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Letter

General formula for the natural width of optical parametric oscillator spectral lines

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Abstract
A new approach to the quantum analysis of the spectral linewidths of optical parametric oscillators (OPOs) is proposed. The approach is based on introducing a Hermitian frequency deviation operator. The spectral linewidths in the triply resonant OPO above-threshold regime have been calculated. It has been found that the width of the generated spectral line depends on the loss at all interacting frequencies, the photon number of frequency under study, and the threshold photon number of pump. The formula obtained for the spectral width is general; in particular cases, it yields the well-known results. The developed simple approach can be applied to the quantum analysis of the spectra of various oscillatory systems, for example, the multifrequency parametric and non-linear optomechanical interactions and lasers.

Keywords: optical parametric oscillator, phase fluctuation, frequency deviation operator, natural spectrum width

1. Introduction

The width of the emission spectrum of optical parametric oscillators (OPOs) is extremely important for their application, for example, to the areas of high-sensitive interferometry and coherent optical communications. The experimental and theoretical study of this problem has attracted close attention since the launch of the OPO (see reviews [1, 2]). Here, we discuss the above-threshold regime of the OPO generation. The subthreshold regime is, as is known, of particular interest for obtaining quadrature-squeezed light (see review [3]) and is beyond our consideration.

In the above-threshold excitation regime of both lasers and parametric oscillators, the finite width of the emission spectrum is related to phase fluctuations [4]. The spectral linewidth was theoretically studied in the OPOs with non-linear crystals located both outside [5–7] and inside a laser resonator [8], the radiation of which is used as pumping. In the calculation [6], the phase fluctuations of the pump radiation were also taken into account.

To analyze the phase fluctuations, some authors [9–11] used a method of linearization with respect to the fluctuations near the exitation threshold, while others [6, 7] used an adiabatic approximation for the pump frequency and one of the generated frequencies. In the latter case, the problem is reduced to analyzing the fluctuations of the van der Pol oscillator [4, 6, 7]. The linearization method has a clear physical justification regarding the linearization by the intensity fluctuations, which are small in the above-threshold generation regime. As for the phase fluctuations, the oscillators do not have a physical mechanism that would limit the phase incursion, the variance of which obeys, as a rule, the diffusion law.

In this paper, the approach developed by us made it possible to get rid of the above-mentioned phase approximation and take into account the quantum nature of broadening of the OPO spectral lines. The introduced Hermitian operator of the frequency deviation allowed us to circumvent the difficulty related to problem of the phase operator (see [12–14] and the references cited there), the fluctuations of which cause a finite width of the spectral lines. We obtained an general expression for the triply resonant OPO spectral linewidths, the particular cases of which were discussed in [6, 7].

In addition, it is shown that the spectral width of the phase difference of the parametrically generated frequencies can be narrower than the linewidth of a single frequency.

2. Basic equations

As in the available studies, we use the OPO model with lumped parameters for analyzing the spectral linewidths. In this approach, the study of the OPO dynamics is reduced to solving time stochastic equations. The conditions for the transition from the equations describing the non-linear-optical interactions in a distributed system in partial derivatives to the equations with ordinary derivatives were considered, for example, in [4, 16].

We consider a triply-resonant OPO whose frequencies satisfy the condition $\omega_3 = \omega_1 + \omega_2$, where ω_3 is the frequency of the monochromatic pump wave and ω_1 and ω_2 are the excited frequencies. We assume that the phase matching condition is fulfilled. Thus, the problem is reduced to calculating the spectral widths of three coupled oscillators under the action of vacuum fluctuations and thermal noise related to the loss in a non-linear crystal and radiation from the resonator.

The Bose operators of the pump frequency and the generated frequencies are denoted by a_3 and a_1, a_2, respectively. The pump field entering the cavity contains a great number of photons; its amplitude and phase fluctuations are ignored and described in a classical manner by introducing the notation α_3. In our formulation, the Heisenberg–Langevin equations describing the investigated three-frequency process have the form (compare with [11])

$$\frac{da_1(t)}{dt} = ga_3(t)a_2^\dagger(t) - \gamma_1 a_1(t) + \hat{\xi}_1(t), \tag{1}$$

$$\frac{da_2(t)}{dt} = ga_3(t)a_1^\dagger(t) - \gamma_2 a_2(t) + \hat{\xi}_2(t), \tag{2}$$

$$\frac{da_3(t)}{dt} = -ga_1a_2 - \gamma_3(a_3 - \alpha_3^{(0)}) + \hat{\xi}_p(t). \tag{3}$$

Here, g is the non-linear wave coupling coefficient, the coefficients $\gamma_j = \kappa_j/T_c$, T_c is the cavity round-trip time, the coefficient κ_j takes into account the loss in the non-linear crystal $\kappa_j^{(cr)}$ and the loss to reflection from mirrors $\kappa_j^{(m)}$:

$$\kappa_j = \kappa_j^{(cr)} + \kappa_j^{(m)}. \tag{4}$$

At the same time, we have

$$\kappa_j^{(cr)} = (1/2)\delta_j L, \kappa_j^{(m)} = (1/2)(1 - R_j), \tag{5}$$

where δ_j is the coefficient of linear loss per crystal, L is the crystal length, and R_j is the mirror reflection coefficients for the intensity.

