ИНСТИТУТ
ЯДЕРНОЙ ФИЗИКИ СОАН СССР

ПРЕПРИНТ И Я Ф 51 - 73

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Новосибирск

1973

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The way to find the intensity of gravitational radiation of ultrarelativistic charge in an external electromagnetic field is presented. The method is applicable to a wide class of problems. The closed expression is obtained for the radiation intensity in the case of circular orbit in the Coulomb field.

The problem of gravitational radiation of an ultrarelativistic charged particle in external electromagnetic field was discussed previously in the works/I-4/. But the consideration made in these works seems to us to be not quite satisfactory due to the reasons discussed in detail below.

In the linear approximation in gravitational field the Einstein equations can be written as

$$\Box \psi_{\mu\nu} = \varkappa \left(T^{p}_{\mu\nu} + T^{f}_{\mu\nu}\right) \tag{I}$$

$$\partial_\mu \psi_{\mu\nu} = 0 \tag{2}$$

Here we put $c=I$, $\varkappa^2=16\pi k$ where k is the Newton gravitational constant, $\psi_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}\delta_{\mu\nu}h_{\lambda\lambda}$ and $\varkappa h_{\mu\nu}$ is the deviation of the metrics of the flat one. $T^{p}_{\mu\nu}$ is the energy-momentum tensor of the particle, $T^{f}_{\mu\nu}$ is the energy-momentum tensor of the electromagnetic field. It is quadric in the field, i.e., $T^{f} \sim FF+2Ff+ff$. The square of external field FF has nothing to do with the particle motion and therefore will not be discussed below. It is unnecessary also to account of the square of the particle field ff since almost everywhere $f \ll F$, and at small distances of the particle where f is large, its contribution is taken into account by mass renormalization, i.e., is contained already in $T^{p}_{\mu\nu}$. Thus only the term 2Ff is essential here and below we shall take just this quantity as $T^{f}_{\mu\nu}$. This term is necessary for the conservation of the energy-momentum tensor since for a particle in an external field $T^{p}_{\mu\nu}$ is not conserved by itself. To account of $T^{p}_{\mu\nu}$ only in the equ. (I) (and just in this way the problem under discussion was considered in the works/I,4/) is inadmissible.

But the contributions of $T^{p}_{\mu\nu}$ and $T^{f}_{\mu\nu}$ into the gravitational wave field are not alike. While the former is the usual outgoing spherical wave, the latter does not at all fall with distance if external field does not vanish at infinity (and $T^{f}_{\mu\nu} \sim \frac{1}{r} e^{i(kr-\omega t)}$). The point is that the particle electromagnetic radiation is resonantly transformed into gravitational one in external field. This effect was for the first time pointed at by Gertzenshtein/5/ (see also /2/).

The problem of gravitational radiation in homogeneous magnetic field was considered in the work/2/. An independent meaning was ascribed there to that part $\psi^{(1)}_{\mu\nu}$ of gravitational field which falls with distance. But as one can check easily, $\partial_\mu \psi^{(1)}_{\mu\nu} \neq 0$ and the non-vanishing divergence of $\psi^{(1)}_{\mu\nu}$ is necessary to cancel out the terms $\sim \frac{1}{r}$

in the divergence of the resonant part of the field. The separation of four-dimensionally transverse part of $\psi^{(i)}_{\mu\nu}$ is by itself an ambiguous procedure. Thus in a problem with infinite homogeneous external field only the resonant part of radiation can be found correctly. And a model with arbitrarily cut off homogeneous field is inadmissible since such cutoff violates the Maxwell equations hence leading to non-conservation of the energy-momentum tensor.

It is natural therefore to consider the problem with inhomogeneous field of the simplest possible kind: the circular motion in the Coulomb field. The characteristic wave-length of the ultrarelativistic particle radiation is much smaller than the orbit radius r_o and the external field varies slowly on the wave-length. Hence in this case one can speak of the resonant conversion of the electromagnetic radiation into the gravitational one. Calculate this effect.

