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WHEN DOES THE DYNAMICAL SYSTEM TURN INTO THE STATISTICAL ONE ?

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This communication will present some results of the analytical and numerical investigations concerning the appearance of stochasticity in conservative system of the type of weakly coupled nonlinear oscillators with Hamiltonian *) :

$$H(I_k, \theta_k) = \sum_{i=1}^{N} H_i^{(0)}(I_i) + \varepsilon H^{(1)}(I_k, \theta_k) \tag{1}$$

where I_k, θ_k are canonical variables and ε stands for a small parameter of perturbation. By the term "stochasticity" I shall briefly denote all the statistical properties of the dynamical system. At present we are still unable to understand completely the nature of statistical laws [3] and hence the conditions of their appearance. Nevertheless, there is apparently no doubt now as to which type of the mechanical motion is the basis for the appearance of the "real" stochasticity or, at least, of its very similar imitation. That is the mixing motion with positive Kolmogorov entropy. **) So when I use the word "stochasticity", I'll mean just this type of the motion. The main problem to be solved is the obtaining of stochasticity criterion, e.g. finding out the conditions under which the statistical laws start

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*) I am considering here only the simplest case of the variables separation at $\varepsilon = 0$. In general, the unperturbed functions $H_i^{(0)}$ may depend on some slow phases θ_m $(\dot{\theta}_m \sim \varepsilon)$. The method presented below can be used in this case as well (see, for example, [1]).

**) In my opinion, this was most clearly shown by Krylov [3].

to act in a dynamical system. From the standpoint of applications the stochasticity is the most dangerous instability of nonlinear oscillations.

For a broad set of mechanical systems there is a simple stochasticity criterion based on the presentation of the motion as a geodesic flow along a certain Riemann surface. Then the everywhere negative curvature of this surface is a sufficient condition for stochasticity. Unfortunately this condition is not necessary at all and, therefore, it does not indicate the real stochasticity-line. Particularly such a criterion cannot be applied at all to the system of weakly coupled oscillators, which is quite important from the standpoint of applications. In the latter case the sign of the curvature for corresponding Riemann surface is always uncertain.

To obtain the stochasticity criterion for this case I am taking into consideration the resonant interaction between the oscillators of the system. It can be performed in accordance with the following brief scheme.

First of all I will reduce the many-dimensional autonomous system (1) to the one-dimensional nonautonomous oscillator. To achieve that I can calculate the dynamical variables of the unperturbed motion ($\varepsilon = 0$) as explicit functions of time: $I_k = const;\ \theta_k(t) = \int \omega_k(I_k)\,dt + \varphi_k$ and substitute them to the perturbation term $H^{(1)}$. As a result, I'll obtain for each oscillator the Hamiltonian of the following type :

$$H_i = H_i^{(0)}(I_i) + \varepsilon H^{(1)}(I_i, \theta_i; t) \qquad (2)$$

In the first approximation with respect to ε one may consider the oscillators (2) as "independent", their coupling being realized via "external" perturbation $H^{(1)}(I_i, \theta_i, t)$. The latter determines the set of resonant frequencies or, briefly, resonances :

$$\omega(I) = \omega_p \qquad (3)$$

It is well known (see, for example [11]) that each separate resonance causes the so-called phase oscillations,

or the pulsation of the oscillator amplitude and frequency around the resonant value ω_p . The maximum swing of this pulsation determines the range of the resonance influence. In frequency units the order of magnitude of this range is as follows :

$$\Omega_\varphi \sim \omega\sqrt{\varepsilon\alpha} \qquad (4)$$

where $\alpha = (I/\omega)(d\omega/dI)$ is the dimensionless factor of nonlinearity. The relation (4) is given in the first approximation with respect to $\sqrt{\varepsilon/\alpha}$, $\sqrt{\varepsilon\alpha}$ and is valid, therefore, under moderate nonlinearity :

$$\varepsilon \ll \alpha \ll 1/\varepsilon \qquad (5)$$

In the case of many resonances the oscillator behavior depends essentially on the quantity :

$$\chi = \left(\frac{\Omega_\varphi}{\Delta}\right)^2 \qquad (6)$$

where Δ is the average interval between the neighbour resonances.

