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REMARKS ON LOW-ENERGY THEOREMS FOR CROSS-SECTIONS OF PHOTOPION PRODUCTION AT THRESHOLD

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A b s t r a c t

The question is discussed on the accuracy of low energy theorems for cross-sections of reaction of pion photoproduction on nucleons. To derive the theorems the amplitudes are expanded in series of 4-momenta of pion and photon and the terms up to quadratic in these momenta are taken into account. It is shown that successive use of the current algebra predictions, of consequences of crossing symmetry, of kinematics of the process allows one to exclude all unknown terms from the ratio of cross-sections of $\pi^{\pm}$-meson production on protons and neutrons as well as from the expressions for cross-sections of π^{0}-meson production. Low energy theorems have, however, different accuracy due to the fact that senior terms in expansion of amplitudes have different order of magnitude. The best accuracy (of the order of a per cent) is for ratio of cross-sections of charged meson production.

In this note the low-energy theorems for the cross-sections of photopion production on nucleons at threshold are considered. The derivation of the theorems is based on the hypothesis of the partial conservation of an axial current and on the assumption on the smallness of the pion mass as compared to some internal mass m_{int}.

Under these assumptions the threshold photoproduction has been studied in a number of papers /1-7/. The well known result is the theorems of Kroll and Ruderman /1/ who starting from the equation $\partial_\mu j_\mu = 0$ have determined the photoproduction amplitude up to the terms linear in the photon momentum κ.

At threshold momentum κ is of order μ and therefore the Kroll-Ruderman determines the amplitude of charged pion photoproduction up to terms linear in μ/m_{int}. In this approximation the amplitudes of neutral pion production is just equal to zero.

The use of the hypothesis of a partially conserved axial current allows one to obtain terms linear in κ /4,3/ and to reduce in this way the uncertainty in the amplitude of charged pion photoproduction to quadratic in μ/m_{int} terms. Simultaneously one obtains the first nonvanishing term in the amplitude of π^0-meson production on protons while the amplitude of π^0-production on neutron remains equal to zero in linear approximation.

In this note we consider the terms of the second order in the pion four-momentum q or the photon four-momentum k. In particular we shall calculate the ratio of the cross sections of π^+-production on proton and π^--production on neutron up to the (neglected) terms of the third order in μ/m_{int}. In other words the expected accuracy of the result obtained is not worse than several per cents. As the usual accuracy of theoretical predictions based on the PCAC hypothesis is (10-20) % this result is of special interest to our mind.

In the case of the threshold photoproduction of neutral pions our consideration improves the accuracy of the prediction for the cross section of π^0-meson production on proton and gives the first nonvanishing term in the amplitude of π^0-meson production on neutrons.

To summarize, three low energy theorems will be obtained. These theorems allow to calculate the ratio of $\pi^\pm$-production cross sections at threshold and the cross sections of π^0-production both on proton and neutron. In the last two cases the results for absolute values of the cross sections may be transformed into the predictions for the ratio of neutral and charged pion production cross sections. But as due to the isotopic symmetry there are only

two independent ratios of photoproduction amplitudes, only two low energy theorems are independent.

The low energy theorems justify the use of the Born approximation with PS-PV πNN-coupling constant to calculate the cross sections and their ratios mentioned above. In this respect our results are analogous to the Kroll-Ruderman theorem /1/ but, as was explained above, we take into account terms of the next orders in μ/m_{int}. Let us note that the Born approximation with PS-PV πNN-coupling constant for the reaction of pion photoproduction was considered in paper /7/.

2. We proceed now to the derivation of the low energy theorems. For the sake of definiteness we shall consider the amplitude of π^+-meson photoproduction on protons $T(\gamma p \to n\pi^+)$

$$T(\gamma p \to n\pi^+) = e\epsilon_\mu M_\mu = -e\epsilon_\mu \langle n\pi^+ | j_\mu(0) | p\rangle \tag{1}$$

where ϵ_μ is the photon polarization vector, $j_\mu(x)$ is the hadron electromagnetic current and $e^2/4\pi = 1/137$.

