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A b s t r a c t

The critical behaviour of a system of N coupled fields with arbitrary nonsymmetric interaction is considered using the renormalization-group techniques. To the first order in $\epsilon = 4-d$ it is shown that there exist both fixed points possessing a symmetry and nonsymmetrical fixed points. Also to the first order the degree of stability of this fixed points is investigated. The existence of fixed points possessing a symmetry is proved to any order in ϵ .

CRITICAL BEHAVIOUR OF THE COUPLED FIELDS MODEL

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As it is well known the renormalization group method suggested by Wilson reduces the problem of critical behaviour of a system to the investigation of the fixed points of the renormalization-group equations (see a review [1]). The features of the critical behaviour are determined by the properties of the corresponding fixed point. In the system of coupled fields the existence of (internal) symmetry in the critical points is connected with the symmetric properties of the corresponding fixed points. In particular, the effectibe Hamiltonian in the fixed point can possess a symmetry even if the initial Hamiltonian doesn't as it was shown in [2] for the system of two coupled fields.

In the present paper we consider a system of N scalar fields with arbitrary (nonsymmetric) interaction. We will show that to the first order in $\epsilon = 4 - d$ (d is the dimension of space) the renormalization group equations possess both symmetric and nonsymmetric fixed points. The degree of stability of the fixed points is also investigated. It is shown that the most stable fixed point for $N \leq 4$ possesses $SO(N)$ symmetry but for $N \geqslant 5$ the most stable fixed point is nonsymmetric. The existence of the fixed points possessing a symmetry is proved also to any order in ϵ .

1. We consider a system of N interacting scalar fields φ_i with Hamiltonian

$$H_0 = \sum_i (\nabla \varphi_i)^2 + \sum_i \tau_i^0 \varphi_i^2 + \sum_{i,k} u_{ik}^0 \varphi_i^2 \varphi_k^2 , \qquad (1)$$

where τ_i^0 and $u_{ik}^0 = u_{ki}^0$ are arbitrary.

To the first order in $\epsilon = 4 - d$ one can obtain by the renormalization-group method [1] the following recursion formulas for τ_i and u_{ik} (see also [3]):

$$\tau_i^{\ell+1} = 4(\tau_i^{\ell} + 2u_{ii}^{\ell} + \sum_n u_{ni}^{\ell}), \qquad (i,k,n = 1,\dots,N)$$

$$u_{ik}^{\ell+1} = 2^{\epsilon}(u_{ik}^{\ell} - 2u_{ik}^{\ell}(u_{ii}^{\ell} + u_{kk}^{\ell}) - \sum_n u_{in}^{\ell} u_{nk}^{\ell} - 4u_{ik}^{\ell 2}) \qquad (2)$$

The fixed points are therefore the solutions of the system of equations:

$$3\tau_i^{*} = -8u_{ii}^{*} - 4\sum_n u_{ni}^{*},$$

$$\epsilon \ln 2 \cdot u_{ik}^{*} = 2u_{ik}^{*}(u_{ii}^{*} + u_{kk}^{*}) + \sum_n u_{in}^{*} u_{nk}^{*} + 4u_{ik}^{*2}. \qquad (3)$$

It is not difficult to show that beside the Gaussian fixed point there exist several types of fixed points*.

I. Trivial fixed point

$$u_{ik}^{*}(i \neq k) = 0, \qquad u_{ii}^{*}(i=1,\dots,N) = \frac{1}{9}\epsilon \ln 2,$$

$$\tau_i^{*}(i=1,\dots,N) = -\frac{4}{9}\epsilon \ln 2.$$

This fixed point corresponds to N independent fields.

The following two type of fixed points correspond to the case when all $u_{ik}^{*} \neq 0$.

* In the work [3] only II and IV type of fixed points have been found.

II. The fixed point with maximal symmetry:

$$u_{ik}^{*}(i,k=1,\dots,N) = \frac{1}{8+N}\epsilon \ln 2, \qquad \tau_i^{*}(i=1,\dots,N) = -\frac{4}{3}\frac{(N+2)}{(N+8)}\epsilon \ln 2.$$

The effective Hamiltonian corresponding to such fixed point possesses the $SO(N)$ symmetry.

III. Nonsymmetric fixed point:

$$u_{ii}^{*}(i=1,\dots,N) = u = \frac{N-1}{9N}\epsilon \ln 2, \qquad \tau_i^{*}(i=1,\dots,N) = -\frac{8(N-1)}{9N}\epsilon \ln 2,$$

$$u_{ik}^{*}(i \neq k;\ i,k=1,\dots,N) = W = \frac{3u}{N-1} = \frac{1}{3N}\epsilon \ln 2.$$

For $N = 2$ this fixed point correspond to two independent fields $\varphi_1 + \varphi_2$ and $\varphi_1 - \varphi_2$ [2].