In equations (1)–(3), the random source operators are determined by the relations

$$\hat{\xi}_j(t) = \sqrt{2\gamma_j^{(cr)}}\hat{b}_j(t) + \sqrt{2\gamma_j^{(m)}}\hat{c}_j(t). \tag{6}$$

Operators $\hat{b}_j(t), \hat{c}_j(t)$ satisfy the conventional commutation relations

$$[\hat{b}_j(t), \hat{b}_k^\dagger(t_1)] = [\hat{c}_j(t), \hat{c}_k^\dagger(t_1)] = \delta_{jk}\delta(t_1 - t),$$
$$\left\langle \hat{b}_k^\dagger(t_1)\hat{b}_j(t)\right\rangle = \left\langle \hat{c}_k^\dagger(t_1)\hat{c}_j(t)\right\rangle =$$
$$\delta_{jk}\langle n_j(T)\rangle\,\delta(t_1 - t). \tag{7}$$

Here, $\langle n_j(T)\rangle$ is the mean thermal photon number at the corresponding frequency:

$$\langle n_j(T)\rangle = \frac{1}{e^{\hbar\omega_j/kT} - 1}, \tag{8}$$

where k is the Bolzmann constant, T is the temperature.

3. Frequency deviation operator

In the process under study, the width of the frequency spectrum is due to the phase fluctuations. However, as noted above, introducing of a phase operator encounters some difficulties. To solve our problem, we will proceed as follows. We introduce the Hermitian operator

$$\hat{\Omega}_j(t) = \frac{i}{2\langle\hat{n}_j(t)\rangle}(\dot{a}_j^\dagger(t)a_j(t) - a_j^\dagger(t)\dot{a}_j(t)), \tag{9}$$

where dots indicate time derivatives, $\langle\hat{n}_j(t)\rangle$ is the mean photon number after a time t and $\hat{n}_j(t) = a_j^\dagger(t)a_j(t)$ is the number operator.

To clarify the physical meaning of the Hermitian operator $\hat{\Omega}_j(t)$, we consider the continuous-mode field Bose operators $a_j(t), a_j^\dagger(t)$ for the coherent state (see, e.g., [21]): $a_j(t)|\{\alpha_j\}>= \alpha_j(t)|\{\alpha_j\}>$. In the narrow-bandwidth approximation used, the eigenvalue is $\alpha_j(t) = |\alpha_j(t)|e^{i\phi(t)}$, where $\phi_j(t)$ is the phase. We assume that the characteristic scale of the phase change $\phi_j(t)$ is much smaller than that of the envelope $|\alpha_j(t)|$.

Therefore, we can put

$$\dot{a}_j(t)|\{\alpha_j\}>= i\dot{\phi}(t)\alpha_j(t)|\{\alpha_j\}>. \tag{10}$$

The mean value of the operator $\hat{\Omega}j(t)$ (9) over the coherent state is

$$\left\langle <\{\alpha_j\}|\hat{\Omega}_j(t))|\{\alpha_j\}>\right\rangle = \dot{\phi}(t). \tag{11}$$

It can be seen that the mean value $\left\langle\hat{\Omega}j(t)\right\rangle$ determines the time derivative of the phase, i.e. the frequency deviation. It should be noted that the Hermitian property of the frequency deviation operator is consistent with the fact that observable is the phase difference [13, 14], rather than the phase itself. The phase difference can be measured, for example, in heterodyning and interferometry.

