \tag{9}$$

Making a change to a variable in the function $f(z)$

$$\eta = \int_0^z W(z_1) dz_1, \tag{10}$$

we leave the function W dependent on z. This leads to the equation

$$\frac{d^2 f(\eta)}{d\eta^2} + W^{-2}\left[\frac{\omega - \omega_0}{\alpha\omega_M} - \frac{\beta_0}{\alpha}V^2)\right] f(\eta) - W^{-4}\left[\frac{1}{2}W\frac{d^2W}{dz^2} - \frac{3}{4}\left(\frac{dW}{dz}\right)^2\right] f(\eta) = 0. \tag{11}$$

Requiring that the coefficient before $f(\eta)$ in last term in this equation was constant, we obtain the additional nonlinear differential equation for the selection of acceptable forms of function $W(z)$

$$W^{-4}\left[\frac{1}{2}W\frac{d^2W}{dz^2} - \frac{3}{4}\left(\frac{dW}{dz}\right)^2\right] = C. \tag{12}$$

Nonlinear Eq. (12) can have several solutions. One of the simplest possible solutions to this equation, as shown in [1,6], has the form

$$W(z) = \frac{1}{1 + \mu p z}, \tag{13}$$

where p is the characteristic wave number of the gradient inhomogeneity, μ is an arbitrary dimensionless parameter. This solution corresponds $C = (\mu p/2)^2$ in Eq. (12). Substituting Eq. (13) into Eq. (10) and performing integration, we obtain

$$\eta = \frac{1}{\mu p}\ln(1 + \mu p z). \tag{14}$$

We make a change to a variable in Eq. (11)

$$\zeta = \exp(\mu p \eta) = 1 + \mu p z. \tag{15}$$

As a result, Eq. (11) takes the form of the Bessel equation

$$\frac{d^2 f}{d\zeta^2} + \frac{1}{\zeta}\frac{df}{d\zeta} + \left[\Omega - \frac{s^2}{\zeta^2}\right] f = 0, \tag{16}$$

where the dimensionless frequency Ω and the index of the Bessel function s are expressed in terms of the parameters of the problem as follows

$$\Omega = \frac{1}{\alpha\mu^2 p^2 \omega_M}[\omega - \omega_0 - \lambda\beta_0\omega_M], \tag{17}$$

$$s^2 = \frac{1}{4} + \frac{\beta_0}{\alpha\mu^2 p^2}. \tag{18}$$

Solutions of Eq. (16) has the form

$$f(\zeta) = Z_s(Q\zeta), \tag{19}$$

where Z_s is the Bessel function of the index s, Q is the dimensionless wavenumber. Substituting Eq. (19) into Eq. (16) leads to the dimensionless dispersion law

$$\Omega = Q^2. \tag{20}$$

Valid values of Q and, respectively, Ω, are determined by the boundary conditions of the problem.

It is convenient to give Eq. (16) for our problem in a different form. Without introducing the function $f(z)$ (9) and the variable η (10), we make the change of variable

$$\zeta = 1 + \mu p z \tag{21}$$

directly in the original Eq. (5). As a result, we immediately obtain the Bessel equation in one of its forms for the desired function $m(\zeta)$

$$\frac{d^2 m}{d\zeta^2} + \left[\Omega - \frac{s^2 - 1/4}{\zeta^2}\right] m = 0, \tag{22}$$

whose solutions are of the form [34]

$$m(z) = \zeta^{1/2} Z_s(Q\zeta). \tag{23}$$

Thus, the representation of the original Eq. (5) in the form of the Bessel equation for potential (8) and (13) can be done with one simple change of variable (21). All the other replacements of functions and variables were needed only to find the kind of potential $V^2(z)$ for which such a transformation of Eq. (5) is possible. Similar to the equation for spin waves with variable magnetic anisotropy (5), the Schrödinger equation for electrons in the potential

$$U(z) = U_0\left(\lambda + \frac{1}{(1 + \mu p z)^2}\right) \tag{24}$$

in space ζ takes the form of the Bessel equation

$$\frac{d^2\psi}{d\zeta^2} + \left[\frac{2\mathfrak{m}_0}{\mu^2 p^2 \hbar}(E - \lambda U_0) - \frac{s^2 - 1/4}{\zeta^2}\right]\psi = 0. \tag{25}$$

Here $\mathfrak{m}_0$ is the mass of the electron and

$$s^2 = \frac{1}{4} + \frac{2\mathfrak{m}_0 U_0}{\mu^2 p^2 \hbar}. \tag{26}$$

The solution of Eq. (25) has the form similar to formula (23). Substitution of this solution into Eq. (25) leads to the dispersion law

$$E = \lambda U_0 + \frac{\mu^2 p^2 \hbar}{2\mathfrak{m}_0} Q^2, \tag{27}$$

where the permissible values of Q are determined by the boundary conditions of the problem.

The general solution of the original Eq. (5) for the potential in the form of Eqs. (8) and (13) is

$$m(z) = A\zeta^{1/2} J_s(Q\zeta) + B\zeta^{1/2} N_s(Q\zeta), \tag{28}$$

where J_s and N_s are the Bessel and Neumann functions, respectively, A and B are arbitrary constants. The index s and argument ζ of these functions are determined by Eqs. (18) and (21).

We consider below even $\beta(z)$ functions, therefore the function $W(z)$ in the expression for the potential will depend on the modulus z:

$$V^2(z) = \lambda + \frac{1}{(1 + \mu p|z|)^2}. \tag{29}$$

Equation (29) models in this case either a potential well ($\mu < 0$, function of the modulus, $V^2(z)$, decreases from the middle of the film to its surfaces) or a potential barrier ($\mu > 0$, function of the module, $V^2(z)$, increases from the middle of the film to its surfaces). Note that the term "barrier" here has a different meaning than in optics or acoustics [1,5–9], where it denotes an obstacle to the wave propagation, regardless of the form of its dependence on z. It is convenient to analyze the spectrum of spin waves for values of the potential well and the height of the potential barrier close to each other. In this case, the parameters λ and μ for the barrier and the well are different. We select the following values for these parameters:

$$\text{potential well,}\quad \lambda = -\frac{3}{4} \text{ and } \mu = -\frac{1}{4}, \tag{30}$$

$$\text{potential barrier,}\quad \lambda = 0 \text{ and } \mu = 1. \tag{31}$$

The value of the gradient wave number for both cases is the same, $p = 2/d$. The shape of the potentials $V^2(z)$ for the selected parameter values is shown in Fig. 1. The parameters of solution Eq. (28) are determined by Eqs. (15), (17) and (18), which have a different form for various forms of gradient profiles: for the solution $m_w(\zeta_w)$ in a potential well,