It is convenient to write explicitly the contribution to the matrix element M_μ of the nucleons' and pion's pole graphs as well as the contribution of the contact term which arises through the substitution $\partial_\mu \to \partial_\mu + ieA_\mu$ in the πNN-vertex. Denoting by N_μ the rest of the matrix element and using the PS-PV πNN- coupling we have

$$M_\mu = -if\sqrt{2}\,\bar{u}_2\left[R^p_\mu + R^n_\mu + R^\pi_\mu + R^c_\mu + N_\mu\right]u_1$$

$$R^p_\mu = \hat{q}\gamma_5 \frac{1}{\hat{p}_2+\hat{q}-m}\left[\gamma_\mu - \frac{\varkappa^p}{2m}\sigma_{\mu\nu}k_\nu\right]$$

$$R^n_\mu = -\frac{\varkappa^n}{2m}\sigma_{\mu\nu}k_\nu \frac{1}{\hat{p}_1-\hat{q}-m}\hat{q}\gamma_5$$

$$R^\pi_\mu = (\hat{q}-\hat{k})\gamma_5 \frac{1}{(q-k)^2-\mu^2}(2q-k)_\mu \qquad (2)$$

$$R^c_\mu = -\gamma_\mu\gamma_5$$

Here u_1 and u_2 are spinors of the initial and final nucleon with momenta p_1 and p_2, respectively, $\varkappa^p$ and $\varkappa^n$ are the proton and neutron anomalous magnetic moments, and the coupling constant f is equal to $1.01\mu^{-1}$.

As the sum of the pole and contact terms satisfy the condition $k_\mu \bar{u}_2[R^p_\mu + R^n_\mu + R^\pi_\mu + R^c_\mu]u_1 = 0$ the rest of the matrix element N_μ is also divergenless. As it is well known it means that N_μ may be expanded in the form

$$N_\mu = \sum_{i=1}^{4} V_i O^i_\mu$$

$$O^1_\mu = \gamma_5 \sigma_{\mu\nu} k_\nu \qquad \eta_1 = +1$$

$$O^2_\mu = \frac{1}{2m}\gamma_5(\hat{q}-\hat{k})\left[(p_1+p_2)_\mu (qk) - q_\mu (k, p_1+p_2)\right] \qquad \eta_2 = +1$$

$$O^3_\mu = \gamma_5\left[\gamma_\mu (qk) - q_\mu \hat{k}\right] \qquad \eta_3 = -1$$

$$O^4_\mu = -i\epsilon_{\mu\nu\rho\sigma}\gamma_\nu k_\rho q_\sigma \qquad \eta_4 = +1 \qquad (3)$$

where V_i (i=1,2,3,4) are invariant functions of $\nu = \frac{1}{4m}(k+q, p_1+p_2)$ and kq. Numbers η_i give the parity of the formfactors $V_i^{\pi^+ n}$, $V_i^{\pi^0 p}$, $V_i^{\pi^- p}$, $V_i^{\pi^+ n} + V_i^{\pi^0 p}$ with respect of the crossing transformation $\nu \to -\nu$, $kq \to kq$ while the parity of the formfactors $V_i^{\pi^+ n} - V_i^{\pi^0 p}$ under this transformation is just an opposite one.

Conservation of the axial current in the limit of the zero pion mass results in the following constrain on formfactor V_1 /2/

$$V_1(\nu = 0, kq = 0, \mu^2 = 0) = 0 \qquad (4)$$

This relation exhausts all the consequences from the PCAC hypothesis for the process considered.

Let us exploit now the smallness of μ as compared to m_{int}. As at threshold q, $\kappa \sim \mu$ the conjectured smallness of μ implies the possibility of expanding formfactors V_i in series of κ, q. The dominant contribution to the amplitude gives the contact term which contains neither q nor κ factors. The nucleon pole term is easily seen to be proportional to μ while the pion pole term vanishes at threshold.

The order of magnitude of the rest of the amplitudes may be readily estimated also. Let us note that invariants O_μ^i are written in such a form as to make explicit their order of magnitude at small κ and q. For example the factor $\frac{1}{2m}(\hat{q}-\hat{k})\gamma_5$ in the invariant O_μ^2 could be transformed into γ_5 but then it would be necessary to keep in mind the smallness of the matrix element of γ_5 in the nonrelativistic limit.

Thus the order of magnitude of any term may be determined by a direct count of factors κ and q, any such factor being associated with an extra smallness of order μ/m_{int}. We keep all the terms in the amplitude whose contribution is of order $(\mu/m_{int})^2$ or larger. Taking into account relation (4) we see that generally speaking three unknown parameters enter the amplitude of any possible reaction ($\gamma p \to n\pi^+, p\pi^0$, and so on).

