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A b s t r a c t

It is shown that Bäcklund transformations of the differential equations integrable by the inverse spectral transform method form infinite abelian groups. Transformation properties of the spectral parameter and Bäcklund transformations under the group of symmetry are considered.

1. Bäcklund transformations play an important role in a study of nonlinear differential equations. By now these transformations were derived for a wide class of equations. It was clarified the relatioship between them and the method of inverse spectral transform (IST method). It was discussed the connection between the transformation of spectral data and the field quantities (see, e.g., refs. [1-7]). But the question on group properties of Bäcklund transformation (BT) is still open.

In the present paper it is shown that the BT of equations integrable by IST method form infinite groups which are tensor product of the infinite-dimensional continuous abelian group of continual (nonsoliton) BT and the infinite discrete abelian group of soliton BT. The transformation properties of the quantities appearing in the IST method (in particular, spectral parameter) under the symmetry group of integrable equations are also discussed.

2. Let us considered differential equations integrable by the following linear spectral problem:

$$\frac{\partial\psi}{\partial x} = -ik\sigma_3\psi + \frac{1}{2}\left[(r+q)\sigma_1 + i(r-q)\sigma_2\right]\psi \qquad (1)$$

where $\sigma_1, \sigma_2, \sigma_3$ - Pauli matrices, $u = \begin{pmatrix} r \\ q \end{pmatrix}$ - is two-component field quantity, k is spectral parameter. These equations have a form [6]

$$\sigma_3 \frac{\partial u(x,t,\vec{y})}{\partial t} + \vec{v}(L,t,\vec{y})\frac{\partial \sigma_3 u(x,t,\vec{y})}{\partial \vec{y}} + \gamma(L,t,\vec{y})u(x,t,\vec{y}) = 0. \quad (2)$$

Here $\vec{v}$ and γ are arbitrary functions, and $L = \Lambda(u,u)$ where, integro-differential operator Λ is equal to

$$\Lambda(\tilde{u},u) = \frac{1}{2i}\left(\sigma_3\frac{\partial}{\partial x} - V_1 I V_1^T \sigma_2 - V_2 I V_2^T \sigma_2\right), \quad (3)$$

$$I f(x,t,\vec{y}) = \int_x^\infty d\xi\, f(\xi,t,\vec{y}); \quad V_1 = \begin{pmatrix} \tilde{r} \\ q \end{pmatrix}, \quad V_2 = \begin{pmatrix} r \\ \tilde{q} \end{pmatrix}.$$

BT : $u \to \tilde{u}$ for eq. (2) are given by the following relation [6]

$$\left(g(\Lambda)+f(\Lambda)\sigma_3\right)\tilde{u} + \left(g(\Lambda)-f(\Lambda)\sigma_3\right)u = 0, \qquad (4)$$

where $f(z)$, $g(z)$ are arbitrary entire functions. Spectral dates $\alpha^{\pm}(k)$ are transform under (4) as follows [6]

$$\alpha^{\pm}(k) \to \tilde{\alpha}^{\pm}(k) = \frac{f(k) \mp g(k)}{f(k) \pm g(k)}\, \alpha^{\pm}(k). \qquad (5)$$

The poles of quantities $\alpha^{\pm}(k)$ correspond to discrete eigenvalues of problem (1) (solitons of eq. (2)) [6].

3. Arbitrary BT (5), in virtue of well known theorems, can be represented in the form

$$\tilde{\alpha}^{\pm}(k) = \left[\exp \omega(k) \cdot \frac{\prod_{i=1}^{N_-}(k-k_{0i}^-)}{\prod_{i=1}^{N_+}(k-k_{0i}^+)}\right]^{\pm 1} \alpha^{\pm}(k), \qquad (6)$$

where $\exp \omega(k)$ - is arbitrary function without poles and zeros. Therefore, arbitrary BT (5) is a combination of BT of two type:

$$\tilde{\alpha}^{\pm}(k) = B^{\pm}\alpha^{\pm}(k) = \left[B_\omega^{(c)} \cdot B^{(d)}\right]^{\pm 1} \alpha^{\pm}(k) \qquad (7)$$

where

$$B_\omega^{(c)} = \exp \omega(k), \qquad (8)$$

and

$$B^{(d)} = \frac{\prod_{i=1}^{N_-}(k-k_{0i}^-)}{\prod_{i=1}^{N_+}(k-k_{0i}^+)}. \qquad (9)$$

The expressions of BT (8) and (9) in the terms of variables u and $\tilde{u}$ can be easily founded from formulae (4) and (5).