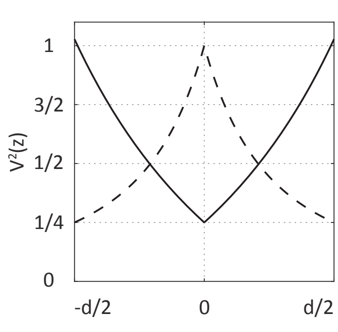

Fig. 1. Function $V^2(z)$ for the cases of a potential well ($\lambda = -3/4$, $\mu = -1/4$, solid curve) and barrier ($\lambda = 0$, $\mu = 1$, dashed curve).

$$\zeta_w = 1 - \frac{1}{2d}|z|, \tag{32}$$

$$\Omega_w = \frac{4d^2}{\alpha\omega_M}\left(\omega - \omega_0 + \frac{3}{4}\beta_0\omega_M\right), \tag{33}$$

$$s_w^2 = \frac{1}{4} + \frac{4\beta_0 d^2}{\alpha}, \tag{34}$$

for the solution $m_b(\zeta_b)$ for a potential barrier,

$$\zeta_b = 1 + \frac{2}{d}|z|, \tag{35}$$

$$\Omega_b = \frac{\omega - \omega_0}{4\alpha\omega_M} d^2, \tag{36}$$

$$s_b^2 = \frac{1}{4} + \frac{\beta_0 d^2}{4\alpha}. \tag{37}$$

The variable ζ in Eq. (28) for the cases of a potential well and a barrier differs not only in the value, but also in the direction of its change: in the first case it decreases with growth $|z|$, in the second case it increases. The values of the index of the Bessel functions, s_b^2 and s_w^2, also differ significantly. Below we omit the indices w and b for all quantities where this does not lead to misunderstandings. It is assumed that in the presence of a potential well, Eqs. (32)–(34) are used, and in the presence of a potential barrier, Eqs. (35)–(37) are used.

The general solution Eq. (28) with a symmetric potential (29) has the following form: for symmetric ($m(z) = m(-z)$) oscillations

$$m(z) = A\zeta^{1/2}[J_s(Q\zeta) + \rho N_s(Q\zeta)] \tag{38}$$

and for antisymmetric ($m(-z) = -m(z)$)

$$m(z) = \mathrm{sign}(z) A\zeta^{1/2}[J_s(Q\zeta) + \rho N_s(Q\zeta)], \tag{39}$$

where $\rho = B/A$. Equations (38) and (39) contain two parameters, Q and ρ, which are determined by the boundary conditions and the symmetry of the oscillations. The parameter A in the absence of force, exciting oscillations, remains arbitrary. We consider standing waves in a thin magnetic film for the conditions of both pinned oscillations on the surfaces of the film

$$m(z)|_{z=\pm d/2} = 0, \tag{40}$$

and unpinned

$$\left.\frac{dm(z)}{dz}\right|_{z=\pm d/2} = 0. \tag{41}$$

Substituting Eqs. (38) and (39) into Eqs. (40) and (41), we obtain the condition for both symmetric and antisymmetric pinned oscillations

$$J_s(Q\zeta_0) + \rho N_s(Q\zeta_0) = 0 \tag{42}$$

and for unpinned

$$\rho = \frac{(1 + 2s)J_s(Q\zeta_0) - 2QJ_{s+1}(Q\zeta_0)}{(1 + 2s)N_s(Q\zeta_0) - 2QN_{s+1}(Q\zeta_0)}, \tag{43}$$

where

$$\zeta_0 = \zeta(z)|_{z=\pm d/2}. \tag{44}$$

To find the two unknowns, Q and ρ, the equations must be added for $z = 0$, corresponding to the conjugation of solutions in the central plane of the film. Symmetric oscillations have an extremum in the center of the film, and this condition has the form

$$\left.\frac{dm(z)}{dz}\right|_{z=0} = 0, \tag{45}$$

and antisymmetric oscillations have a nodal point at $z = 0$ and

$$m(z)|_{z=0} = 0. \tag{46}$$

Substituting Eq. (38) into Eq. (45) and Eq. (39) into Eq. (46), we obtain the conjugation equations for symmetric

$$\rho = \frac{(1+2s)J_s(Q) - 2QJ_{s+1}(Q)}{(1+2s)N_s(Q) - 2QN_{s+1}(Q)}, \tag{47}$$

and for antisymmetric oscillations

$$J_s(Q) + \rho N_s(Q) = 0. \tag{48}$$

The results of solving Eqs. (38) and (39) are conveniently presented in the form of a matrix of the 2nd order m_n^{ij}, where the rows correspond to pinned ($i = p$) and unpinned ($i = u$), and the columns to symmetric ($j = s$) and antisymmetric ($j = a$) oscillations

$$m_n^{ij} = \begin{pmatrix} m_n^{ps}(z) & m_n^{pa}(z) \\ m_n^{us}(z) & m_n^{ua}(z) \end{pmatrix}. \tag{49}$$

The systems of equations for Q and ρ for pinned oscillations m_n^{ps} and m_n^{pa} is Eqs. (42), (45) and Eqs. (42), (46), respectively, and for unpinned oscillations m_n^{us} and m_n^{ua} – Eqs. (43), (45) and (43), (46), respectively.

The results obtained here for gradient films will be compared with the results of the theory of spin-wave resonance in a film with uniform magnetic anisotropy, the value of which is equal to the average value of the anisotropy of a gradient film. In this case, the equation for magnetic oscillations is

$$\frac{d^2m}{dz^2} + \frac{\omega - \omega_0}{\alpha\omega_M}m - \frac{\beta_0}{\alpha}\langle V^2(z)\rangle m = 0. \tag{50}$$

Eigenfunctions of this equation are harmonic oscillations, antisymmetric and symmetric, respectively

$$m_n^a \propto \sin k_n z, \quad m_n^s \propto \cos k_n z, \tag{51}$$

and the discrete spectrum of eigenvalues is determined by the dispersion equation

$$\omega_n = \omega_0 + \beta_0\langle V^2(z)\rangle\omega_M + \alpha\omega_M k_n^2, \tag{52}$$

where

$$k_n = n\frac{\pi}{d}, \tag{53}$$

where $n = 1, 3, 5, \ldots$ for symmetric and $n = 2, 4, 6, \ldots$ for antisymmetric oscillations. Here

$$\langle V^2(z)\rangle = \frac{1}{d}\int_{-d/2}^{d/2} V^2(z)dz. \tag{54}$$

For a potential of the form (29) we obtain

$$\langle V^2(z)\rangle = \lambda + \frac{1}{1+\mu pd/2}. \tag{55}$$

With the chosen values of the parameters λ, μ and p we have for the potential well and the potential barrier, respectively

$$\langle V^2(z)\rangle_w = \frac{7}{12}, \quad \langle V^2(z)\rangle_b = \frac{1}{2}. \tag{56}$$

