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A b s t r a c t

The general form of the differential equations integrable by the general linear spectral matrix problem of the order N X N and corresponding Backlund-transformations are found.

The inverse spectral method allow to integrate the great number of various differential equations (see, e.g. [1,2]). All these eqs. are united in the classes of eqns. integrable by the same linear spectral problem. The general form of the eqns. integrable by linear spectral problem of the order 2 x 2 was found in papers [3,4]. This class of eqns. is characterized by $n-1$ arbitrary functions (n is the number of independent variables) and some integro-differential operator. In the papers [4,5] a wide class of Bäcklund transformations for the eqns. of this type was obtained.

In the present paper we construct Bäcklund transformations and find the general form of the eqns. integrable by the general linear spectral problem of the order N X N

$$\frac{\partial \psi}{\partial x} = i\lambda A\psi + P\psi, \qquad (1)$$

where λ is the spectral parameter, A is diagonal matrix N X N ($A_{ik} = a_i \delta_{ik}$; $i,k = 1,\dots,N$), "potentials" $P(x)$ is the N X N matrix ($P_{ii} = 0$; $i = 1,\dots,N$). The examples of eqns. integrable by (1) were considered in the papers [6-8].

Let us consider the arbitrary transformation $P \to P'$, $\psi \to \psi'$ conversing the form of the eqns. (1). It isn't difficult to satisfy that (for N = 2 see [5])

$$\psi' - K\psi = -\psi \int_x^{\infty} dy\, \psi^{-1}(P' - P)\psi', \qquad (2)$$

where constant matrix K is determined by asymptotic properties of ψ.

Following to [7] we introduce for linear problem (1) matrix-solutions $F^{\pm}$ ($F^{\pm} \xrightarrow[x \to \pm\infty]{} \exp(i\lambda Ax)$) and the transition matrix S ($F^{+} = F^{-}S$). Passing in the formula (2) to the limit $x \to -\infty$ one can obtain (choosing $\psi = F^{+}$)

$$S' - S = -S \int_{-\infty}^{+\infty} dx\, F^{+-1}(x)\big(P'(x) - P(x)\big)F^{+\prime}(x). \qquad (3)$$

Let us to suppose now that transition matrix S' is transformed as follows

$$(S^{-1}S')_{in} = -\frac{1}{i}(S^{-1}BS')_{in}, \quad i\neq n \qquad (4)$$

where B is some diagonal matrix. Then equating the right-hand parts of (3) and (4) and using the relation

$$(S^{-1}BS')_{in} = -\int_{-\infty}^{+\infty} dx \frac{\partial}{\partial x}\left(F^{+^{-1}} B F^{+}\right)_{in} = \int_{-\infty}^{+\infty} dx \{F^{+^{-1}}(PB-BP')F^{+}\}_{in} \qquad (i\neq n)$$

we find

$$\int_{-\infty}^{+\infty} dx \{F^{+^{-1}}(x)[P'(x)-P(x)+i(P(x)B(\lambda)-B(\lambda)P'(x))]F^{+}(x)\}_{in} = 0 \quad (5) \qquad (i\neq n)$$

Rewritting formula (5) in the components and designating $\Phi^{(in)}_{ke} = (F^{+^{-1}})_{ik}(F^{+})_{en}$ we obtain ($B_{ik}(\lambda) = B_i(\lambda)\delta_{ik}$)

$$\int_{-\infty}^{+\infty} dx \sum_{k,e}\{P'_{ke} - P_{ke} + i(B_e(\lambda)P_{ke} - B_k(\lambda)P'_{ke})\}\Phi^{(in)}_{ke} = 0; \quad (i\neq n). \qquad (6)$$