This conclusion forms a basis for further calculations of the variance of the phase incursion fluctuations. In the OPO above-threshold regime, the relative fluctuations of the photon number at the excited frequencies are much less than unity. Therefore, in the above-threshold generation regime, we can replace the mean value $\langle \hat{n}_j(t) \rangle$ by the stationary value $\left\langle \hat{n}_j^{(st)} \right\rangle = \bar{n}_j$. As a result, the formula (9) is simplified:

$$\hat{\Omega}_j(t) = \frac{i}{2\bar{n}_j}(\dot{a}_j^\dagger(t)a_j(t) - a_j^\dagger(t)\dot{a}_j(t)). \qquad (12)$$

In order to obtain the relations for the photon numbers at the interacting frequencies in the stationary regime, we omit the fluctuation terms in the system of equations (1)–(3) [7]:

$$\gamma_1\bar{n}_1 = \gamma_2\bar{n}_2,\ \bar{n}_3 = \bar{n}_3^{(thr)} = \frac{\gamma_1\gamma_2}{g^2},$$
$$\bar{n}_1 = \frac{\gamma_2\gamma_3}{g^2}\left(\sqrt{n_3^{(0)}/\bar{n}_3^{(thr)}} - 1\right), \qquad (13)$$

where $\bar{n}_3^{(thr)}$ is the pump threshold. The stationary phase relation is $\phi_3^{(st)} - \phi_2^{(st)} - \phi_1^{(st)} = 0$ or 2π. As expected, the parametric generation process is implemented when the pump photon number $n_3^{(0)}$ exceeds the threshold. Moreover, the entire photon number excess over the generation threshold goes to the excitation of parametric frequencies. The stationary value of the photon pump number remains at the threshold level.

Using equations (1)–(3), for example, for the operator $\hat{\Omega}_1$ at frequency ω_1, we obtain the equation

$$\hat{\Omega}_1(t) = \frac{i}{\bar{n}_1}(g\Sigma(t) + a_1(t)\hat{\xi}_1^\dagger(t) - a_1^\dagger(t)\hat{\xi}_1)(t). \qquad (14)$$

The function $\Sigma(t) = a\dagger_3 a_2 a_1 - a_3 a_2^\dagger a_1^\dagger$ satisfies the equation

$$\dot{\hat{\Sigma}}(t) = -\Gamma^{-1}\hat{\Sigma}(t) + a_3^\dagger a_2\hat{\xi}_1 - a_3 a_2^\dagger\hat{\xi}_1^\dagger +$$
$$a_3^\dagger a_1\hat{\xi}_2 - a_3 a_1^\dagger\hat{\xi}_2^\dagger + a_1 a_2\hat{\xi}_3^\dagger - a_1^\dagger a_2^\dagger\hat{\xi}_3, \qquad (15)$$

where $\Gamma = \gamma_1 + \gamma_2 + \gamma_3$. To simplify the notation, the temporary argument for the operators a_j and $\hat{\xi}_j$ is hereinafter omitted.

First of all, we note that, after averaging of equation (15) over the state of random sources, we obtain the equation

$$\left\langle \dot{\hat{\Sigma}}(t) \right\rangle = -\Gamma^{-1}\left\langle \hat{\Sigma}(t) \right\rangle, \qquad (16)$$

since the fluctuation part of the operators a_j, $a_j^\dagger$ functionally depends only on the fluctuations $\hat{\xi}_j$, $\hat{\xi}_j^\dagger$. The average $\left\langle \hat{\Sigma}(t) \right\rangle \to 0$ at $\Gamma t \to \infty$, actually in the stationary state. The Γ coefficient in (15) limits the effect of the delta-correlated noise to the frequency band $\Delta\Omega = \Gamma$. Since the OPO generated spectral lines are always narrower than the lines in the passive resonator spectrum, the fluctuations $\hat{\Sigma}(t)$ can still be considered delta-correlated and, for simplicity, we use the adiabatic approximation. As a result, we have

$$\hat{\Sigma}(t) = \Gamma^{-1}[a_3^\dagger a_2\hat{\xi}_1 - a_3 a_2^\dagger\hat{\xi}_1^\dagger +$$
$$a_3^\dagger a_1\hat{\xi}_2 - a_3 a_1^\dagger\hat{\xi}_2^\dagger + a_1 a_1\hat{\xi}_3^\dagger - a_1^\dagger a_2^\dagger\hat{\xi}_3]. \qquad (17)$$

We draw attention to the fact that, here, the adiabatic approximation is used for the operator $\hat{\Sigma}(t)$ whose dynamics depends on the loss at all the parametrically interacting frequencies. In addition, it is worth noting that the solution of (17) presented below can be made without the adiabatic approximation (see [15]) ; however, the results obtained are not quite clear to compare them with the available ones.

4. Variance of the phase fluctuations and natural spectrum width

Substituting (17) into (14) and integrating over the time interval $[t-\tau, t]$, we arrive at the expression for the phase incursion operator in this interval

$$\Delta\hat{\phi}_1(\tau) = \int_{t-\tau}^{t} \hat{\Omega}_1(t')dt'. \qquad (18)$$

When averaging over random forces $\hat{\xi}_j^\dagger$ and $\hat{\xi}_j$, the operators $a_j^\dagger$ and a_j are replaced by stationary values (see the remark above). Thus, we find $\left\langle \Delta\hat{\phi}(\tau) \right\rangle = 0$.