BT (7) with $B^{(d)} = 1$ is BT which does not change the number of poles of $\alpha^{\pm}(k)$. Let us call it continual BT. These BT form as it follow from (7) and (8) the infinite dimensional Lie group. The function $\omega(z)$ is the "parameter" of transformations: $\omega = 0$ correspond to the identical transformation, in virtue of relation $\Lambda(\tilde{u}, u) = \Lambda(u, \tilde{u})$ function $(-\omega)$ correspond to the inverse transformation. Infinite-dimensional group of continual BT can be considered as an infinite parameterical abelian group. Indeed, let expand an arbitrary function $\omega(z)$ in a Taylor series: $\omega(z) = \sum_{m=0}^{\infty} \omega_m z^m$, ($\omega_m$ are arbitrary numbers). Then arbitrary BT (8) can be represented as the infinite product of the following transformations:

$$\tilde{u}(x,t) = \exp\left(\omega_m \sigma_3 \Lambda^m(\tilde{u},u)\right) u(x,t),$$
$$\tilde{\alpha}^{\pm}(k) = \exp\left(\pm \omega_m k^m\right) \alpha^{\pm}(k), \qquad (10)$$

where ω_m $(-\infty < \omega_m < \infty)$ are parameters of the transformations and m takes the values 0,1,2,..., ∞. It is clear, that the transformations with different m commute each with other. Under the infinitesimal transformation (10)

$$B_m(k): \ \delta u(x,t) = \omega_m \sigma_3 L^m u(x,t). \qquad (11)$$

BT(7) with $B = B^{(d)}$ $(B^{(c)} \equiv 1)$ have essentially discrete character. Arbitrary BT of this type can be represented in the form

$$B^{(d)} = \prod_{i=1}^{N_-} B_{k_{0i}^-}^{-\frac{1}{2}} \prod_{i=1}^{N_+} B_{k_{0i}^+}^{\frac{1}{2}} \qquad (12)$$

where $B_{k_0}^{-\frac{1}{2}} \equiv \left(B_{k_0}^{\frac{1}{2}}\right)^{-1}$ and $B_{k_0}^{\frac{1}{2}}$ is the following BT

$$\alpha^{+}(k) \to \tilde{\alpha}^{+}(k) = (k-k_0)^{-1}\alpha^{+}(k). \qquad (13)$$

In the variables u and $\tilde{u}$ BT (13) is given by relation (4) with $f = \Lambda - k_0 + 1$ and $g = \Lambda - k_0 - 1$ (see also ref. [8]). Let us call BT $B_{k_0}^{\frac{1}{2}}$ the elementary BT (EBT). EBT $B_{k_0}^{\frac{1}{2}}$ commute each with other and each of them add one soliton to initial solution.

It is not difficult to satisfy that the number k_0 in BT $B_{k_0}^{\frac{1}{2}}$ cannot be considered as the group parameter of the EBT. It is a index numerating different (with different k_0) EBT. So EBT $\{B_{k_0}^{\frac{1}{2}}\}$ itself does not form a group. But if one consider EBT $\{B_{k_0}^{\frac{1}{2}}\}$ together with the identical transformation E and all possible by-product BT (i.e. BT (12) with all possible $k_{0i}^{\pm}$, N_+ and N_-) then the whole set of these BT form a group. This infinite group of all soliton BT is abelian and discrete.