For $\chi \ll 1$ the range of resonance influence is relatively small, so an oscillator is found either far from all the resonances or under the influence of only one of them and, hence, it performs stable phase oscillations. But if $\chi \gtrsim 1$, then the influence ranges of the neighbour resonances overlap and an oscillator gets the possibility to transit from one resonance to the other, changing its frequency and energy. From here the following hypothesis is naturally arising [1] : the condition

$$\chi \sim 1 \qquad (7)$$

determines the stochasticity-line.

This hypothesis proved to be quite fruitful and gave the possibility to investigate some applied problems including : particle motion in magnetic traps [1,4,5] , Fermi stochastic acceleration [6] , nonlinear waves [7] .

Numerical integration of motion equations for some specific examples confirms that the condition (7) determines a real physical line of motion stability. This line is not sharp, on the contrary, it forms an intermediate zone penetrating deeply both into the region of stability and into that of stochasticity. One can show that in the intermediate zone there exist, depending on the initial conditions, quite various types of motion, including even a monotone change of oscillator's energy as for a linear resonance.

The stable region $(\chi \ll 1)$ corresponds, at least asymptotically, to the region for which everywhere dense set of invariant tori, with almost-periodic motion on them, exists as it was discovered by Kolmogorov and Arnold [8,9]. It's possible and even quite likely that for this region there exists, nevertheless, a very slow mixing [10], so slow that no one was able till now to find it out by numerical calculations. However, this does not belittle the importance of the stochasticity-line (7) behind which a relatively fast mixing (in the first order with respect to ε) takes place. I would like to emphasize that in the region of stochasticity the mixing is not merely fast but as fast as possible for a given perturbation.

It is important to mention that the quantity χ depends not only on parameters of the system such as ε, but also on some dynamical variable I, θ. Hence, the stochasticity-line not only determines the critical value of perturbation, but also separates the phase space of the system and even, in general, the each energy surface. This leads to a very interesting situation, quite different from the classical statement of the problem in the ergodic theory; the latter considers stochasticity on the whole energy surface of conservative system. Particularly the problem arises about the transition from the statistical description to the dynamical one, inside the intermediate zone (7). In the first approximation this transition can be described as a boundary condition like a reflecting wall for the distribution function in the region of stochasticity; yet such an approximation is quite rough [6].

I wish to consider, as an example, the problem Fermi-Pasta-Ulam [12] about the statistical properties of nonlinear

string with fixed ends which is obeing the equation *):

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 x}{\partial z^2}\left(1 + 3\beta\left(\frac{\partial x}{\partial z}\right)^2\right) + \gamma^2 \frac{\partial^4 x}{\partial z^4} \qquad (8)$$

Here x is displacement; z - a co-ordinate along the string; β, γ stand for coefficients of nonlinearity and dispersion, respectively. The dispersion term is included in the equation, which is necessary for a correct formulation of the problem that was cleared up by Zabusky [13].

It is known that Fermi, Pasta, Ulam did not find any statistical properties of the motion as a result of their numerical integration of the equation (8). This has provoked a lively discussion which is still going on now. From my point of view such a result may be explained by the use of unlucky (or rather lucky) initial conditions that were found in the region of stability. An estimation of stochasticity-line position for this problem was obtained in common with Israelev [7], it can be given in the form

$$3\beta w \sim \frac{\sqrt{m}}{k} \qquad (9)$$

where $w = E/L$ is the density of oscillation energy per unit length of the string, E is the total energy, m - quantity of excited modes with the mean number k. The mode means space Fourier transform of displacement, number of which $k = 2L/\lambda$, where λ and L are the length of wave and string, respectively.

The left-hand side of criterion (9) is equal to the average value of the nonlinear term $(\langle 3\beta(\partial x/\partial z)^2\rangle)$. Hence, for a weak nonlinearity $(3\beta(\partial x/\partial z)^2 \ll 1)$, the stochasticity is possible only if high modes are excited. Their mean number is

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*) I consider here only a simpler case with cubic nonlinearity (term $\sim (\partial x/\partial z)^3$); quadric nonlinearity ($\sim \partial x/\partial z$) shows a similar picture.

communicated either with the initial conditions $(\kappa \sim \kappa_o)$ or with the formation of a shock wave (or more correctly, solitons [17]). In the last case the following estimation is valid [18]:

$$\kappa \sim \frac{L}{\pi \gamma} \sqrt{3 \beta w} \tag{10}$$

At $\gamma \to 0$ the quantity κ is increasing infinitely and the stochasticity criterion (9) is always realized.