Let $-Q$ be the charge of the centre, and e,m be the charge and the mass of the particle. Then $eQ \approx m\gamma r_o = \mathcal{E} r_o$ where $\gamma = (I - v^2)^{-1/2}$. In the ultrarelativistic case ($\gamma \gg I$) the synchrotron radiation is directed into narrow cone tangentially to the trajectory. Inside the cone the external field depends only on the coordinate x directed along the ray. Therefore, look for the solution of the equ. (I) as

$$\psi^{f}_{\mu\nu} = \frac{1}{x}\sum_{\omega} a_{\mu\nu}(\omega,x)e^{i\omega(x-t)}$$

where $a_{\mu\nu}(\omega,x)$ is a slowly varying function of x. Substituting the expression for $T^{f}_{\mu\nu}$ in the wave zone of the synchrotron radiation

$$T^{f}_{\mu\nu} = \frac{1}{x}\sum_{\omega} \tau_{\mu\nu}(\omega,x)e^{i\omega(x-t)}$$

into the right-hand side of the equ. (I), obtain the equation (for $\omega x \gg I$)

$$2i\omega\frac{da_{\mu\nu}}{dx} + \frac{d^2 a_{\mu\nu}}{dx^2} = \varkappa\,\tau_{\mu\nu} \qquad (3)$$

Neglect the second derivative of the slowly varying function $a_{\mu\nu}(\omega,x)$. Then the solution at infinity is

$$a_{\mu\nu}(\omega) = \frac{\varkappa}{2i\omega}\int_0^{\infty} dx\,\tau_{\mu\nu}(\omega,x) \qquad (4)$$

The following components only of $a_{\mu\nu}$: a_{yy}, a_{yz}, a_{zz} contribute to the gravitational radiation. Taking into account the explicit structure of $\tau_{\mu\nu}$, one can easily see that the resonant conversion takes place only due to the components of the external field orthogonal to the wave vector. After simple calculations we come to the following expression for the intensity of the resonant gravitational radiation I_{res} :

$$I_{res} = \frac{\kappa Q^2}{r_0^2} I_{em} = \frac{2}{3}\frac{\kappa \mathcal{E}^2}{r_0^2}\gamma^4 \qquad (5)$$

where

$$I_{em} = \frac{2}{3}\frac{e^2}{r_0^2}\gamma^4 \qquad (6)$$

is the intensity of the synchrotron radiation. Evidently, the angular and spectral distributions of I_{res} and I_{em} coincide.

It should be noted that the resonant and non-resonant parts of gravitational radiation do not interfere. This fact was pointed out in the work/2/ for the case of homogeneous magnetic field. In the Appendix we present the general proof of this statement. It is clear then that I_{res} is at any rate the lower bound for the total intensity of gravitational radiation.

Estimate now the intensity of non-resonant gravitational radiation I_{gr}. For this purpose compare it with I_{em}. These quantities differ, firstly, in coupling constants: $k\varepsilon^2$ in I_{gr} and e^2 in I_{em}. Secondly, the component of the vector-potential contributing to electromagnetic radiation is three-dimensionally transverse: $A_\perp \sim \theta \sim \gamma^{-1}$; and the potential of gravitational field is doubly transverse: $\psi_{\perp\perp} \sim \theta^2 \sim \gamma^{-2}$. (We use here the purely kinematical fact that any radiation of the ultrarelativistic particle lies inside a cone with the angle $\theta \sim \gamma^{-1}$.) Accounting of these two distinctions and of the formula (6), we find

$$I_{gr} \sim \frac{K\mathcal{E}^2}{r_0^2}\gamma^2 \qquad (7)$$

This result was obtained in the work/6/ at the consideration of the problem of ultrarelativistic rotator.

Now it is clear that the expression (5) is not only the lower bound, but also the correct ultrarelativistic approximation for the total intensity of the gravitational radiation of the charge moving in a circular orbit in the Coulomb field.

The problem considered was being solved by numerical method in the work/3/. The result agrees qualitatively with ours, but differs in the numerical coefficient. We have no doubts whatsoever in our result, the more so, as it is confirmed by the direct calculation (see the Appendix).

It is quite clear that I_{res} can be computed for the motion in arbitrary slowly varying external field once I_{em} is known. Find in general case the conditions under which I_{res} constitutes the main part of gravitational radiation. Consider at first the case when the deflection angle α of the particle in the field is much larger than the angle of the cone to which the radiation is confined $\theta \sim \gamma^{-1}$. Then the estimates (6), (7) for I_{em} and I_{gr} are valid (r_0 means now the characteristic impact parameter). In this case

$$I_{res} \sim \kappa F^2 a^2 I_{em} \sim \kappa F^2 a^2 \frac{e^2}{r_0^2} \gamma^4 \tag{8}$$

where F is the characteristic external field strength, a is the path of the light in this field. From (7), (8) follows

$$I_{res}/I_{gr} \sim \frac{a^2}{R^2} \gamma^2 \tag{9}$$

where $R = \mathcal{E}/eF$ is the radius of curvature of the particle's trajectory. If the paths of the light and the particle in the external field are of the same order of magnitude, then $a/R \sim \alpha \gg \gamma^{-1}$ and hence the resonant radiation is dominating. This is the general solution of the problem of gravitational radiation in a field sufficiently strong and slowly varying.