Continual and soliton BT commute each with other. Therefore the total group B of BT of equations (2) is the tensor product $B = B^{(c)} \otimes B^{(d)}$ where $B^{(c)}$ is the infinite-dimensional continuous abelian group of continual BT and $B^{(d)}$ is the infinite discrete abelian group of soliton BT with a continual set of EBT $\{B_{k_0}^{\frac{1}{2}}\}$.

Under the reduction of the general system (2) i.e. if it is exist some relation between r and q the group of BT is reduced. So, if $r = const \cdot q$ [1], then $N_- = N_+$, $\omega(-z) = -\omega(z)$ and $k_0^+ = -k_0^- = k_0$. EBT is the following transformation

$$B_{k_0} = B_{-k_0}^{-\frac{1}{2}} \cdot B_{k_0}^{\frac{1}{2}} \qquad (14)$$

In the case $r = const \cdot q^*$ we have $N_- = N_+$, $\omega^*(z^*) = \omega(z)$, $k_0^+ = k_0^{-*} = k_0$

and EBT is

$$B_{k_0} = B_{k_0^*}^{-\frac{1}{2}} \cdot B_{k_0}^{\frac{1}{2}} \qquad (15)$$

Well known soliton BT for sine-gordon (SG) equation, modified Korteweg-de-Vries (mKdV) equation and nonlinear Shroedinger (NLS) equations have the form (14), (15) [5].

Note that for BT (14)

$$B_{k_0}^{-1} = B_{-k_0} \qquad (16)$$

and for BT (15)

$$B_{k_0}^{-1} = B_{k_0^*}. \qquad (17)$$

4. Let consider now transformation properties of spectral parameter k and quantities ψ under the transformation of the symmetry group of eqns (2). We restricted here with the case of two independent variables (x, t) and functions γ of the form const$\cdot L^n$. In particular, for $n = -1$ we have S-G eq., for $n = 2$, - N L S eq. and for $n = 3$ - KdV and mKdV eqs. [1].

It isn't difficult to see that this type of eqs. are invariant under the translations alone x and t and transformation $\Lambda(\lambda)$:

$$x \to x' = \lambda x,\ t \to t' = \lambda^n t,\ u(x,t) \to u'(x',t') = \lambda^{-1} u(x,t). \quad (18)$$

Linear spectral problem (1) is invariant under the space--time translations with k and ψ remain unchanged. For invariance (1) under the transformations (18) the following transformation properties must be satisfied

$$k \to k' = \lambda^{-1} k,\quad \psi(x,t,k) \to \psi'(x',t',k') = \psi(x,t,k) \quad (19)$$

Spectral data $\alpha^{\pm}(k)$ is invariant under transformation (18) too.

For the concrete reductions the eqs. (2) can posses more wide symmetry group. For example, NLS eq. ($n = 2, r=-q^*$) as known is invariant also under the Galilei transformations

$$\Gamma(v):\quad q(x,t)\to q'(x-vt,t)=\exp\left(-\frac{ivx}{2}+\frac{iv^2t}{4}\right)q(x,t)$$

and gauge transformations $q(x,t)\to q'(x,t) = \exp i\alpha\cdot q(x,t)$. Spectral problem (1) is invariant under these transformations if under the Galilei transformations

$$\begin{aligned} &\psi_1(x,t,k)\to\psi_1'(x',t',k')=\exp\left(-\frac{ivx}{4}+\frac{iv^2t}{8}\right)\psi_1(x,t,k),\\ &k\to k'=k+\frac{v}{4},\\ &\psi_2(x,t,k)\to\psi_2'(x',t',k')=\exp\left(\frac{ivx}{4}-\frac{iv^2t}{8}\right)\psi_2(x,t,k), \end{aligned}\tag{20}$$

and under the gauge transformations

$$\begin{aligned} &\psi_1(x,t,k)\to\psi_1'(x,t,k)=\exp\left(\frac{i\alpha}{2}\right)\cdot\psi_1(x,t,k),\\ &k\to k'=k,\\ &\psi_2(x,t,k)\to\psi_2'(x,t,k)=\exp\left(-\frac{i\alpha}{2}\right)\cdot\psi_2(x,t,k). \end{aligned}\tag{21}$$