Numerical integration of the equation (8) was carried out in common with Israelev and Khisamutdinov in order to obtain "experimentally" the stochasticity-line position. The local motion instability *) proves to be a very convenient and sensitive indication of stochasticity. We used the property of space symmetry of the solution (8) according to which the even modes cannot appear during the motion if they were not excited initially [12]. But if they are given very little energy ($\sim 10^{-15} E$ in our case), then this energy will remain at about the same level in the region of stability, but in the region of stochasticity it will be growing exponentially in time, achieving quickly the energy level of the main excited modes. An example of such a behavior of the even modes is given in Fig 1, which demonstrates the crossing of stochasticity-line.

Dependence of the rise time τ for local instability on the perturbation parameter β (Fig 2) shows the existence of a broad intermediate zone which is extending far into the region of stability. It is interesting to note that even in the case of exciting only the first mode (the main initial conditions for Fermi, Pasta, Ulam calculations [12]) there is a very weak stochasticity as well. It is clearly seen from the

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*) In my opinion, it is the most important property of stochastic motion which determines probably the very physical nature of "real" stochasticity. The local instability is widely used in analytical study of stochasticity [14-16].

curve of the second mode energy increasing (upper curve in Fig 3), although the lower curve ($E_1(t)$) seems to give a convincing indication of the almost-periodic character of the motion.

For equidistant set of resonances the theory leads to the linear relation between τ^{-1} and $\ln \beta$ (both quantities are proportional to Kolmogorov entropy):

$$1/\tau = \Omega \cdot \ln(\beta/\beta_{\kappa p}) \tag{11}$$

where $\beta_{\kappa p}$ corresponds to the stochasticity-line and the factor of proportionality $\Omega \sim \Delta/\pi$. For large β this law is realized quite well (Fig 2) and it gives a possibility to get the upper stochasticity-line point (by the extrapolation to $\tau^{-1} = 0$). The upper line is the main stochasticity-line which should be compared with the analytical estimations [7]. Furthermore one can distinguish, more or less certainly, in the intermediate zone in Fig 2, the other linear plots which apparently correspond to a more dense set of resonances.

All the critical perturbation values obtained in such a way are plotted in Fig 4 and they agree in the order of magnitude with analytical estimations [7]. Hence, it can be concluded that we rightly understand the main mechanics of stochasticity for nonlinear waves of the type (8), though, of course, there remain many interesting and unclear details in their behavior. The only serious contradiction is connected with a wonderful stability of solitons discovered by Zabusky and Kruskal [17, 18] which seemed to exclude the stochasticity. It's possible that this contradiction may be explained via non-equivalence of the origin wave equation (8) and the first order Korteweg - de Vries type equation from which solitons were deduced [17, 18].