The calculation was carried out for the following parameters of the gradient films: $M = 1000\,G$, $\alpha = 2\times10^{-12}$ cm^2, $\beta_0 = 6$, H = 13 kOe, d = 200 nm. The index values of the Bessel functions corresponding to these parameters are $s = 69.28$ for the potential well and $s = 17.33$ for the potential barrier. The shapes of the pinned (a) and unpinned (b) spin-wave oscillations for the case of a potential well are conventionally shown at the corresponding levels of the discrete frequency spectrum (Fig. 2). Black-covered modes correspond to symmetric and green dashed curves to antisymmetric oscillations. The amplitudes of oscillations at all levels n are normalized to the same value. The shape of potential well $V^2(z)$ is also given in the correspondent units of measurement. Outside the potential well, there are no oscillations, except for the tails of internal oscillations penetrating through the surface $V^2(z)$ as a result of tunneling. The critical frequency ω_c corresponds to

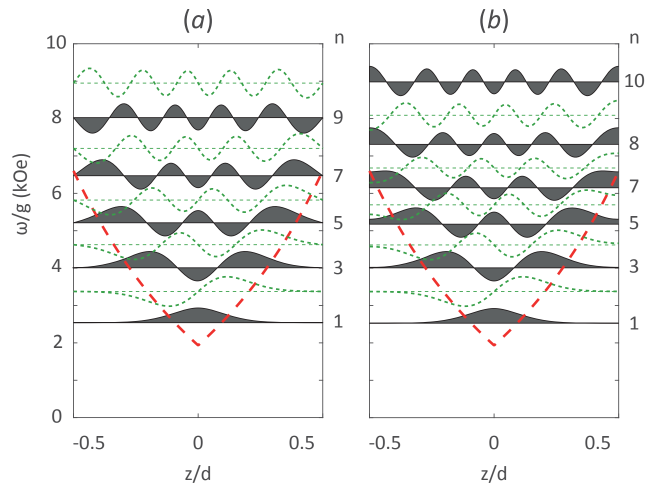

Fig. 2. Discrete spectral levels and normalized shapes of pinned (a) and unpinned (b) oscillations $m_n(z)$ for the case of a potential well. Black-covered modes correspond to symmetric and green dotted curves – to antisymmetric oscillations. The shape of the potential well in units of ω/g is also shown (thick red dashed curve).

the upper edge of the gradient potential well. The properties of oscillations at frequencies $\omega_n < \omega_c$ differ significantly from the properties of oscillations at frequencies $\omega_n > \omega_c$ occurring in a rectangular potential well formed by the surfaces of the film. At $\omega_n < \omega_c$ the ends of the magnetic oscillations are pinned on the "surface" of the potential well $V^2(z)$. This effect was found in Ref. [14] for films with a variable magnetization $M(z)$ and was called "dynamic pinning" there. It was studied also for electromagnetic and elastic waves [1]. Due to this effect, the shape of the oscillations at frequencies $\omega_n < \omega_c$ does not depend on the boundary conditions on the film surface (compare Fig. 2a and b). Due to the same effect, the odd oscillation modes at $\omega_n < \omega_c$ will be well excited by an external alternating field h, independent of z, for films with both pinned, Eq. (40), and unpinned, Eq. (41), oscillations. For these modes, the high-frequency magnetic susceptibility

$$\chi = \frac{1}{hd}\int_{-d/2}^{d/2} m(z)dz \tag{57}$$

is not equal to zero.

We will call critical the energy level $n = n_c$, the frequency of which is closest to ω_c from the side of low frequencies. As can be seen from Fig. 2, the mode of oscillations at $n \leqslant n_c$ differs significantly from the harmonic one: the effective "wavelength" of the oscillations increases from the center of the film to its surfaces. The form of oscillations at the levels $n > n_c$ with growth n closer to harmonic ones, the excitations of which by an external alternating field h are possible only for spins pinned on the film surface (Fig. 2a). The susceptibility of antisymmetric oscillations is zero for any boundary conditions and any n: for the excitation requires a high-frequency field, the amplitude of which depends on z. Note that the parity of the levels n of unpinned oscillations for $n > n_c$ changes to the opposite. In contrast to oscillations in a uniform film, in a gradient film, symmetric and antisymmetric oscillations are not generally described by different functions (coskz and sinkz), but by the product of a function $\zeta^{1/2}(z)$ by the sum of the Bessel J and Neumann N functions. The function $\rho_n(n)$, which is found from the corresponding system of equations for ρ and Q, characterizes the mathematical structure of the solution: the ratio between the amplitudes A and B functions J and N at each level n. This function is shown in Fig. 3 for pinned symmetric (circles) and antisymmetric (crosses) oscillations. At $n < 6$, the parameter ρ_n is close to zero. In this region, the same function J (with different phases) describes both symmetric and antisymmetric oscillations. The amplitude B in front of the function N increases when approaching the critical level $n_c = 7$. The conditional

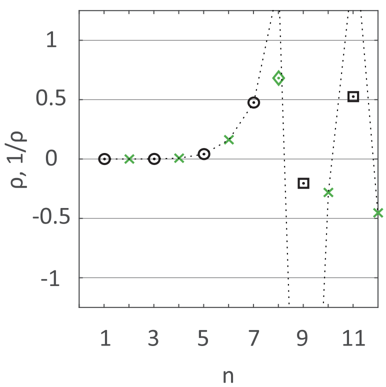

Fig. 3. Function ρ_n (black circles for symmetric and green crosses for antisymmetric oscillations) and $1/\rho_n$ (black squares for symmetric and green rhombuses for antisymmetric oscillations) vs n for the case of a potential well. The dotted curve is the conditional continuation of the function $\rho_n(n)$ between the points of its existence.

continuous dashed curve connecting the discrete values of the function ρ_n in this figure has the form of oscillations, the amplitude of which increases with growth n. Sharp rises and falls of a function $\rho_n(n)$ with a change in its sign correspond to drastic changes in the ratio between functions J and N at neighboring levels n. When the magnitude of the function ρ_n exceeds one, it is convenient to use the reciprocal $1/\rho_n$, which is also shown in Fig. 3. The function $\rho_n(n)$ for unpinned oscillations (not shown here) is qualitatively similar to the function in Fig. 3, differing from it in details: it increases when approaching the critical level $n_c = 7$ in the direction of negative, but not positive values ρ_n, and has smoother fluctuations about the axis $\rho_n = 0$. The basic characteristics of the oscillations calculated by us, which can be measured experimentally, are shown in Fig. 4: discrete frequency spectrum ω_n (a) and relative susceptibility χ_n/χ_1^0 (b), where χ_1^0 is the susceptibility of the first peak of a homogeneous film. For comparison, the same features for a uniform film are also shown. The function $\omega_n(n)$ for a gradient film has a complex shape. For $n < n_c$, the function $\omega_n(n)$ increases more slowly with n increasing than n in the first degree, has an inflection point near $n = n_c$, and for $n > n_c$ it acquires the dependence $\omega_n \propto n^2$ which is typical for a uniform film. The function $\omega_n(n)$ has a different physical meaning in a gradient and homogeneous medium. A packet of waves with different k, which is determined by the Fourier transform of the oscillation shape $m_n(z)$, corresponds to each discrete frequency ω_n in a gradient medium. One wave number k_n corresponds to each value n of function ω_n in a homogeneous medium, and the function $\omega_n(n)$ in this case reproduces the discrete points of the continuous dispersion law of spin waves of a homogeneous medium. From Fig. 4 (b), it can be seen that the susceptibility of the first peak of gradient films is less than the susceptibility of homogeneous films. However, it decreases with growth n much slower than the susceptibility of homogeneous films, and exceeds the latter by several times for peaks in the range from $n = 2$ to $n = n_c$.