For the phase incursion variance

$$\left\langle (\Delta\hat{\phi}_1(\tau))^2 \right\rangle = \int\int_{t-\tau}^{t} \left\langle \hat{\Omega}_1(t'')\hat{\Omega}_1(t') \right\rangle dt''dt', \qquad (19)$$

we obtain

$$\left\langle (\Delta\hat{\phi}_1(\tau))^2 \right\rangle = D_1\tau. \qquad (20)$$

Here, D_1 is the so-called diffusion coefficient of the spectral line phase at frequency ω_1. It is determined as

$$D_1 = \frac{2(\gamma_2+\gamma_3)^2}{(\gamma_1+\gamma_2+\gamma_3)^2}\frac{\gamma_1}{\bar{n}_1} + \frac{2\gamma_1^2(\gamma_2+\gamma_3)}{(\gamma_1+\gamma_2+\gamma_3)^2}\frac{1}{\bar{n}_3^{(thr)}}. \qquad (21)$$

The expression for the diffusion coefficient D_2 at the radiation frequency ω_2 is obtained from (21) via replacing index 1 by index 2 and vice versa. In (21), we ignored the contribution of the thermal fluctuations, the mean photon number of which at the optical frequency is much less than unity at room temperature ($\langle n_j(T) \rangle$, see (8)).

The diffusion coefficients $D_j (j = 1.2)$ determine the natural width $\Delta\omega_j$ of the Lorentz spectral distribution, $\Delta\omega_j = D_j$.

According to formula (21), the width of the spectrum at the parametrically excited frequencies depends not only on the photon number of the generated radiation at the investigated frequency and loss at the generated frequencies, but also on the threshold photon number and loss on it. Near the generation threshold, i.e. at $\bar{n}_1 \ll \bar{n}_3^{(thr)}$, the contribution of the first term to the spectrum width will prevail. With an increase in the pump photon number and, therefore, in the number of generated photons, the contribution of the second term in (21) to the spectrum width increases.

The formula (21) can be rewritten in terms of power P_j by replacing the mean photon number: $P_j = 0,5\gamma_j\bar{n}_j$.

Let us estimate the spectral linewidth $\Delta\omega_j = D_j$ in order of magnitude. Suppose that a parametrically generated power equal to $P_1 = 30$ mW is twice the pump threshold. Let the generated frequency be equal $\nu_1 = \omega_1/2\pi = 3\times10^{14}$ Hz and close to the degenerate regime. For estimation, we take the coefficients γ_j the same $\gamma_j \cong 1\times10^8$ c^{-1}. We obtain $\Delta\omega_1 \cong$ 1 Hz. It should be noted that two-thirds of this quantity is associated with the second term in (21).

5. Difference frequency spectrum

Let us now consider the spectral width of the difference between the frequencies ω_2 and ω_1 generated in the OPO. Due to the symmetry of the problem, the frequency deviation operator for a wave with the frequency ω_2 is determined by the expression similar to (14).

$$\hat{\Omega}_2(t) = \frac{i}{\bar{n}_2}(g\Sigma(t) + a_2(t)\hat{\xi}_2^\dagger(t) - a_2^\dagger(t)\hat{\xi}_2)(t). \qquad (22)$$

The operator of the frequency deviation difference is $\Delta\hat{\Omega}(t) = \hat{\Omega}_2(t) - \hat{\Omega}_1(t)$. The variance of the phase difference incursion at the difference frequency $\omega_d = \omega_2 - \omega_1$ is expressed as

$$\left\langle (\Delta\hat{\phi}_d(\tau))^2 \right\rangle = \int\int_{t-\tau}^{t} \left\langle \Delta\hat{\Omega}(t^{''})\Delta\hat{\Omega}(t^{'}) \right\rangle dt^{''}dt^{'}. \qquad (23)$$

If we make the assumption $\gamma_1 \cong \gamma_2$, expression (23) yields the simple result

$$\left\langle (\Delta\hat{\phi}_d(\tau))^2 \right\rangle = D_d\tau, D_d = 2(\frac{\gamma_1}{\bar{n}_1} + \frac{\gamma_2}{\bar{n}_2}) \cong 4\frac{\gamma_1}{\bar{n}_1}. \qquad (24)$$

In the investigated case, the spectral width of the difference frequency is only determined by the intrinsic loss at the OPO generated frequencies and independent of the loss at the pump frequency and the threshold value. Obviously, this width can be smaller than the width of a single spectral line, since the correlated phase fluctuations at the difference frequency are subtracted.