Analogous estimates at $\alpha \ll \gamma^{-1}$ (here both I_{em} and I_{gr} are γ^2 times smaller than at $\alpha \gg \gamma^{-1}$) show that in this case the contribution of resonant radiation is small.

Unfortunately, the problem considered seems to be of purely methodical interest since in all the real situations which we can imagine, the radiation intensity turns out negligible.

We are sincerely grateful to V.V.Flambaum, E.V.Shuryak and A.I.Vainshtein for valuable discussions.

APPENDIX

Present now the straightforward calculation of gravitational radiation of ultrarelativistic particle moving in the Coulomb field in a circular orbit. Split the gravitational field $\psi_{\mu\nu}$ into the parts $\psi^{p}_{\mu\nu}$ and $\psi^{f}_{\mu\nu}$ whose sources are $T^{p}_{\mu\nu}$ and $T^{f}_{\mu\nu}$ respectively. As to the $\psi^{p}_{\mu\nu}$, it can be computed in a routine way (see, e.g., /2/). We do not present these calculations here, the more so, as $\psi^{p}_{\mu\nu}$ proves to be of higher order in γ^{-1} than $\psi^{f}_{\mu\nu}$. The computation of all the components of $\psi^{f}_{\mu\nu}$ is useful since it permits of independent check-up by means of the condition (2), accounting of course of the $\psi^{p}_{\mu\nu}$ as well. But we present here the computation of those components of $\psi^{f}_{\mu\nu}$ only which contribute to the intensity of radiation, omitting systematically in ψ^{f}_{mn} the terms $\sim\delta_{mn}$ and $\mathbf{k}_m\mathbf{b}_n$ (m,n=I,2,3) where $\vec{k}$ is the wave vector of the radiation and the vector $\vec{b}$ is an arbitrary one.

The tensor T^{f}_{mn} is

$$T^{f}_{mn} \rightarrow -\frac{1}{4\pi}\left(\mathcal{E}_m E_n + \mathcal{E}_n E_m\right) \qquad \text{(A.I)}$$

where $\vec{\mathcal{E}}$ is the external field and $\vec{E}$ is the field of the particle. The arrow here and below means that the terms $\sim \delta_{mn}$, $k_m b_n$ are omitted.

Analyze ψ^{t}_{mn} into Fourier components

$$\psi^{t}_{ij} = \sum_n \psi^{n}_{ij} e^{-in\omega_0 t}$$

where ω_0 is the rotation frequency. Solving the wave equation in a standard way, we obtain the following expression for ψ^{n}_{ij} in the wave zone

$$\psi^{n}_{ij} = \frac{\varkappa}{(2\pi)^3} \frac{e^{ikR}}{R} \int_0^T \frac{dt}{T} e^{ikt} \int d\vec{r}\, \delta(\vec{r} - \vec{r}_0(t)) \times \tag{A.2}$$

$$\times \int \frac{d\vec{q}\, e^{-i\vec{q}\vec{r}}}{(\vec{k}-\vec{q})^2 (k^2 - q^2 + i\varepsilon)} \left\{ (\vec{k}-\vec{q})_i (k\vec{v} - \vec{q})_j + (\vec{k}-\vec{q})_j (k\vec{v} - \vec{q})_i \right\}$$

Here $\vec{r}_0(t)$ and $\vec{v}(t)$ are the particle's coordinate and velocity, $k = n\omega_0$, $\vec{k} = k\vec{R}/R$. In (A.2) the Fourier-transform of the source T^t is substituted for by its expression via the contraction of the fields $\vec{\mathcal{E}}$ and $\vec{E}$.

The integral

$$\int d\vec{q}\, e^{-i\vec{q}\vec{r}} \frac{(\vec{k}-\vec{q})_i (k\vec{v} - \vec{q})_j}{(\vec{k}-\vec{q})^2 (k^2 - q^2 + i\varepsilon)} \tag{A.3}$$

by means of the Feynman parametrization

$$a^{-1} b^{-1} = \int_0^1 dx \left[ax + b(1-x) \right]^{-2}$$

is reduced to the form

$$-\pi^2 \int_0^1 dx\, e^{ixkr - i(1-x)\vec{k}\vec{r}} \frac{r_i}{r} \left[ikv_j + \frac{r_j}{r} \left(ixk - \frac{1}{r} \right) \right] \tag{A.4}$$

After the substitution of (A.4) into (A.2) the integration over $\vec{r}$ is carried out trivially.