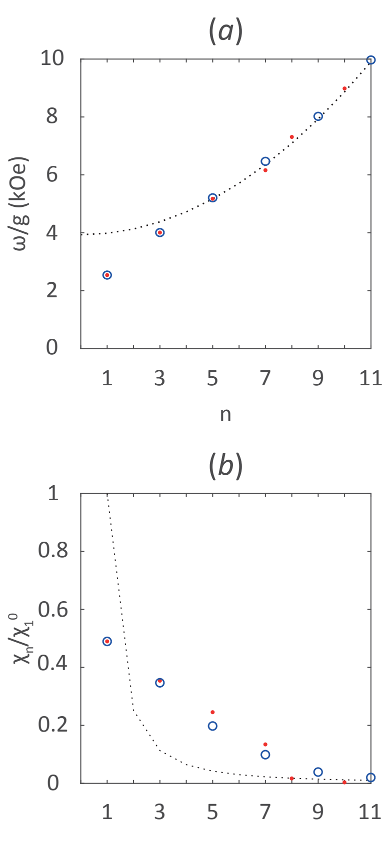

Fig. 4. Frequency spectrum $\omega_n(n)$ (a) and relative high-frequency susceptibility χ_n/χ_1^0 (b) of pinned (blue circles) and unpinned (red dots) oscillations for the case of a potential well. Black dashed curves show the same values for a uniform film.

The shapes of the spin-wave pinned (a) and unpinned (b) spin-wave oscillations for the case of a potential barrier are conventionally shown at the corresponding levels of the discrete frequency spectrum (Fig. 5). It can be seen that at $n < n_c$ the oscillations occur in two potential wells $V^2(z)$, bounded by the boundaries of the barrier and the corresponding film surfaces. There are no oscillations inside the barrier, except for the tails of external oscillations penetrating through the surface of the barrier as a result of tunneling. In the thin part of the barrier ($n = 5 - 7$), these tails can merge, forming transparency windows in the barrier. Symmetric oscillations with $n > n_c$ have noticeable distortions in the region above the top of the barrier. The numbering of the levels is chosen so that it corresponds to the number of half-waves of oscillations in the system of two interacting potential wells. Therefore, the parity of the number n varies not only for the unpinned oscillations, as in Fig. 2, but also for pinned ones. The sharp difference in the spectrum Fig. 5 from the spectrum for the potential well (Fig. 3) lies in the fact that the degeneracy of the spectral levels of symmetric and antisymmetric oscillations occurs at levels from $n = 1$ to $n = 5$. This degeneration is lifted for higher levels. Also, as in the case of a potential well, the oscillations have the form of the product of the function $\zeta^{1/2}(z)$ by the sum of the Bessel J and Neumann N functions, the relationship between the contributions of which is described by the parameter ρ_n.

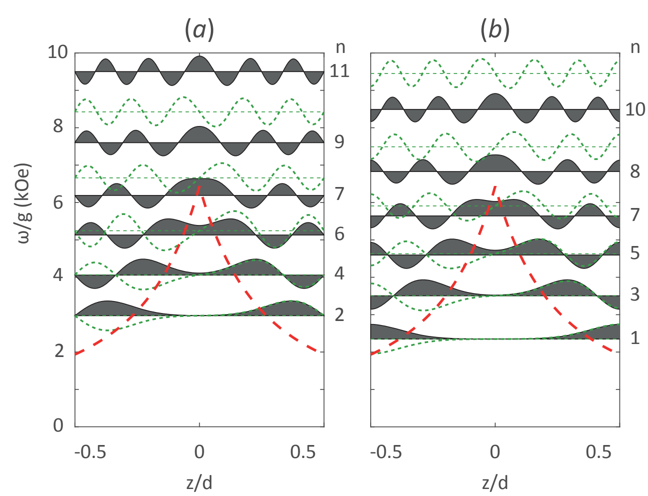

Fig. 5. Discrete spectral levels n and normalized modes for pinned (a) and unpinned (b) oscillations $m_n(z)$ for the case of a potential barrier. Designations correspond to Fig. 2.

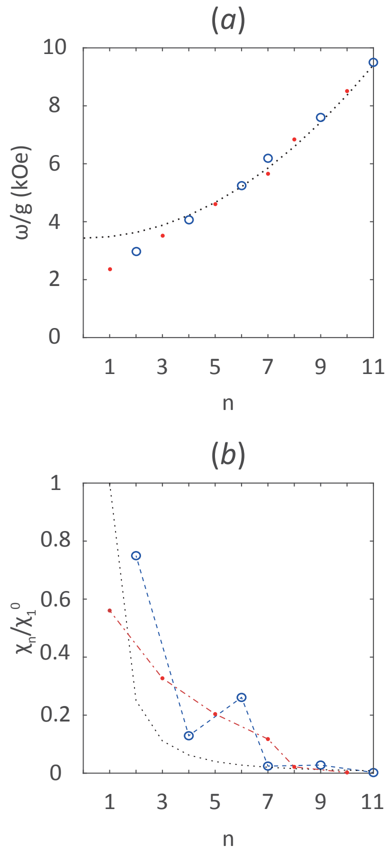

Fig. 6. Frequency spectrum $\omega_n(n)$ (a) and the relative high-frequency susceptibility χ_n/χ_1^0 (b) for the case of a potential barrier. Auxiliary lines (b) connect circles (blue dotted line) and dots (red dash-dotted line). The remaining designations correspond to Fig. 4.

The dependence of a parameter ρ_n on n (not shown here) a qualitatively similar to Fig. 3. When $n \ll n_c$, the oscillations are described by a function J, and with increasing n, both functions J and N, alternately make a primary contribution to the oscillation form. The frequency spectrum $\omega_n(n)$ and relative susceptibility χ_n/χ_1^0 for the potential barrier are shown in Fig. 6 (a) and (b), respectively. It can be seen that the function $\omega_n(n)$ for the potential barrier, in contrast to the case of the potential well (Fig. 4a), increases with n for $n < n_c$ according to a law that is close to linear. The susceptibility for pinned oscillations for a potential barrier differs from the susceptibility of oscillations in a potential well. Sharp dips of the function $\chi_n(n)$ occur at levels 4 and 8, which correspond to an even number of half-waves in each potential well on both sides of the barrier.