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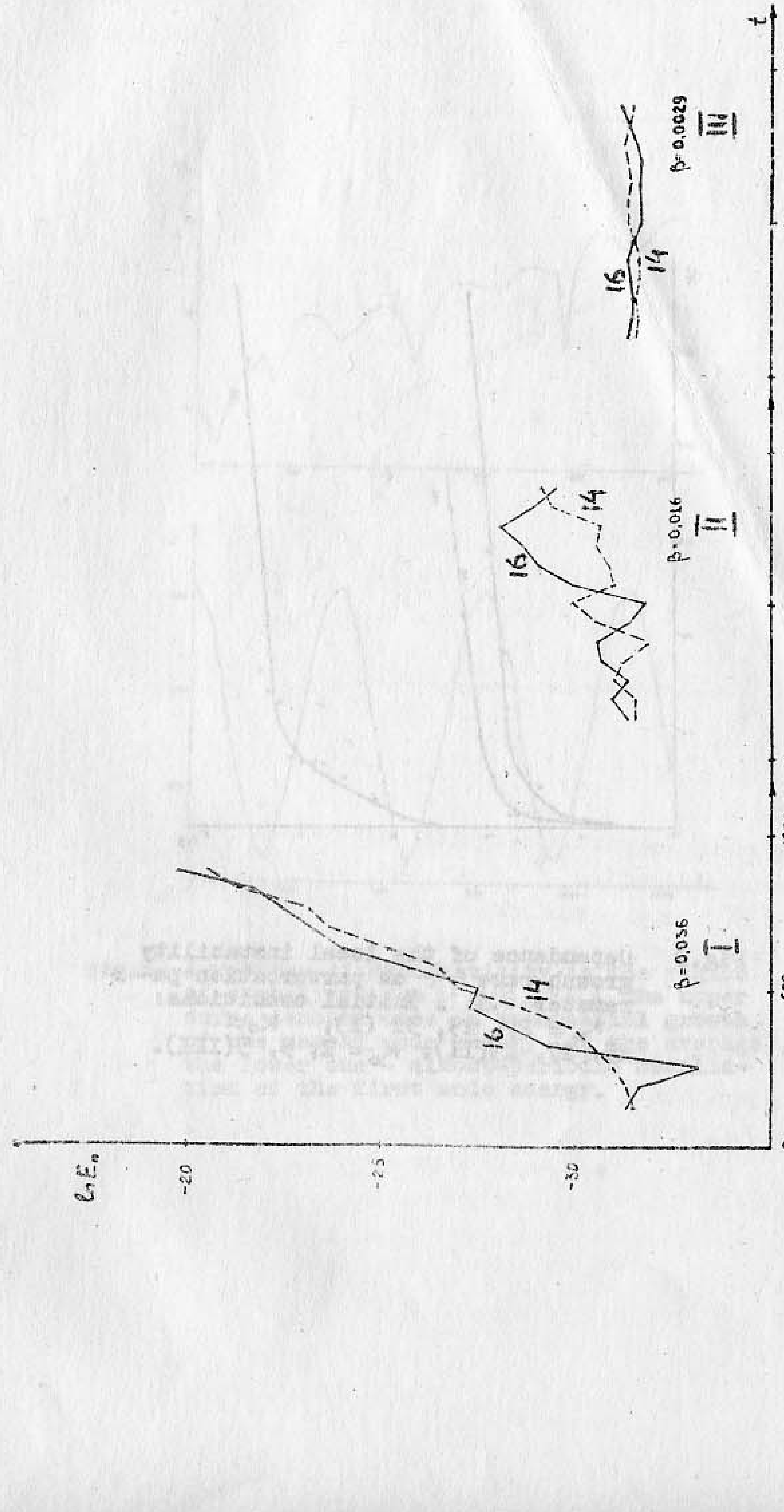

Fig. 1 The local instability of the string: I - the region of stochasticity; II - the stochasticity-line; III - the region of stability; initial conditions : K_o = 15, 17, 19; numbers at curves are mode numbers.