It is easy to see that the sum in the formula (6) does not contain the diagonal quantities $\Phi^{(in)}_{kk}$ $(k=1,\dots,N)$. Expressing $\Phi^{(in)}_{kk}$ $(i\neq n)$ with the help of the formula (1) through nondiagonal $\Phi^{(in)}_{ke}$ $(k\neq e, i\neq n)$ one can satisfy that in the subspace with the basis $\{\Phi^{(in)}_{ke};\ k\neq e, i\neq n\}$ the following relation is valid

$$\sum_{q,p}\Lambda_{keqp}\Phi^{(in)}_{qp} = \lambda\Phi^{(in)}_{ke}, \quad (i\neq n) \qquad (7)$$

where ($\rho_{ke} = (a_k - a_e)^{-1}$)

$$\Lambda_{keqp} = i\rho_{ke}\{\delta_{kq}\delta_{ep}\frac{\partial}{\partial x} - \delta_{kq}P'_{ep}(x) + \delta_{ep}P_{qk}(x) - P'_{ek}(x)\int_x^{\infty} dy(\delta_{kp}P_{qk}(y) - \delta_{kq}P'_{kp}(y))\cdot + P_{ek}(x)\int_x^{\infty}dy(\delta_{ep}P_{qe}(y) - \delta_{eq}P'_{ep}(y))\cdot . \qquad \begin{matrix}(q\neq p)\\(k\neq e)\end{matrix}$$

In virtue to (7) formula (6) can be represented as follows

$$\int_{-\infty}^{+\infty}dx\sum_{q,p}\{P'_{qp}(x) - P_{qp}(x) + i\sum_{k,e}[P_{ke}(x)(B_e(\Lambda))_{keqp} - P'_{ke}(x)(B_k(\Lambda))_{keqp}]\}\Phi^{(in)}_{qp}(x) = 0. \quad (i\neq n) \qquad (8)$$

At last, integration by parts and changing the order of the integration in (8) gives $(i\neq n)$

$$\int_{-\infty}^{+\infty}dx\sum_{q,p}\Phi^{(in)}_{qp}(x)\cdot\{P'_{qp}(x) - P_{qp}(x) + i\sum_{k,e}[(B_e(\Lambda^+))_{qpke}P_{ke} - (B_k(\Lambda^+))_{qpke}P'_{ke}]\} = 0, \qquad (9)$$

where

$$\Lambda^+_{qpke} = i\rho_{ke}\{-\delta_{kq}\delta_{ep}\frac{\partial}{\partial x} - \delta_{kq}P'_{ep}(x) + \delta_{ep}P_{qk}(x) - (\delta_{kp}P_{qk}(x) - \delta_{kq}P'_{kp}(x))\int_{-\infty}^{x}dy\, P'_{ek}(y)\cdot + (\delta_{ep}P_{qe}(x) - \delta_{eq}P'_{ep}(x))\int_{-\infty}^{x}dy\, P_{ek}(y)\cdot . \qquad \begin{matrix}(q\neq p)\\(k\neq e)\end{matrix}$$

The equality (9) is fulfilled if the expression in the round brackets is equal to zero.

Thus, the transformations $P \to P'$ conserving (1) have the form

$$P'_{qp}(x) - P_{qp}(x) + i\sum_{k,e}\Big[\big(B_e(\Lambda^+)\big)_{qpke} P_{ke} - \big(B_k(\Lambda^+)\big)_{qpke} P'_{ke}\Big] = 0 \qquad (10)$$

$$(q,p = 1,\dots,N;\ q \neq p)$$

The arbitrary functions $B_e(\lambda)$ can depend also on any number of the parameters ($t, \vec{y}$).