3. Coordinate-dependent exchange parameter

Consider the case when the gradient inhomogeneity in the expression for energy, Eq. (1), has only the exchange parameter $\alpha = \alpha(z)$. The wave equation according to Eqs. (2) and (3) in this case has the form

$$\alpha(z)\frac{d^2m}{dz^2} + \frac{d\alpha(z)}{dz}\frac{dm}{dz} + \frac{\omega - \omega_0}{\omega_M}m = 0, \tag{58}$$

where $\omega_0 = g[H - (4\pi - \beta)M]$ is the frequency of the uniform ferromagnetic resonance. Eq. (58), in contrast to Eq. (5), also contains the first derivative of the function $m(z)$. By its mathematical structure, this equation is equivalent to the equation for elastic waves in a medium with a variable shear modulus $G(z)$

$$G(z)\frac{d^2u}{dz^2} + \frac{dG(z)}{dz}\frac{du}{dz} + \frac{G_0\omega^2}{v_0}u = 0, \tag{59}$$

where v_0 is the velocity of elastic waves and G_0 is the shear modulus in a homogeneous medium. The profiles of the shear modulus and the corresponding exact solutions of Eq. (59) were found in Refs. [5,7]. In Ref. [5], the following profile model was investigated

$$G(z) = G_0(1 + az)^{2q}, \quad 0 < q < 1, \tag{60}$$

and the solution was obtained

$$u = A(1 + az)^{1/2-q}J_s[Q(1 + az)^{1-q}], \tag{61}$$

where the dimensionless wave number Q and the index of the Bessel function s, respectively,

$$Q = \frac{\omega}{av_0(1 - q)} \text{ and } s = \frac{q - 1/2}{1 - q}. \tag{62}$$

In Ref. [7], the authors used the auxiliary barrier method developed by them to find the profile $G(z)$ and solve Eq. (59). The solution was obtained by them in the space η

$$u(\eta) = AF^{1/2}(\eta)\exp\left(iQ\int_0^\eta F(\eta_1)d\eta_1\right), \tag{63}$$

where

$$d\eta = \frac{dz}{W^2(z)}, \quad W^2(z) = G(z)/G_0. \tag{64}$$

In this approach, the auxiliary profile in η space has the simple form

$$F^2(\eta) = (1 + s_1a_1\eta + s_2a_2^2\eta^2)^{-2}, \tag{65}$$

where a_1 and a_2 are arbitrary parameters, s_1 and s_2 can take values $\pm$ 1. The formula for the desired profile $W^2(z)$ after the transition to z space is cumbersome, so it is not given here. Our purpose is to find the simplest symmetric increasing and decreasing profiles of the function $\alpha(z)$ that allows us to obtain the exact solution of Eq. (58) or (59). Our approach allows us to obtain a simple expressions for both the profile $\alpha(z)$ and the exact solution of the wave equation, Eq. (58). This makes it possible to study spin-wave oscillations with a variable exchange parameter by analytical methods. We use the standard substitution of the function [34] to get rid of the first derivative in Eq. (58)),

$$\nu(z) = m(z)\exp\left\{\frac{1}{2}\int\frac{1}{\alpha(z_1)}dz_1\right\}. \tag{66}$$

Equation for $\nu(z)$ has the form

$$\frac{d^2\nu}{dz^2} + \frac{1}{\alpha(z)}\left[\frac{\omega - \omega_0}{\omega_M} - \left(\frac{1}{2}\frac{d^2\alpha}{dz^2} - \frac{1}{4\alpha(z)}\left(\frac{d\alpha}{dz}\right)^2\right)\right]\nu = 0. \tag{67}$$

We require that the exchange function $\alpha(z)$ satisfies the equation

$$\frac{1}{2}\frac{d^2\alpha}{dz^2} - \frac{1}{4\alpha(z)}\left(\frac{d\alpha}{dz}\right)^2 = C, \tag{68}$$

where C is a constant. One of the possible solutions of this equation is

$$\alpha(z) = \alpha_0(b + az)^2. \tag{69}$$

Substituting this solution causes the constant C to vanish, and Eq. (67) takes the form

$$\frac{d^2\nu}{dz^2} + \frac{\omega - \omega_0}{\alpha_0\omega_M}V^2(z)\nu = 0, \tag{70}$$

where

$$V(z) = \frac{1}{b + az}. \tag{71}$$

It can be seen that the mathematical structure of Eq. (70) corresponds to the structure of Eq. (2.9) in Ref. [1] for the vector potential component of electromagnetic waves in a medium with a gradient refractive index n. Therefore, we carry out transformations of Eq. (70) similar to the transformations of Eq. (2.9) in Ref. [1]. We introduce a new function F and a new variable η:

$$\nu = V^{-1/2}(z)F;\, V(z) = \Phi^{-1}(z);\, \eta = \int V(z_1)dz_1. \tag{72}$$

In this case, Eq. (70) takes the form

$$\frac{d^2F}{d\eta^2} + \frac{\omega - \omega_0}{\alpha_0\omega_M}F - \left[\frac{1}{2}\Phi\frac{d^2\Phi}{dz^2} - \frac{1}{4}\left(\frac{d\Phi}{dz}\right)^2\right]F = 0, \tag{73}$$

where $F = F(\eta)$ and $\Phi = \Phi(z)$. Require the function Φ to satisfy the equation

$$\frac{1}{2}\Phi\frac{d^2\Phi}{dz^2} - \frac{1}{4}\left(\frac{d\Phi}{dz}\right)^2 = C_1, \tag{74}$$

where C_1 is a constant. The authors of Ref. [1] found a solution to this equation in the form

$$\Phi(z) = \frac{1}{V(z)} = b + az + a_1z^2. \tag{75}$$

However, in our case, besides Eq. (74), Eq. (68) must be satisfied to eliminate the first derivative in Eq. (58). Both of these conditions are satisfied if $a_1 = 0$, $C_1 = -(a/2)^2$, and the function $\alpha(z)$ is determined by Eq. (69). Equation (73) in this case takes the form of an equation of oscillations with constant coefficients

$$\frac{d^2F}{d\eta^2} + \left[\frac{\omega - \omega_0}{\alpha_0\omega_M} + \left(\frac{a}{2}\right)^2\right]F = 0, \tag{76}$$

whose solutions are of the form

$$F_\pm(\eta) = \exp(\pm iq\eta), \tag{77}$$

$$m(z) = (b + az)^{-1/2}F(z), \quad \eta = \frac{1}{a}\ln(b + az). \tag{78}$$

The solution of a similar problem for elastic waves in a medium with a variable shear modulus $G(z)$ was found [5] for a profile, Eq. (60), in the interval $0 < q < 1$. That solution diverges at the point $q = 1$ (see Eq. (62)). Thus, the solution of the problem for the profile, Eq. (60), is Eq. (61) with $q < 1$ and our Eq. (78) with $q = 1$.