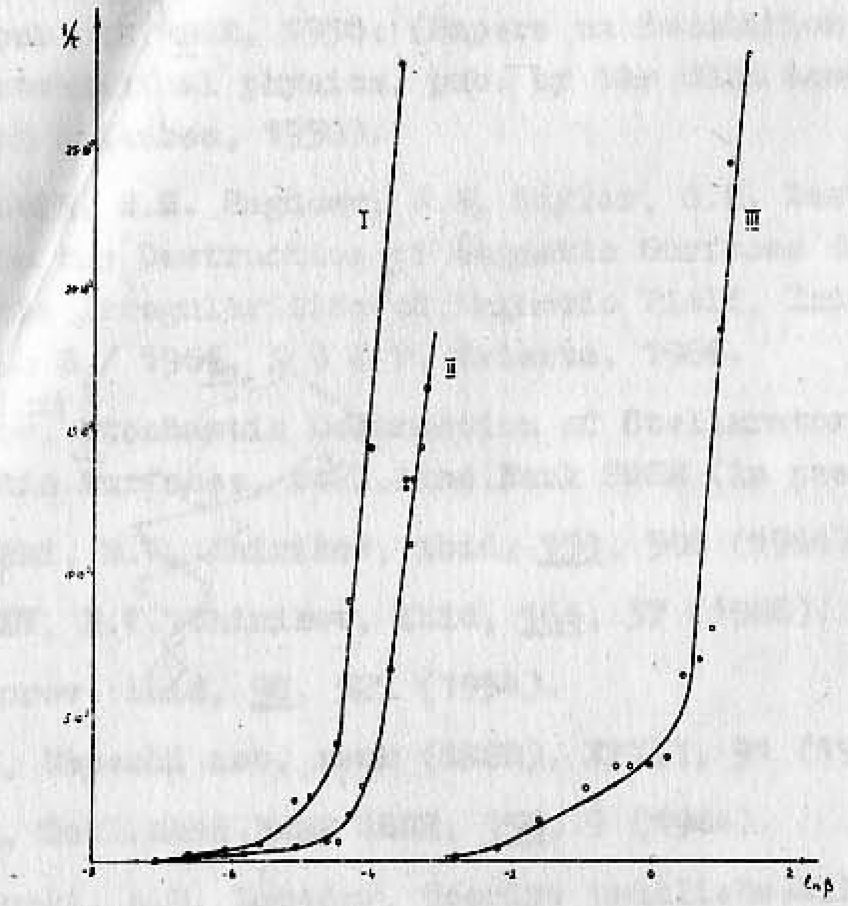

Fig.2 Dependence of the local instability growth rate $1/\tau$ on perturbation parameter β. Initial conditions: $K_0 = 27, 29, 31$ (I); $K_0 =$ 15, 17, 19(II); $K_0 = 1, 3, 5$(III).