Then there exist fixed points for which some $u_{ik}^{*}(i \neq k) = 0$.

IV. The fixed points with partial symmetry:

$$u_{ik}^{*}(i \neq k) = 0, \qquad \text{for } i=1,\dots,N;\ k=n+1,\dots,N.$$

$$u_{ii}^{*}(i=1,\dots,n) = u = \frac{1}{8+n}\epsilon \ln 2,$$

$$u_{ik}^{*}(i \neq k;\ i,k=1,\dots,n) = W = u = \frac{1}{8+n}\epsilon \ln 2,$$

$$u_{ii}^{*}(i=n+1,\dots,N) = \frac{1}{9}\epsilon \ln 2,$$

$$\tau_i^{*}(i=1,\dots,n) = -\frac{4}{3}\frac{(n+2)}{(n+8)}\epsilon \ln 2,$$

$$\tau_i^{*}(i=n+1,\dots,N) = -\frac{4}{9}\epsilon \ln 2.$$

The effective Hamiltonian possesses the $SO(n)$ symmetry. The fields φ_i $(i=n+1,\dots,N)$ do not interact neither one with others nor with the fields φ_i $(i=1,\dots,n)$.

V. The fixed point analogous to III

$$\mathcal{U}^{*}_{ik}\,(i\neq k)=0 \qquad \text{for} \quad i=1,\dots,N; \quad k=n+1,\dots,N.$$

$$\mathcal{U}^{*}_{ii}\,(i=1,\dots,n)=\mathcal{U}=\frac{n-1}{9n}\,\epsilon\ln 2,$$

$$\mathcal{U}^{*}_{ik}\,(i\neq k;\ i,k=1,\dots,n)=W=\frac{3\mathcal{U}}{n-1}=\frac{1}{3n}\,\epsilon\ln 2,$$

$$\mathcal{U}^{*}_{ii}\,(i=n+1,\dots,N)=\frac{1}{9}\,\epsilon\ln 2,$$

$$\tau^{*}_{i}\,(i=1,\dots,n)=-\frac{8(n-1)}{9n}\,\epsilon\ln 2,$$

$$\tau^{*}_{i}\,(i=n+1,\dots,N)=-\frac{4}{9}\,\epsilon\ln 2.$$

It should be noted that there do not exist fixed points which are intermediate between II and IV (III and V) in the sence that some $\mathcal{U}_{ik}\,(i=1,\dots n;\ k=n+1,\dots N)$ are not equal to zero.

VI. Finally the general case, when among N fields φ_i there exist q uncoupled groups of fields containing correspondingly n_q fields (it is possible that $n_q=1$). In every group of coupled fields it is possible (for $n_q\geqslant 2$) either the symmetric case ($SO(n_q)$ group of symmetry) or nonsymmetric case (of type V).

The critical exponents for each enumerated type of fixed points can be calculated by the standart method [1].

It should be noted that symmetric (II) and nonsymmetric (III) fixed points have a different behaviour in the limit $N\to\infty$. For symmetric fixed points we obtain in this limit the agreement with the spherical-model results [4]:

$$\mathcal{U}^{*}_{ik}\,(N\to\infty)\to 0, \quad \tau^{*}_{i}\,(N\to\infty)=-\frac{4}{3}\,\epsilon\ln 2$$

For nonsymmetric fixed points (of type III) we obtain in the

limit $N\to\infty$ a system of N uncoupled but selfinteracting fields with

$$\mathcal{U}^{*}_{ii}\,(N\to\infty)=\frac{1}{9}\,\epsilon\ln 2; \quad \tau^{*}_{i}\,(N\to\infty)=-\frac{8}{9}\,\epsilon\ln 2.$$

2. Now we will find the ranges of initial values of $\mathcal{U}^{\circ}_{ik}$ corresponding to each fixed point and discuss the degree of stability of fixed points.

Let us consider a system of n coupled fields. As we have seen there exist three fixed point (we denote $\mathcal{U}_{ii}=\mathcal{U}$, $\mathcal{U}_{ik}\,(i\neq k)=W$): 1) $\mathcal{U}\neq 0$, $W=0$ (type I), 2) $W=\mathcal{U}$ (type II) - fixed point with $SO(n)$ symmetry, 3) $W=\frac{3\mathcal{U}}{n-1}$ (type III) - nonsymmetric fixed point.