We consider below symmetric and antisymmetric spin-wave oscillations in a film with a gradient inhomogeneity symmetric along the z axis, therefore the function $\eta(z)$ will depend on the modulus z:

$$\eta = \frac{1}{a}\ln\zeta,\, \zeta = b + a|z|. \tag{79}$$

The general solution of Eq. (56) in this case is conveniently represented as

$$m(z) = A\zeta^{-1/2}[\sin(Q\ln\zeta) + \rho\cos(Q\ln\zeta)] \tag{80}$$

for symmetric oscillations and

$$m(z) = \mathrm{sign}(z)A\zeta^{-1/2}[\sin(Q\ln\zeta) + \rho\cos(Q\ln\zeta)] \tag{81}$$

for antisymmetric oscillations. Here, as in the case of Eqs. (38) and (39), $\rho = B/A$, $Q = q/a$ is the dimensionless wavenumber. The values Q and ρ are determined from the Eqs. (40) and (41) for the boundary conditions and the conjugation Eqs. (45) and (46), for symmetric and antisymmetric oscillations, respectively. The boundary condition for the pinned oscillations, the Eq. (38), leads to the equation

$$\rho = -\tan(Q\ln\zeta_0), \tag{82}$$

where $\zeta_0 = b + ad/2$. The boundary condition for unpinned oscillations, Eq. (39), leads to the equation

$$\rho = -\frac{\sin\varphi - 2Q\cos\varphi}{\cos\varphi + 2Q\sin\varphi}, \tag{83}$$

where $\phi = Q\ln\zeta_0$. We transform this equation by making a replacement

$$2Q = \tan\psi. \tag{84}$$

Then Eq. (83) takes the form

$$\rho = -\tan(Q\ln\zeta_0 - \psi), \tag{85}$$

where

$$\psi = \arctan(2Q). \tag{86}$$

The conjugation condition for symmetric oscillations, Eq. (45), leads to the equation

$$\rho = -\tan(Q\ln b - \psi), \tag{87}$$

and Eq. (46) for antisymmetric oscillations leads to the equation

$$\rho = -\tan(Q\ln b). \tag{88}$$

Equating the right-hand sides of Eqs. (82) and (85) to the right-hand sides of each of Eqs. (87) and (88), we obtain, taking into account the periodicity of the functions, all the elements of the matrix Q_n^{ij}

$$Q_n^{ij} = \frac{1}{\sigma}\left\|\begin{matrix} k\pi - \psi_n(Q^{ps}) & k\pi \\ k\pi & k\pi + \psi_n(Q^{ua}) \end{matrix}\right\|, \tag{89}$$

where $\sigma = \ln(1 + ad/2b)$, $k = 1, 2, 3, \ldots, n = 2k - 1$ for odd modes, and $n = 2k$ for even ones. The matrix (89) corresponds to the representation of oscillations m_n^{ij} in the form of a matrix (49). The elements of the matrix Q_n^{pa} and Q_n^{us} are equal to each other and are expressed through the parameters of the problem. The elements of the matrix Q_n^{ps} and Q_n^{ua} are transcendental equations that require solutions for a specific value of the parameters.

As a specific model, we consider an even function $\alpha(z)$ that depends on the module z, and choose the form of constants $a = \mu p$ and $b = 1 - \mu/2$ in which this function takes the form

$$\alpha(z) = \alpha_0\left(1 - \frac{1}{2}\mu + \mu p|z|\right)^2. \tag{90}$$

Here $p = 2/d$ is the wave number of the gradient inhomogeneity and μ is a dimensionless parameter. Equation (90) describes both the increasing ($\mu > 0$) and decreasing ($\mu < 0$) functions (Fig. 7) with the same basic characteristics: the depth of the well $2|\mu|$ coincides with the height of the barrier and the average value of both functions is the same:

$$\langle\alpha(z)\rangle = \alpha_0\left(1 + \frac{1}{12}\mu^2\right). \tag{91}$$

For this model, the components of the matrix Q_n^{ij} are expressed in the terms of μ:

$$Q_n^{ij} = q_n^{ij}\frac{d}{2\mu}, \quad \psi_n^{ij} = \arctan\left(q_n^{ij}\frac{d}{\mu}\right), \quad \sigma = \ln\frac{1 + \mu/2}{1 - \mu/2}. \tag{92}$$

The components of the matrix of discrete frequencies are

$$\omega_n^{ij} = \omega_0 + \alpha_0\omega_M[(\mu/d)^2 + (q_n^{ij})^2], \tag{93}$$

where q_n^{ij} is determined from the equations for the components of the matrix Q_n^{ij}. Formulas for q_n^{pa} and q_n^{us} are the same:

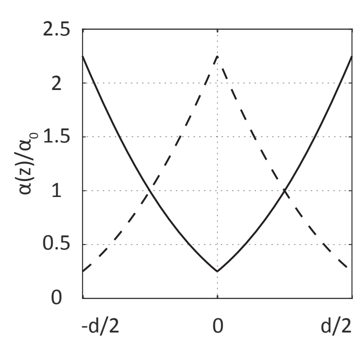

Fig. 7. Exchange parameter for increasing ($\mu = 1$, solid curve) and decreasing ($\mu = -1$, dashed curve) from the middle of the film to its surfaces of the function $\alpha(z)$.

$$q_n^{pa} = q_n^{us} = \frac{\mu}{d}\frac{n\pi}{\sigma}, \tag{94}$$

where $n = 2, 4, 6, \ldots$. Transcendental equations for finding q_n^{ps} and q_n^{ua} have the form

$$q_n^{ij} = \frac{\mu}{\sigma d}[(n+1)\pi \mp 2\arctan(q_n^{ij} d/\mu)], \tag{95}$$

where $n = 1, 3, 5, \ldots$, the sign "$-$" in the numerator corresponds q_n^{ps} and the sign "$+$" corresponds q_n^{ua}. Expanding in this equation $\arctan x$ for large x to the first two terms and solving the resulting equation, we get in the first approximation

$$q_n^{ps} = \frac{\mu n\pi}{\sigma d}\left(1 + \frac{2}{n(n+1)\pi^2}\right). \tag{96}$$

The shape of symmetric and antisymmetric pinned oscillations is shown in Fig. 8 for the cases of the increasing (a) and decreasing (b) function $\alpha(z)$. The square root of the frequency is in Fig. 8 along the ordinate axis, in contrast to the frequency, which was on similar graphs for the case of variable anisotropy, Fig. 2 and 5. This allows you to make the space between the spectral levels n approximately the same and sufficient to demonstrate the shape of the oscillations at each level. It is seen that the form of oscillations differs sharply from the harmonic. The effective wavelength decreases towards the middle of the film for increasing $\alpha(z)$ and increases for decreasing $\alpha(z)$. The functions ρ_n and $1/\rho_n$ (not shown here), characterizing the ratios of the contributions of the sine-like and cosine-like functions to the oscillation form, do not have a smooth section for small n, as in Fig. 3. These functions experience more frequent and sharp jumps when changing n than similar functions that characterize the relationship between the Bessel and Neumann functions in the case of variable anisotropy. The frequency spectrum $\omega_n^{ij}(n)$ for m_n^{pa} and m_n^{us} is quadratic, as well as for a homogeneous medium. For m_n^{ps} and m_n^{ua} there are deviations from the quadratic law in the region of small n. However, these deviations are small. From Eq. (96) it follows that the position of the level $n = 1$ for q_n^{ps} differs by $1/\pi^2$ from a quadratic law, and the positions of the subsequent levels $n = 3, 5$ differ by less than one hundredth of its magnitude. Therefore, discrete frequencies for m_n^{ij} in the coordinate system ω_n, n^2 are located along straight lines (Fig. 8a). The effective exchange parameter has the form

$$\alpha_{\text{eff}} = \alpha_0 \varkappa^2, \qquad \varkappa = \frac{\mu}{\ln\frac{1+\mu/2}{1-\mu/2}}, \tag{97}$$

where the coefficient $\varkappa$ characterizes the ratio of the angle of inclination