6. Conclusion

In conclusion, to solve the problem of the spectral linewidth of OPOs outside the limitations of previous works, we introduced the Hermitian frequency deviation operator. The resulting expression (21) obtained for the triply resonant OPO natural linewidth is more general from the viewpoint of the influence on it of various sources of quantum fluctuations during parametric generation.

First, the available theories lack the second term. Therefore, the width of the spectral line is determined not only by the generated power at the frequency under consideration, but also by the threshold pump power. Moreover, if the generated power exceeds several times the threshold power, the latter can mainly determine the spectral linewidth.

Second, according to (21) the OPO spectral linewidth depends on loss at all the interacting frequencies (coefficients γ_j). In special cases, formula (21) yields the known results.

In (21), the first term leads to the result of study [5] at $\gamma_3 \gg \gamma_1, \gamma_2$. At γ_3 and/or $\gamma_2 \gg \gamma_1$, we obtain the formula from study [7].

The simple approach developed in this work is in progress of an application for quantum spectral analysis of coupled multifrequency parametric interactions [17, 18] and parametric self-frequency conversion [19]. This approach can also be used in spectral analysis of non-linear optomechanical interactions [20] and other vibrational quantum systems.

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References

[1] Harris S E 1969 Tunable optical parametric oscillators *Proc. IEEE* **57** 2096–113
[2] Fischer R and Kulevskii L A 1977 Optical parametric oscillators (review) *Soviet J. Quant. Electron.* **7** 135–59
[3] Loudon R and Knight P L 1987 Squeezed light *J. Mod. Opt.* **34** 709–59
[4] Akhmanov S A, D'yakov Yu E and Chirkin S A 1981 *Introduction to Statistical Radiophysics and Optics* (Moscow: Nauka) (in Russian)
[5] Graham R and Haken H 1968 The quantum-fluctuations of the optical parametric oscillator *I Z. Phys.* **210** 276–302
[6] Graham R 1968 The quantum-fluctuations of the optical parametric oscillator *II Z. Phys.* **210** 319–36
[7] White D R and Louisell H 1970 Noise Calculations for parametric generation *Phys. Rev.* A **1** 1347–56
[8] Emel'yanov V I 1973 Phase fluctuations in a parametric light source operating inside a laser resonator *Sov. J. Quant. Electron.* **2** 35–43
[9] Golubev Y M and Gorbachev V N 1988 Squeezed field states of optical generation *Opt. Spectroscop.* **64** 638–42
[10] Courtois J Y, Smith A, Fabre C and Reynaud S 1991 Phase diffusion and quantum noise in the optical parametric oscillator: a semiclassical approach *J. Mod. Opt.* **38** 177–91
[11] Walls D F and Milburn G J 2008 *Quantum Optics* 2-nd ed (Berlin: Springer)
[12] Carruthers P and Nieto M 1968 Phase and angle variables in quantum mechanics *Rev. Mod. Phys.* **40** 411–40
[13] Alodjants A P and Arakelian S M 1999 Quantum phase measurements and non-classical polarization states of light *J. Mod. Opt.* **46** 475–507
[14] Luis A and Sanchez-Soto L L 2000 *Quantum phase difference, phase measurementsd and Stokes operators Progress in*

Optics ed Wolf E vol 41 (Amsterdam: Elsevier) pp 421–81
[15] Chirkin A S and Chipouline A V 2011 Generalized expression for the natural width of the radiation spectrum of quantum oscillators *JETP Letters* **93** 114–18
[16] Saygin M Yu and Chirkin A S 2016 Theory of the optical parametric oscillator in the quadratic and cubic approximations *Laser Phys.* **26** 015402
[17] Chirkin A S and Shutov V I 2007 On the possibility of the nondegenerate parametric amplification of optical waves at low-frequency pumping *JETP Letters* **86** 693–7
[18] Chirkin A S and Shutov V I 2009 Parametric amplification of light waves at low-frequency pumping in aperiodic nonlinear photonic crystals *JETP* **109** 547–56
[19] Novikov A A and Chirkin A S 2004 Nonclassical light generation in quasi-phase-matched parametric self-frequency conversion *JETP* **99** 947–57
[20] Bowen W P and Milburn G J 2016 *Quantum Optomechanics* (Boca Raton, FL: CRC Press)
[21] Loudon R 2000 *The Quantum Theory of Light* (Oxford: Oxford Scientific Publications)