Let us firstly consider the case $n\leq 4$. Using the recursion formulas (2) it is not difficult to show that the symmetric fixed point gives the critical behaviour for any initial condition with W_0 in the range $0<W_0<\frac{3\mathcal{U}_0}{n-1}$. The fixed point 1) gives the critical behaviour only for initial condition $W_0=0$ and nonsymmetric fixed point gives the critical behaviour only for initial $W_0=\frac{3\mathcal{U}_0}{n-1}$. So the symmetric fixed point is the most stable fixed point for $n\leq 4$.

The ranges $W_0<0$ and $W_0>\frac{3\mathcal{U}_0}{n-1}$ are anomalous; for this type of initial conditions the recursion formulas give values of $\mathcal{U}_{ik}$ which tend to infinity. As a result one go outside the range of the validity of the recursion formulas (2).

The situation is changed for $n\geqslant 5$. In this case the most stable fixed point is a nonsymmetric one. The nonsymmetric fixed point (with $W=\frac{3\mathcal{U}}{n-1}$) gives the critical behaviour for any initial conditions with $0<W_0<\mathcal{U}_0$. Symmetric fixed point gives the critical behaviour only for initial condition $W_0=\mathcal{U}_0$, and therefore a symmetric fixed point is less stable than nonsymmetric one.

In the case $n\geqslant 5$ the ranges $W_0<0$ and $W_0>\mathcal{U}_0$ are anomalous. We see also that for $n \geqslant 4$ symmetric fixed point lies on boundary of the range of validity of the recursion formulas (2).

The fixed points of type VI (and in particular of types IV and V) are less stable than fixed points of types II and III. Indeed, if in the space of values of u_{ik} we go out of the point $u_{ik}\,(i \neq k) = 0$ and move along a line for which all $u_{ik}\,(i \neq k) \neq 0$, then we will get either into symmetric fixed point (for $n \leq 4$) or into the nonsymmetric one ($W = \frac{3u}{n-1}$) (for $n \geqslant 5$). In the fixed point of type VI we will get only if we will move along the lines for which some of the $u_{ik}\,(i \neq k)$ are equal to zero.

Analogous for fixed points of type VI the degree of stability is in general the less the more quantities $u_{ik}\,(i \neq k)$ for these fixed points are equal to zero. For the given fixed point the degree of stability it isn't difficult to determine by reasoning used above.

Thus, the fixed points of the recursion formulas (2) can be classified not only by the types of symmetry, but also by degree of stability and fixed points with higher symmetry are more stable.

As it was shown in [1] thedegree of stability of fixed point is connected with a number of thermodynamic parameters that are fixed at the corresponding critical point. Namely, for critical point corresponding less stable fixed point more thermodynamic parameters are fixed.

So, for the system of N coupled fields thefixed points II ($N \leq 4$) and III ($N \geqslant 5$) give the critical behaviour with the least number of fixed thermodynamic parameters and fixed point I corresponds critical point with the most number of fixed thermodynamic parameters. For fixed points of type VI we have large number of intermediate cases.

Thus, for critical point with higher internal symmetry the less number thermodynamic parameters are fixed and vice versa.

For the field theory the degree of instability of a fixed point determines the number of free parameters in renormalized theory associated with the fixed point [1] . So the more symmetric renormalized field theory (renormalized theory associated with a more symmetric fixed point)of N interacting scalar fields has less number free parameters than the less symmetric renormalized field theory.

3. We investigated above the fixed points for the system of N coupled fields to the first order in $\epsilon = 4 - d$. Now we consider the question of existence of fixed points possessing a symmetry to any order in ϵ .

As it is known [1] to the higher order in ϵ an effective Hamiltonian H^* has the Landau-Ginsberg form in which all quantities $u^{(2n)}$ are important ($u^{(2n)}$ are the coefficient corresponding to the term φ^{2n} in H^*). The recursion formulas for $u^{(2n)}$ to any order in ϵ can be obtained by standart method [1] . Following [1] we suggest that $u^{(2n)}$ (excepting $u^{(2)}$) are constants. Therefore the "Diagrams" (see Lecture IV in [1]) are sums of all one-particle-irreducible graphs. The equations which determine the fixed point have a form:

$$u_i^{(2)*} = f_i\left(u_i^{(2)*},\, u_{ik}^{(4)*}\right), \tag{4}$$

$$u_{ik}^{(4)*} = f_{ik}\left(u_i^{(2)*},\, u_{ik}^{(4)*}\right), \tag{5}$$

$$u_{ike}^{(6)*} = f_{ike}\left(u_i^{(2)*},\, u_{ik}^{(4)*}\right), \tag{6}$$

$$\ldots = \ldots\ldots\ldots$$

The functions $f_i, f_{ik}, \ldots$ are polynomiales in $u_i^{(2)*}$ and $u_{ik}^{(4)*}$ of the degree m and $m+1$, where m is the order in ϵ which we are dealing with. There is a correspondence between each monom in the polynomials in r.h.s. of (4-6) and the graph with definite set of internal and external lines.