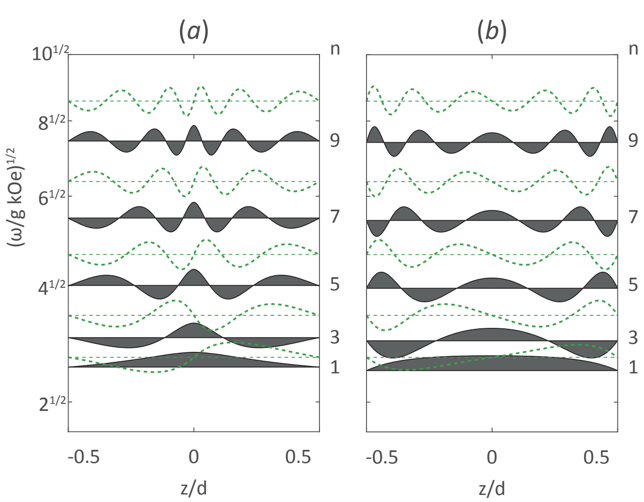

Fig. 8. Discrete spectral levels n and the normalized forms of the pinned oscillations for the case of increasing, $\mu = 1$, (a) and decreasing, $\mu = -1$, (b) function $\alpha(z)$. Designations correspond to Fig. 2.

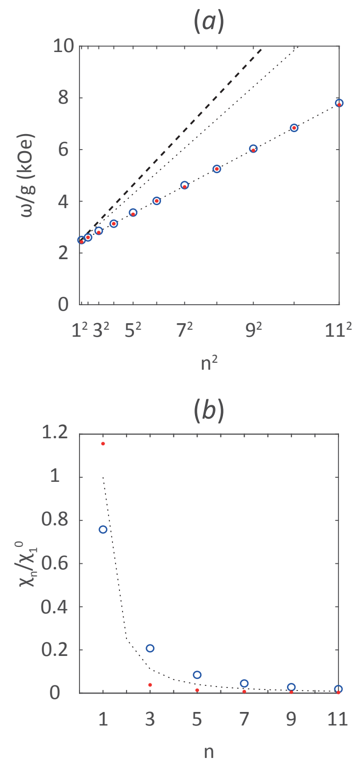

Fig. 9. Frequency spectrum $\omega_n(n)$ vs n^2 (a) and relative high-frequency susceptibility χ_n/χ_1^0 vs n (b) for the cases of increasing ($\mu = 1.5$, blue circles) and decreasing ($\mu = -1.5$, red dots) function $\alpha(z)$. The same values for a homogeneous film with $\mu = 0$ (dashed curves) and μ, corresponding to $\langle\alpha(z)\rangle$ (thick dashed curve).

of this straight line to the angle of inclination of the dotted curve (Fig. 9a), corresponding to $\mu = 0$, for which $\varkappa = 1$. The coefficient $\varkappa \leqslant 1$ and does not depend on the sign of μ: variations of the exchange parameter lead to a decrease in α_{eff} for both the increasing and decreasing functions $\alpha(z)$. The effect of gradient inhomogeneity is more accurately characterized by the ratio of the angle of inclination of the straight line corresponding to α_{eff} to the angle of inclination of the dashed curve in Fig. 9a, corresponding to $\langle\alpha(z)\rangle$. The frequency spectrum in Fig. 9a is shown for $\mu = 1.5$, which corresponds to the gradient change in function $\alpha(z)$ in *50* times along the film thickness. The high-frequency susceptibility of the film with a variable exchange parameter differs little from the susceptibility of a homogeneous film (Fig. 9b).

At the end of this section, we will briefly deviate from magnetic topics. The model of the Hamiltonian with z-dependent effective mass is being intensively studied for the Schrödinger equation at present [35,36]. This model is closest to the model with a varying interaction constant considered here, differing from it in that the potential U also depends on z. We consider here only a simpler version of this model when U is constant. The Schrödinger equation in this case is

$$\frac{d}{dz}\left[\frac{1}{\mathfrak{m}(z)}\frac{d\psi}{dz}\right] + \frac{2}{\hbar}(E - U)\psi = 0. \tag{98}$$

Replacing the gradient function $\mathfrak{m}(z)$ with the inverse function

$$\mathfrak{m}(z) = \frac{\mathfrak{m}_0}{\upsilon(z)}, \tag{99}$$

leads Eq. (98) to form

$$\upsilon(z)\frac{d^2\psi}{dz^2} + \frac{d\upsilon(z)}{dz}\frac{d\psi}{dz} + \frac{2\mathfrak{m}_0}{\hbar}(E - U)\psi = 0. \tag{100}$$

The structure of Eq. (100) coincides with the structure of Eq. (58), in which the function $\alpha(z)$ plays the role of the variable coefficient $\upsilon(z)$.

We carry out transformations for Eq. (100), similar to those that were made for Eq. (58). Using Eq. (69) for the variable coefficient $\upsilon(z)$ obtained in this paper, we write the variable coefficient $\mathfrak{m}(z)$ in the form

$$\mathfrak{m}(z) = \frac{\mathfrak{m}_0}{\zeta^2(z)}, \tag{101}$$

where $\zeta = b + az$, and we find the exact solution of Eq. (98) with this coefficient

$$\psi^{\pm}(z) = \frac{A}{\sqrt{\zeta(z)}}\exp[\pm iQ\ln\zeta(z)]. \tag{102}$$

When modeling the effective mass $\mathfrak{m}(z)$ of a function even in z, the form of Eqs. (101) and (102) does not change, but ζ depends on the module z (see Eq. (90))

$$\zeta(z) = 1 - \frac{1}{2}\mu + \frac{2\mu}{d}|z|, \tag{103}$$

where z varies in the interval

$$-d/2 \leqslant z \leqslant d/2. \tag{104}$$

Equation (103) describes an even increasing function $\mathfrak{m}(z)$ with $\mu < 0$ and even decreasing function with $\mu > 0$. Substitution of the solution (102) into Eq. (98) leads in this case to the dispersion equation

$$E = U + \frac{2\mu^2\hbar}{\mathfrak{m}_0 d^2}\left(\frac{1}{4} + Q^2\right). \tag{105}$$

The dimensionless dispersion parameter Q is determined by the boundary conditions of the problem. According to the same considerations, two more exact solutions of the Schrödinger equation, Eq. (98), can be written: this is function (61) if $\mathfrak{m}(z)$ is a function inverse to function (60) found in [5], and also function (63) if $\mathfrak{m}(z)$ is a function inverse to function (65) found in [7].