The equations integrable by the linear problem (1) are obtained from (10) if one consider in variations of P resulting from the infinitesimal translations alone t and $\vec{y}$ ($\Delta P = \frac{\partial P}{\partial t} + \vec{H}(\lambda,t,\vec{y})\frac{\partial P}{\partial \vec{y}}$). The general form of these eqns. is the following

$$\frac{\partial P_{qp}(x,t,\vec{y})}{\partial t} + \sum_{k,e}\Big\{\big[\vec{H}(L^+,t,\vec{y})\big]_{qpke}\frac{\partial P_{ke}}{\partial \vec{y}} + i\big[Y_e(L^+,t,\vec{y}) - Y_k(L^+,t,\vec{y})\big]_{qpke} P_{ke}\Big\} = 0, \quad (q \neq p) \qquad (11)$$

where $L^+ = \Lambda^+(P'_{ke} = P_{ke})$ and $\vec{H}(\lambda,t,\vec{y})$, $Y_e(\lambda,t,\vec{y})$ $(e = 1,\dots,N)$ are arbitrary functions.

Thus, the class of the eqns. integrable by linear spectral problem (1) are characterized by the integrodifferential operator L^+ and by $N+n-3$ arbitrary functions $Y_e - Y_k$ $(e,k = 1,\dots,N)$, $\vec{H}(\lambda,t,\vec{y})$. Some concrete eqns. of the type (11) are well known. The models of the resonantly interacting wave envelopes [6-8] correspond to linear functions $Y_e(\lambda)$ $(Y_e(\lambda) = Y_e\cdot\lambda;\ e = 1,\dots,N)$ and $\vec{H}$ = const. Many-component nonlinear Schrödinger eqn. [9] correspond to quadratic functions $Y_e(\lambda)$ $(Y_e(\lambda) = Y_e\cdot\lambda^2)$, $\vec{H} = 0$ and some special reduction of matrix $P(x)$. If $Y_e(\lambda) = Y_e\cdot\lambda^{-1}$ we have various generalizations of sine - Gordon equation.

The evolution of the transition matrix S (scattering data) is determined by the eq.

$$\Delta S = \frac{1}{i}[S, Y]. \qquad (12)$$

So, if $\vec{H} = 0$ then $S(\lambda,t) = \exp\big(i\int_{-\infty}^{t} dt'\, Y(\lambda,t')\big)\times S(\lambda,0)\exp\big(-i\int_{-\infty}^{t} dt'\, Y(\lambda,t')\big)$. In particular, $S_{ii}(\lambda,t) = S_{ii}(\lambda,0)$ $(i = 1,\dots,N)$.

Using the relation (4) it isn't difficult to show that the transformations (10) with matrix B which does not depend on t and $\vec{y}$, transform the solutions of any eq. of the type (11) into solutions of the same eq.. Therefore, such transformations (10) are Bäcklund-transformations for eqns (11). If B depend on t and $\vec{y}$ then the transformations (10) are generalized Bäcklund transformations - it transform one to another the solutions of different eqns of the type (11) (with different functions Y_e and $\vec{H}$).

The analysis of Bäcklund transformations for the eqns. integrable by (1) with $N \geqslant 3$ is of a great interest due to the existence of the nontrivial processes in these cases.

R e f e r e n c e s

1. A.C.Scott, F.Y.F. Chu, D.W. McLaughlin, Proc. IEEE, 61, 1443 (1973).
2. "Solitons", Ed. R. Bullough, Springer-Verlag (1978).
3. M.S.Ablowitz, D.S.Kaup, A.C.Newell, H.Segur, Stud. Appl. Math., 53, 249 (1974).
4. F.Calogero, A.Degasperis, Nuovo Cim. 32B, 201 (1977).
5. R.K.Dodd, R.K.Bullough, Phys. Lett., 62A, 70 (1977).
6. V.E.Zakharov, A.B.Shabat, Func. Anal. and Appl., 8, 226 (1974).
7. V.E.Zakharov, S.V.Manakov, Zh. Eksp. Teor. Fiz., 69, 1654 (1975); preprint Institute of Nuclear Phys. 74-41, Novosibirsk (1974) (in Russian).
8. M.J.Ablowitz, R.Haberman, J. Math. Phys., 16, 2301 (1975).
9. S.V.Manakov, Zh. Eksp. Teor. Fiz., 65, 505 (1973) (in Russian).

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