These parameters are $\frac{\partial V_1}{\partial \nu}$, V_3, V_4 formfactors taken at $\nu = \kappa q = 0$.

Further simplifications arise in the following way. In the non-relativistic limit the contribution of V_4 formfactor vanishes at threshold. For this reason its contribution contains an extra factor μ/m_{int} and may be neglected. For the same reason the amplitude depends in fact only on the sum of $\partial V_1/\partial\nu$ and V_3. Moreover, due to the crossing relations the same sum enters both the amplitude of π^+ and π^- production. In the case of π^0 production $\partial V_1/\partial\nu$ and V_3 are the odd functions of ν and their contribution may be disregarded.

In the next section we shall give predictions for the cross sections which follow from the procedure of constructing the amplitude describing above. Here it is noteworthy to make two remarks on the possibility of expanding the amplitude in momenta κ and q.

First of all one should consider separately the contribution of isobar N* (1236) because in this case the internal mass is itself small, $m_{int} \sim \mu$. However, the isobar intermediate state in the s-channel of the reaction give the resonant contribution in p-wave amplitude only which unessential at threshold. It may be shown that the contribution the graph with isobar in u-channel is numerically negligibly small.

Secondly, expanding V_i in powers of κ and q we implicitly assumed that V_i are analytic functions near the threshold. This is in fact not true. The contribution of rescattering in the s- and u-channels (see Fig.1,2) cannot be expanded in powers of ν and κq and should be treated more carefully. We estimate the contribution of the graphs in Figs.1 and 2 in Appendix and show that it is small.

3. We give now an expression for the threshold cross section of the reaction $\gamma p \to n\pi^+$

$$\sigma_p^+ = \frac{\alpha}{2\pi}\frac{f^2m^2}{(m+\mu)^2}\left[1-\frac{\mu}{m}+\frac{\mu^2}{2m^2}(2+\varkappa^p+\varkappa^n)+2\mu^2\gamma+O\left(\frac{\mu^3}{m^3_{int}}\right)\right] \qquad (5)$$

where we have introduced the following notations

$$\sigma_p^+ = \lim_{q\to 0}\frac{k}{|\vec{q}|}\frac{d\sigma}{dO}(\gamma p \to n\pi^+)$$

$$\gamma = \left(\frac{\partial V_1}{\partial \nu}+V_3\right)\Big|_{\nu=\kappa q=0}$$

We see that an unknown term enters this expression. Due to the presence of this term we do not obtain in the approximation considered any

prediction for the cross section σ_p^+. If one neglects the terms proportional to γ then σ_p^+ is determined uniquely /4,3/. As was already mentioned the same parameter γ enters the expression for the cross section of π^- production on neutrons

$$\sigma_n^- = \frac{\alpha}{2\pi}\frac{f^2m^2}{(m+\mu)^2}\left[1+\frac{\mu}{m}-\frac{\mu^2}{2m^2}(\varkappa^p+\varkappa^n)+2\mu^2\gamma+O\left(\frac{\mu^3}{m^3_{int}}\right)\right] \qquad (6)$$

For the ratio of σ_n^- and σ_p^+ we obtain a relation without any unknown parameters

$$R = \frac{\sigma_n^-}{\sigma_p^+} = 1+2\frac{\mu}{m}+\frac{\mu^2}{m^2}(1-\varkappa^p-\varkappa^n)+O\left(\frac{\mu^3}{m^3_{int}}\right)\approx 1.32 \qquad (7)$$

This relation is our final result for the charged pions photoproduction cross sections.

Let us note that equality of the ratio R to 1.3 was predicted first within the framework of a crude model /8/. A more reliable foundation for this prediction was given in papers /4,3/ where linear in μ/m_{int} terms were taken into account. Here we have determined the terms of the second order in μ/m_{int} and found them to be small. This result implies high accuracy of relation (7). Possible violation of this is in fact of order of electromagnetic corrections and is expected not to exceed several per cents.

Experimentally, ratio R is equal to 1.25±0.15 /9/, 1.265±0.065 /10/. Because of high accuracy of theoretical calculation of R further improvements of experimental accuracy seems to be desirable.