4. Conclusions

The method of searching for the profiles of the gradient dependence of the material parameters of a substance on the coordinate, allowing the exact solution of the wave equations, developed earlier for electromagnetic and elastic waves [1], is generalized to spin waves in gradient ferromagnets. The derivation and solution of additional nonlinear equations for previously unknown profiles of the gradient coefficients of wave equations is the basis of this method. The equation for spin waves with variable magnetic anisotropy $\beta(z)$ is equivalent to the equation for electromagnetic waves in a plasma with variable density previously studied by this method [1,6]. A brief repetition of the derivation of the additional equation introduced there is given in our work for the case $\beta(z)$ to demonstrate the method. For further development of the theory of spin-wave oscillations, we will use a slightly modified solution of the additional equation found in [1,6], replacing z with $|z|$. This made it possible to consider spin-wave oscillations for the cases of both increasing and decreasing function $\beta(z)$. The function of the module z, $\beta(z)$, increasing from the middle of the film to its surfaces, describes a potential well in the energy (frequency) spectrum of oscillations, the decreasing function $\beta(z)$ describes a potential barrier with a top in the central plane of the film. The discrete frequency spectrum ω_n in both cases contains the critical level $n = n_c$, the frequency of which $\omega_n \leqslant \omega_c$ is most closely located to the frequency ω_c of the upper edge of the gradient potential well (or the top of the potential barrier) from the low frequencies. Oscillations at levels $n < n_c$ with an increasing function $\beta(z)$ occur inside a gradient potential well, at levels $n > n_c$ – in a rectangular potential well bounded by the film surfaces. This case differs from the previously investigated case of spin-wave resonance in a ferromagnet with a parabolic well [10–16] only in the shape of this well. Therefore, the main qualitative characteristics for it are known, there are only quantitative differences. In both cases, dynamic pinning of oscillations [14] takes place on the "surface" of the gradient potential and their damping tunnelling through this surface. Due to the effect of dynamic pinning, the shape of the oscillations and the possibility of their excitation by an external alternating field at $n < n_c$ does not depend on the boundary conditions on the film surface. The function $\omega_n(n)$ for oscillations in the potential well for $n < n_c$ increases with increasing n more slowly than n in the first degree and has an inflection point near $n = n_c$ (Fig. 4*a*). The theory of spin-wave resonance in a ferromagnet with a decreasing function $\beta(z)$ (potential barrier) was developed for the first time in this work. Oscillations at levels $n < n_c$ in this case occur in two potential wells bounded by the boundaries of the barrier and the surfaces of the film. There are no oscillations inside the barrier, with the exception of external oscillation tails penetrating the barrier surface as a result of tunneling. In the thin upper part of the barrier, these tails may merge, forming transparency windows in the barrier. The function $\omega_n(n)$ for oscillations in the potential barrier, in contrast to the case of the potential well, increases with n for $n < n_c$ according to a law close to linear (Fig. 6*a*). The susceptibility of gradient films in the interval $1 < n < n_c$ exceeds the susceptibility of homogeneous films by several times. This may be of interest to practical application of gradient films. For $n > n_c$, the functions ω_n and χ_n, with increasing n, approach the functions characteristic of homogeneous films: $\omega_n \propto n^2$ and $\chi_n \propto 1/n^2$. Changing the shape of the function $\omega_n(n)$ when $n = n_c$, as well as a sharp decrease in χ_n at the same point, makes it possible to experimentally determine the frequency ω_c of the upper edge of the gradient potential well (height of the potential barrier).

The equation for spin waves with a variable exchange parameter $\alpha(z)$ is in its mathematical structure equivalent to the equation for elastic waves in a medium with a variable shear modulus $G(z)$. The additional equation and the following solution for $G(z)$ obtained earlier [1,7] were rather cumbersome. Therefore, we found another additional equation that allowed us to obtain a simple expression for the profile of the gradient $\alpha(z)$ and to carry out an analytical study of the exact solution of the wave equation. The theory of spin-wave resonance in a ferromagnet with a gradient (both concave and convex) of exchange $\alpha(z)$ is developed for the first time in this work. The coordinate-dependent exchange $\alpha(z)$, in contrast to the coordinate-dependent $\beta(z)$, does not change the shape of the energy potential. Spin-wave oscillations in a film with a variable exchange occur in the same rectangular potential well created by the film surfaces, as in a uniform film, regardless of the profile of the function $\alpha(z)$. The form of oscillations is different from harmonic one. The discrete frequency spectrum ω_n, depending on the symmetry and boundary conditions of the oscillations, is either quadratic, as in a homogeneous film, or has small deviations from the quadratic. The main effect of the variable exchange $\alpha(z)$ is that it leads to a decrease in the effective exchange parameter in the law of ω_n versus n. The analytical formula for the effective exchange parameter is derived.

The authors of Ref. [30], who for the first time created ferromagnetic films with predefined profiles of variable magnetic parameters, also investigated spin-wave resonance in films with an artificially created parabolic profile of the magnetization $M(z)$ and a wedge-shaped profile of exchange $\alpha(z)$. The strong deviations from the quadratic dependence of ω_n on n, which they obtained for the case of changing $\alpha(z)$, and the results of our theory, which allows only negligible deviations, diverge from each other. The theoretical result follows from the general statement that the form of the energy potential is independent of the form of the function $\alpha(z)$. Therefore, it can be assumed that discrepancy between theory and experiment is caused by to the fact that some gradient parameter, besides $\alpha(z)$, was present in the samples of Ref. [30].

Exact solutions of the wave equations with the coordinate-dependent coefficients found in the work can be used in the theory of electromagnetic and elastic waves in gradient metamaterials, as well as in the theory of electrons in condensed media. Examples of this application of the results are given in the work: the exact solution of the wave equation with variable magnetic anisotropy $\beta(z)$ was used to find the exact solution of the Schrödinger equation, Eq. (25), for electrons in the potential $U(z)$, proportional to $\beta(z)$; the exact solution of the wave equation with the variable exchange parameter $\alpha(z)$ was used to find the exact solution of the Schrödinger equation, Eq. (98), with the effective mass profile $\mathtt{m}(z)$, proportional to the function, inverse to the function $\alpha(z)$.

CRediT authorship contribution statement

V.A. Ignatchenko: Conceptualization, Methodology, Writing - original draft, Visualization. **D.S. Tsikalov:** Software, Investigation, Writing - review & editing, Visualization, Validation.

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