It is important that a numerical factor before each monom is purely combinatorial, i.e. this numerical factor is completely determined by the structure of corresponding graph.

Let us prove now that to any order in ϵ the system of N coupled fields have a fixed point possessing $SO(N)$ symmetry i.e. the system of equations (4-5-6) have a solution $U_i^{(2)*}=U^{(2)}$, $U_{ik}^{(4)*}=U^{(4)}$ $(i,k=1,...,N)$.

Let us first consider equation (4) and (5). Let us divide all equations (5) into two groups; first - for quantities $U_{ii}^{(4)*}$ and second - for nondiagonal quantities $U_{ik}^{(4)*}$ $(i\neq k)$. Comparing two equations for two different nondiagonal $U_{ik}^{(4)*}$, for example $U_{mn}^{(4)*}$ and $U_{pq}^{(4)*}$ we note that these equations are the results of summing of two sets of graphs which are topologically equivalent one to another. Therefore, to each monom from one equation it is possible to compare such monom of the another equation that these monoms will be connected by substitution $m\leftrightarrow p$, $n\leftrightarrow q$. The numerical factors (in view of there combinatorial nature) before such monoms in different equations coincide. Thus, the equation for $U_{mn}^{(4)*}$ turns into the equation for $U_{pq}^{(4)*}$ under substitution $m\leftrightarrow p$, $n\leftrightarrow q$. Analogous, we see that the equations for diagonal quantities $U_{ii}^{(4)*}$ also transform one into another under substitution of indexes. Therefore, the conditions $U_{ii}^{(4)*}(i=1,...N)=U^{(4)}$, $U_{ik}^{(4)*}(i\neq k; i,k=1,..,N)=W^{(4)}$ do not contradict to the system of equations for $U_{ik}^{(4)*}$. Under these conditions this system reduces to two equations for $U^{(4)}$ and $W^{(4)}$.

Futher, it is not difficult to see that the polynomial in the equation for $U_{ii}^{(4)*}$ is possible to present in the form of sum of the monoms with coefficients which coincide with coefficients in the polynomial for $U_{ik}^{(4)*}(i\neq k)$ (to the first order in ϵ see equations (3)). Therefore, if one take $U^{(4)}=W^{(4)}$ then the r.h.s. in the equations for $U_{ii}^{(4)*}$ and $U_{ik}^{(4)*}(i\neq k)$ will coincide and thus, the system of equations for $U_{ik}^{(4)*}$ has a solution $U^{(4)}=W^{(4)}$. As a result the whole system (4-5) reduces to one

equation for $U^{(4)}$.

Above we have suggested that all $U_i^{(2)*}=\tau$. It is possible to see that this condition doesn't contradict to the equations (4). Indeed, the equations (4) transform one into another under substitution of indexes, and therefore have a solution $U_i^{(2)*}=\tau$, $U_{ik}^{(4)*}=U^{(4)}$.

Thus, the system of equations (4-5) have the solution $U_i^{(2)*}(i=1,...N)=\tau$ and $U_{ik}^{(4)*}(i,k=1,..N)=U^{(4)}$ that correspond to $SO(N)$ symmetry of quadradic and biquadradic term in effective Hamiltonian H^*.

Futher, since the quantities $U_{ik\ldots}^{(2n)*}$ $(n\geq 3)$ are expressed only through $U_i^{(2)*}$ and $U_{ik}^{(4)*}$ then taking the values $U_i^{(2)*}=\tau$, and $U_{ik}^{(4)*}=U^{(4)}$ one obtain, as it is not difficult to show the values of $U_{ik\ldots}^{(2n)*}$ corresponding to $SO(N)$ symmetry.

Thus, we have shown that for the system of N interacting fields there exist (to any order in ϵ) fixed point for which effective Hamiltonian possesses the $SO(N)$ symmetry.

It is not difficult to see that to any order in ϵ there also exist the fixed point of type VI for which H^* possesses partial symmetry. It is obvious that these fixed points are less stable than those with $SO(N)$ symmetry (for $N\leq 4$).

Thus we have shown that the system of N scalar fields with arbitrary (nonsymmetric) interaction possesses a large set of fixed points with a different type of symmetry; from $SO(N)$ group of internal symmetry (fixed point with maximal symmetry) to the absence of continuous group of symmetry (fixed point with independent fields).

R E F E R E N C E S

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