In the amplitude of π^0-meson production the senior contact term is absent. The terms of the first and second order in $\mu/m_{int.}$ are uniquely determined by pole contributions. Other possible terms of this order vanish due to eq.(4) and requirements of the crossing symmetry of the amplitude. Thus for the threshold cross section of the reaction $\gamma p \to p \pi^0$ we have

$$\sigma_p^0 = \frac{\alpha}{4\pi} \frac{f^2 \mu^2}{(m+\mu)^2} \left[1 - \frac{\mu}{m}(\varkappa^p + 1) + O\left(\frac{\mu^2}{m_{int}^2}\right)\right] \approx 0.11 \frac{\mu b}{st} \tag{8}$$

Because of the absence of the senior term the relative accuracy of this relation is worse than in the case of eq.(7). Within the same approximation relation (8) may be transformed into the prediction for the magnitude of the ratio of σ_p^0 and σ_p^+

$$\frac{\sigma_p^0}{\sigma_p^+} = \frac{\mu^2}{2m^2} \left[1 - \frac{\mu}{m} \varkappa^p + O\left(\frac{\mu^2}{m_{int}^2}\right)\right] \approx 0.080 \tag{9}$$

Experimentally, $\sigma_p^0 = (0.07 \pm 0.02) \frac{\mu b}{st}$ [11]; $(0.10 \pm 0.01) \frac{\mu b}{st}$ [12].

For cross section of the π^0-meson production on neutrons we come to the following expression

$$\sigma_n^0 = \frac{\alpha}{4\pi} \frac{f^2 \mu^2}{(m+\mu)^2} \left(\varkappa^n \frac{\mu}{2m}\right)^2 \left[1 + O\left(\frac{\mu}{m_{int}}\right)\right] \approx 0.003 \frac{\mu b}{st} \tag{10}$$

or for the ratio of σ_n^0 and σ_n^-

$$\frac{\sigma_n^0}{\sigma_n^-} = \frac{\mu^2}{2m^2} \left(\varkappa^n \frac{\mu}{2m}\right)^2 \left[1 + O\left(\frac{\mu}{m_{int.}}\right)\right] \approx 1.7 \cdot 10^{-4} \tag{11}$$

As it was discussed above due to isotopic symmetry only two of three equations (7,10,11) are independent.

Appendix

In this appendix we shall consider the contribution to the amplitude of the non-analytic terms due to rescattering in the s- and u-channels of the reaction (see graphs on Fig.1 and Fig.2, respectively). These terms should be treated separately because they are not expanded in series in variables ν, κq.

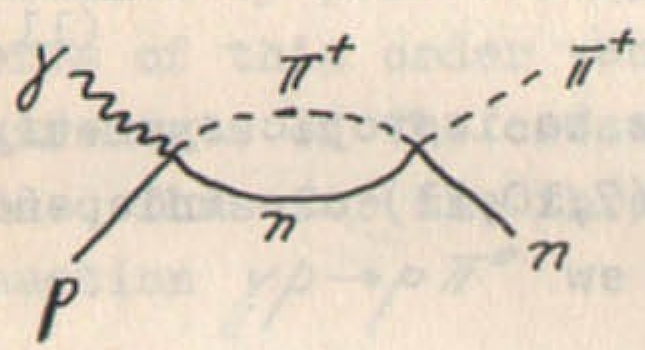

Fig. 1.

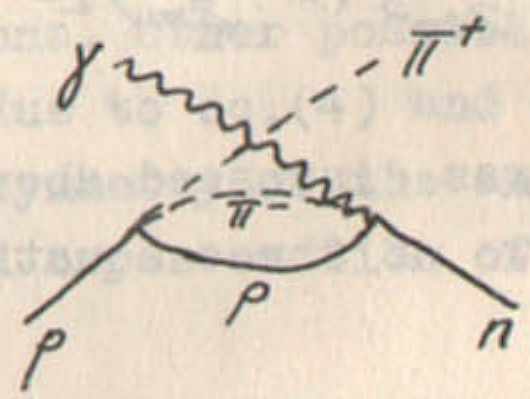

Fig. 2.

One of the reasons to believe that the contribution of graphs 1 and 2 is suppressed is rather apparent and lies in the fact that the amplitude of πN-scattering is small near the threshold. We keep however the terms of second order in μ/m_{int} and the mentioned argument is not sufficient to justify the neglegence of non-analytic terms. We shall see below that nonanalytic terms contain more factors μ/m than it

follows from a naive count and for this reason they are extremely small.