KdV eq. ($n = 3$, $r = -1$) together with transformations (18) is also invariant under Galilei transformations $q(x,t)\to q'(x-vt,t)=q(x,t)+\frac{v}{6}$. The corresponding linear problem (1) is invariant under these transformations if

$$\begin{aligned} &\psi_1(x,t,k)\to\psi_1'(x',t',k')=\psi_1(x,t,k)+i\left(k-\sqrt{k^2+\tfrac{v}{6}}\right)\psi_2(x,t,k),\\ &k^2\to k'^2=k^2+\frac{v}{6},\\ &\psi_2(x,t,k)\to\psi_2'(x',t',k')=\psi_2(x,t,k). \end{aligned}\tag{22}$$

For linear problems, which correspond to another classes of the integrable eqs., the spectral parameter as a rule transform in nontrivial way too [10]. Transformation properties of the spectral parameter guarantee the right transformation laws for dependent variables (for soliton solutions).

5. BT for eqs (2) posses nontrivial transformation properties under the symmetry group too. Using (4), (5), (13) one

can to show that for EBT (14) and (15) is valid the following relation

$$\Lambda(\lambda)\, B_{k_0}\, \Lambda(\lambda) = B_{\lambda^{-1}k_0} \tag{23}$$

where $\Lambda(\lambda)$ is the transformation (18). For SG eq. the analogous relation is known for a long time (see for example [11]).

For KdV and NLS eqs. we also have

$$\Gamma(v)\, B_{k_0}\, \Gamma^{-1}(v) = B_{k_0'} \tag{24}$$

where $\Gamma(v)$ is Galilei transformation and k_0' is given by formulae (20) and (22).

In virtue of (23) an arbitrary EBT B_{k_0} can be represented in the form

$$B_{k_0} = \Lambda^{-1}(k_0)\, B_1\, \Lambda(k_0). \tag{25}$$

For NLS eq. EBT B_{k_0} can be expressed in virtue to (23) and (24) from the BT B_1 and transformation (18) or Galilei transformation.*

The relation of the type (23), (24) take place in the general case too. Namely, if G is the symmetry group of the integrable eq. and B_{k_0} is EBT for this eq. then

$$G\, B_{k_0}\, G^{-1} = B_{k_0'=Gk_0}. \tag{26}$$

The validity of the relation (26) follow from the obvious diagram

$$\begin{tikzcd} u \arrow[r, "B_{k_0}"] \arrow[d, "G"'] & \tilde{u} \arrow[d, "G"] \\ u' \arrow[r, "B_{k_0'}"] & \tilde{u}' \end{tikzcd}$$

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* Analogous result for NLS eq. was obtained by Dr. H.Steudel [12]

In the conclusion note the following. For EBT (14) from formulae (16) and (23) we have

$$B_{K_0}^{-1} = \Lambda(-1)\, B_{K_0}\, \Lambda(-1) . \tag{27}$$

Transformation $\Lambda(-1)$ is discrete one. This transformation map component of connection $\Lambda^{+}(\lambda > 0)$ of group of transformation (18) on the component $\Lambda^{-}(\lambda < 0)$ and conversely. For SG eq. $\Lambda(-1) = I_{xt}$ (I_{xt} is the reflection of coordinates x and t), for mKdV $\Lambda(-1) = I_x$. Therefore the transformations I of discrete symmetries of the eqs. (2) transform the operators of creation and annihilation of solitons one to other

$$B_{K_0}^{-1} = I\, B_{K_0}\, I .$$

The complex group (18) (group of the transformations (18) with the complex λ) contain Λ^{+} and Λ^{-} and it is simple connected. Continuous transformations of this group connect B_{K_0} and $B_{K_0}^{-1}$ each with other. Transformations of the complex group (18) act transitively on the complex plane of the spectral parameter k and as a result one can identify the spectral parameter k with the group parameter of these transformations.

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