Fix now the system of coordinates. Let the trajectory lie in the plane X,Y. Put the observation point $\vec{R}$ into the plane Y,Z so that $\vec{R}=R(0,\sin\theta,\cos\theta)$. The integration over **t** leads then to Bessel functions and we obtain the next formulae for those components of ψ^n_{ij} through which the intensity is expressed

$$\psi^n_{xx}-\psi^n_{yy}\cos^2\theta=\frac{\varkappa}{4\pi}\frac{\kappa e^{i\kappa R}}{R}\int_0^1 dx\, e^{in v(1-x)}\Big\{J'_n(\xi)\frac{1+\cos^2\theta}{x\sin\theta}\Big[1-\frac{1}{nv}\Big(i(1-x)-\frac{1}{nv}\Big)\Big]+J_n(\xi)\Big[\Big(\frac{1+\cos^2\theta}{x^2v^2\sin^2\theta}-\cos^2\theta\Big)\Big(i(1-x)-\frac{1}{nv}\Big)-\frac{1+\cos^2\theta}{x^2nv\sin^2\theta}\Big]\Big\} \tag{A.5}$$

$$\psi^n_{xy}\cos\theta=\frac{\varkappa}{4\pi}\frac{\kappa e^{i\kappa R}}{R}\int_0^1 dx\, e^{in v(1-x)}\Big\{J'_n(\xi)\frac{i\cos\theta}{xv\sin\theta}\Big[-\frac{v}{n}+\Big(i(1-x)-\frac{1}{nv}\Big)\Big]-i\cos\theta J_n(\xi)\Big[\frac{v}{2}-\frac{1}{x^2v\sin\theta}+\frac{1}{nx^2v^2\sin^2\theta}\Big(i(1-x)-\frac{1}{nv}\Big)\Big]\Big\} \tag{A.6}$$

Here ξ=**nvx**$\sin\theta$.

The integrals of the type

$$\int_0^1 dx\, e^{inv(1-x)}J_n(nvx\sin\theta)$$

can be computed at large **n** (in the ultrarelativistic limit just $n\gg 1$ are essential) if one expands $J_n(nvx\sin\theta)$ in the vicinity of $x=1$ in powers of $x-1$. Then the integration leads to the series in powers of $n^{-1/3}$. Since the major contribution to the radiation is given by $n\sim\gamma^3$, we obtain the gravitational field in the wave zone as an expansion in terms of γ^{-1}.

The intensity of the radiation is reduced after all substitutions to the form (only the leading in γ terms are presented)

$$\frac{dI}{d\Omega}=\frac{\kappa\varepsilon^2}{2\pi r_0^2}\sum_{n=1}^{\infty}n^2\left[J_n'^2(nv\sin\theta)+ctg^2\theta J_n^2(nv\sin\theta)\right] \quad (A.7)$$

or

$$dI=\frac{\kappa Q^2}{r_0^2}dI_{em} \quad (A.8)$$

in complete agreement with the result presented in the text.

Prove now that resonant and non-resonant parts of gravitational radiation do not interfere. The field $\psi_{\mu\nu}^{n}$ is proportional to

$$\int d\vec{q}\,F(\vec{k}-\vec{q})f(\vec{q}) \quad (A.9)$$

where

$$f(\vec{q})\sim\frac{j(\vec{q})}{k^2-q^2+i\varepsilon}=j(\vec{q})\left[P\frac{1}{k^2-q^2}-i\pi\delta(k^2-q^2)\right] \quad (A.10)$$

The second term in the curly brackets is the solution of the free wave equation, so that its contribution corresponds to the resonant part of the radiation. This term is shifted in phase by $\pi/2$ relatively to the non-resonant part of ψ^f. The latter in its turn should have the same phase as ψ^p to guarantee the transversality of their sum. Hence the made assertion follows.

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Ответственный за выпуск С.Н.РОДИОНОВ
Подписано к печати 11.У1-73г. МН 08326
Усл. 0,7 печ.л., тираж 200 экз. Бесплатно.
Заказ №51. ПРЕПРИНТ

Отпечатано на ротапринте в ИЯФ СО АН СССР, вг