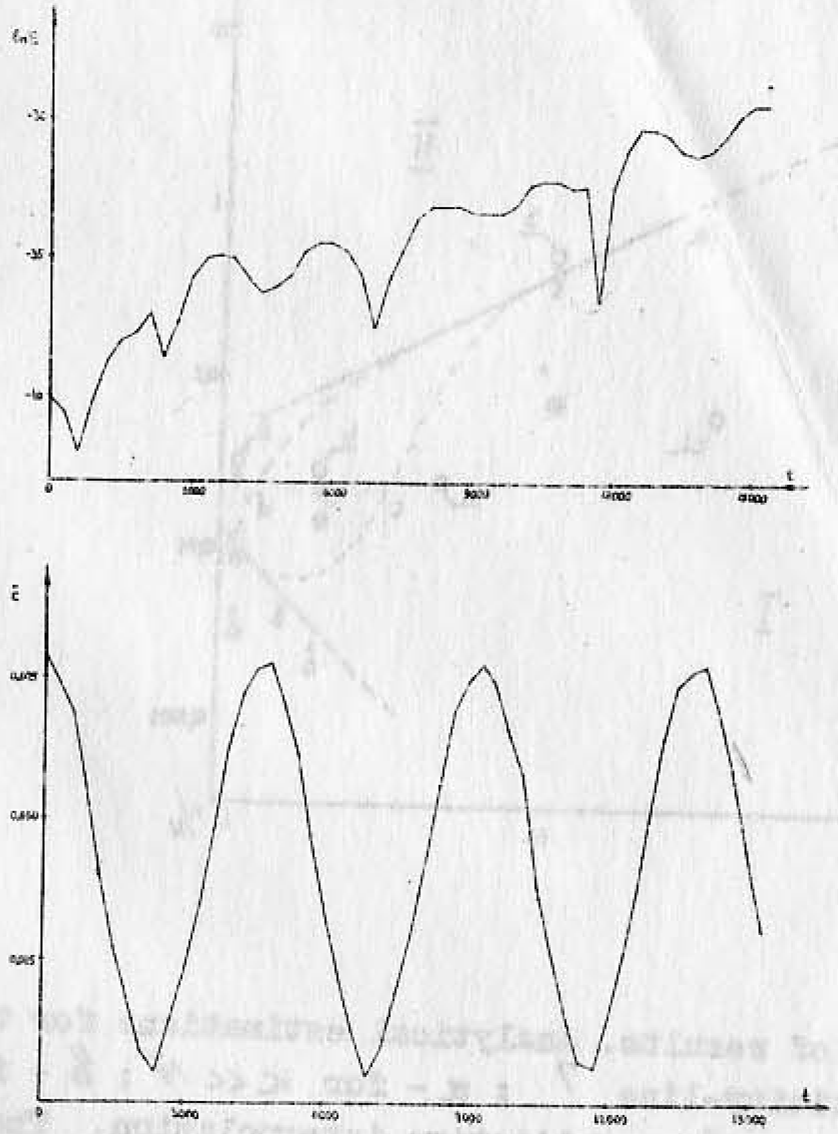

Fig.3 A vestige of stochasticity in the region of stability: $K_0=1$; $\beta/\beta_{K_0}\approx 0.1$. The upper curve demonstrates an exponential growth of the second mode energy (at the average), the lower one - almost-periodic oscillation of the first mode energy.

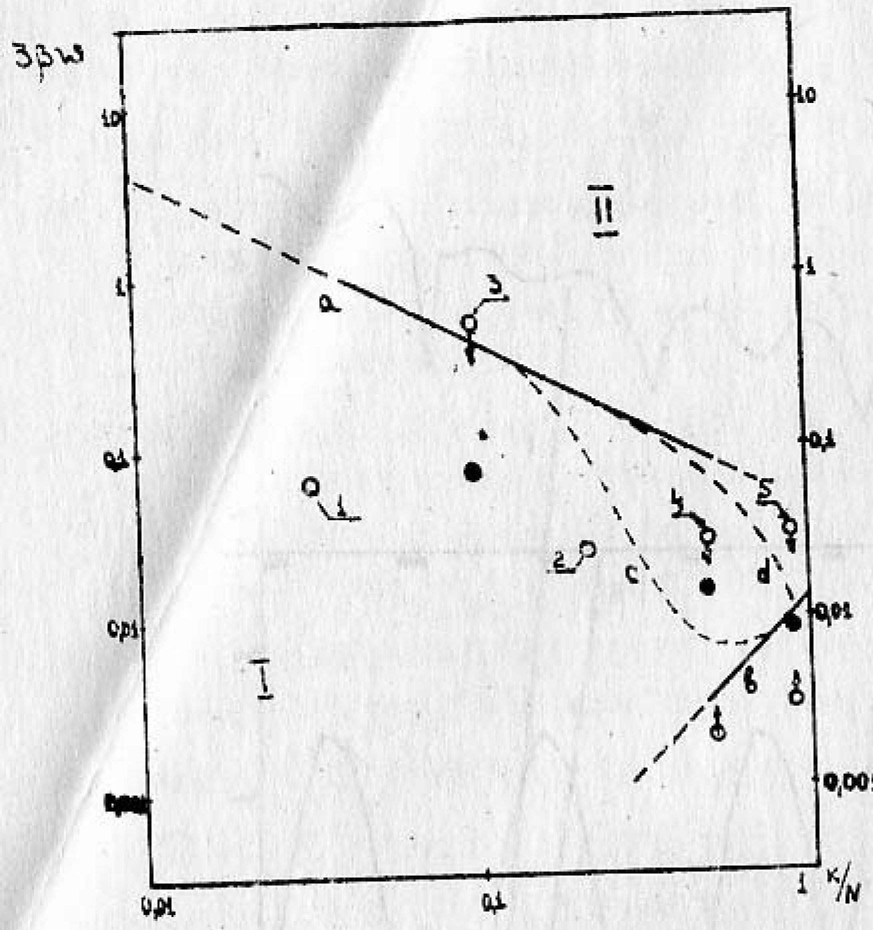

Fig. 4 Summary of results. Analytical estimations for the stochasticity-line [7] : a - for $K \ll \mathcal{N}$; b - for $K \approx \mathcal{N}$; c, d - qualitative interpolation. The results of numerical calculations [12] : $\mathcal{N} = 32$; $X_m = 1$; $K_o = 1$; $\beta = 8$ (1); $K_o = 7$, $\beta = 1/16$ (2). The results of numerical determination of the stochasticity-line - upper (♀), lower (♂), intermediate (●): $\mathcal{N} = 31$; $X_m = 1$; $K_o = 1,3,5$ (3); $K_o = 15, 17, 19$ (4); $K_o = 27, 29, 31$ (5).