Let us now estimate the contribution of graphs 1,2 to the amplitude of photoproduction at threshold. The amplitude of photoproduction which enters the corresponding matrix element we shall approximate by the contact term. As we have seen this contribution indeed dominates the amplitude so that this approximation seems to be reasonable. The πN-scattering amplitude we shall describe by the expression following from current algebra $T_{\pi^- p} = -T_{\pi^+ p} = \frac{1}{2} c^2 \bar{u}_2 (\hat{q}_1 + \hat{q}_2) u_1$ where $c = \frac{f\sqrt{2}}{g_A} = 1.2\mu^{-1}$ and q_1, q_2 are momenta of initial and final π-mesons.

Then for the imaginary parts of the matrix elements M_1 and M_2 corresponding to graphs 1 and 2 we have

$$\operatorname{Im} M_1 = M^{(0)} \frac{c^2 (s - m^2)}{8\pi} \frac{q_s}{\sqrt{s}} \, \theta(s - (m+\mu)^2)$$

$$\operatorname{Im} M_2 = M^{(0)} \frac{c^2 (u - m^2)}{8\pi} \frac{q_u}{\sqrt{u}} \, \theta(u - (m+\mu)^2)$$

where $M^{(0)} = -ief\sqrt{2}\,\bar{u}_2 \gamma_5 \hat{\epsilon} u_1$ corresponds to the contribution of the contact term to the photoprocess amplitude and q_s and q_u are expressed through variables s,u and masses

$$q_s = \frac{1}{2\sqrt{s}} \sqrt{[s-(m+\mu)^2][s-(m-\mu)^2]}$$

$$q_u = \frac{1}{2\sqrt{u}} \sqrt{[u-(m+\mu)^2][u-(m-\mu)^2]}$$

At threshold the imaginary part of the amplitude vanishes and we are interested not in the $\mathrm{Im}\,M$ itself but in the rapidly varying term in the real part associated with rescattering. Therefore we introduce functions h(s) and h(u) which satisfy conditions Imh(s)= $\frac{q_s}{\sqrt{s}}$ and Im h(u)= $=\frac{q_u}{\sqrt{u}}$, respectively. In the explicit form

$$h(s) = -\frac{1}{\pi}\frac{q_s}{\sqrt{s}} \ln \frac{\sqrt{s-(m+\mu)^2}+\sqrt{s-(m-\mu)^2}}{\sqrt{s-(m+\mu)^2}-\sqrt{s-(m-\mu)^2}}$$

Now for nonanalytic terms in the amplitude of photoproduction near threshold we have the following expression

$$M_1 + M_2 = M^{(0)}\frac{c^2}{8\pi}[(s-m^2)h(s)+(u-m^2)h(u)]$$

It is a straightforward matter now to estimate the nonanalytic terms. At threshold their relative contribution is proportional to μ^3/m^3 and negligibly small numerically.

R e f e r e n c e s

1. N.M.Kroll and M.A.Ruderman. Phys.Rev. 93, 233 (1954).
2. S.Fubini, G.Furlan and C.Rossetti. Nuovo Cim. 40, 1171 (1965).
3. S.L.Adler and Y.Dothan. Phys. Rev. 151, 1267 (1966).
4. C.W.Gaffney. Phys. Rev. 161, 1599 (1967).
5. K.Yamamoto. Bull. Amer. Phys. Soc. 13, 362 (1968).
6. D.L.Weaver. Phys. Lett. 26B, 451 (1968).
7. M.S.Bhatia and P.Narayanaswamy. Phys. Rev. 172, 1742 (1968).
8. M.Cell-Mann and K.Watson. Ann. Rev. Nucl. Sci. 4, 449 (1954).
9. M.I.Adamovich, V.G.Larionova, S.P.Kharlamov, F.R.Yagudina. Yadernaya Fizika, 8, 1181 (1968).
10. J.P.Burq. Ann. Phys.(Paris) 10, 363 (1965).
11. B.B.Govorkov, S.P.Denisov, E.V.Minarik. Yadernaya Fizika, 4, 371 (1965).
12. B.B.Govorkov, E.V.Minarik. Yadernaya Fizika, 6, 1046 (1967).