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DYNAMICAL STOCHASTICITY IN CLASSICAL MECHANICS

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A b s t r a c t

Some recent results of investigations of the stochastic motion in classical dynamical systems are presented. The questions connected with the deriving of stochasticity border and finding out of the statistic properties of stochastic motion are considered. The conditions for arising of the modulation diffusion as well as peculiarities of the latter are discussed in detail.

1. Introduction

The term "dynamical stochasticity" in the title of this survey paper emphasizes that we are dealing with a specific case of motion of a completely deterministic (dynamical) system. The fact that the motion in this case turns out to be extremely irregular, complicated and unpredictable, is determined exclusively by the internal dynamics of the system, and is not related to any effect of external random perturbations. This type of dynamical motion has been called stochastic or chaotic, the terms whose vagueness reflects the enormous variety of the different special cases of such motions.

The ever increasing interest in the study of stochastic motion is caused mainly by two factors. First, in various branches of physics, engineering and other sciences, more and more particular problems arise whose solution requires a well developed theory of stochastic motion. Secondly, this unusual (nontrivial) regime of motion builds a bridge between dynamical and statistical laws of physics, which were considered once as contradictory, and gives the possibility of understanding and deriving the latter from the former.

From the view point of applications, stochastic motion is a widespread (and dangerous) instability of nonlinear oscillations, which leads to diffusion in the phase space and to other unpleasant consequences. An example is the loss of charged particles in accelerators /1/, plasma traps /2 , 3/, or in the Earth's radiation belts/4/. Still, sometimes such stochastic instability may also prove to be useful, e.g., for the heating of a plasma by a high-frequency field (cf., for example, /5/) or for realizing stochastic acceleration of charged particles (cf. /6/).

Within the framework of applications of the theory of stochasticity, the problems of main interest are the following:

1. To determine the conditions for stochastic instability of a motion.

2. To find out the statistical properties of stochastic motion, and primarily the rate of diffusion in the phase space of the system. These are the problems that will mainly be discussed in this paper.

In regard to the more fundamental problem of the relation between dynamical and statistical laws of physics, the most "acute" question is: can a strictly deterministic motion (of a dynamical system) be at the same time a random one (in the intuitive sense of the word)? Any detailed discussion of this (partly philosophic) question goes beyond the range of this paper. We would however like to make some brief remarks, since it seems to us important to help overcome the still existing psychological barrier that has been built up by the centuries-long tradition of opposing the deterministic to the random. The development of contemporary ergodic theory, and also the Kolomogorov algorithmic theory of complexity have recently reached such a level that one can give a quite definite answer to the question put above, namely, there exist dynamical (completely deterministic) systems, whose motion is in principle indistinguishable from a "random" motion, whatever the precise meaning that we attribute to the latter term. The validity of this "global" assertion is related to an interesting feature of the motion of a random dynamical system: in the language of symbolic dynamics, the set of its trajectories is <u>complete</u>, i.e., contains all the trajectories. We shall come back to this interesting question later in Sec. 4.2. A systematic and relatively readable (for physicists!) presentation of this circle of questions can be found, for example, in the reviews /7-9/. A fairly realistic example of such random dynamics is the elastic collision of balls in billiards, as was shown in the classic papers of Sinai (cf./9/). Thus now the example of a truly random process is not the traditional coin, nor the roulette wheel at Monte Carlo but rather the balls of "Sportloto".

In ergodic theory such systems are termed Bernouilli systems; we shall call them simply random. We recall that such systems can be very simple; in particular, two degrees of

freedom are sufficient for random dynamics (cf., for example, /9/).

We shall restrict ourselves to treating Hamiltonian (non-dissipative) systems. Such systems are sometimes called conservative, but this term seems to us misleading, since it gives rise to a confusion with the conservation of energy. Actually, the energy of a Hamiltonian system may not be conserved if the Hamiltonian depends explicitly on time. It is however important that even in this case the phase density is conserved (Liouville theorem). This significantly simplifies the statistical analysis of the stochastic motion in Hamiltonian systems, since they possess a simple and well known invariant measure.

Until recently it was supposed that including dissipation simplifies the dynamics, since it seemed that all the trajectories then approach either stable equilibrium or a stable periodic solution (limit cycle), which are the simplest examples of attractors in a dissipative system. We now know that this is not always the case. The first example of a nontrivial (stochastic) attractor was studied in the classic paper of Lorenz /10/. The nontriviality of the Lorenz attractor is related both to the fact that the motion is stochastic and that its geometric structure in the dynamical space of the system is highly singular (Cantor structure). It should be noted that such a structure of a stochastic attractor is typical for so-called Anosov (/11/) systems with dissipation, which are also Bernouilli systems. One of the types of stochastic attractor was investigated in detail by Smale /12/. So, the recent popularity of the term "strange attractor" is surprising (strange to whom?).

The fundamental difficulty of the theoretical analysis of a motion on a stochastic attractor is related to the fact that one must beforehand find an invariant measure, and it may prove to be highly singular in the original dynamical space. We note, however, that in the case of an Anosov system with a weak dissipation one can make approximate use of the simple invariant measure corresponding to the system <u>without</u> dissipa-

tion, for example, the phase density of a close Hamiltonian system. This follows from the fact that all Anosov systems are structurally stable /11/. But then another difficulty arises, connected with the fact that in many cases of practical interest one has to deal with more general systems, which, in particular, are not structurally stable. For such systems weak dissipation leads, as a rule, to a degeneration of the stochastic motion into a periodic one /13/. True, this degeneration disappears with increasing dissipation, but the invariant measure may then turn out to be quite complicated.

It should be mentioned that a dissipative system is not, strictly speaking, purely dynamical, at least if we are dealing with a real physical system. In fact, the dissipation describes (in a very simplified form) some stochastic process at the molecular level, and is therefore necessarily accompanied by fluctuations, i.e., by some random perturbation external to the dynamical system, which must also generally be taken into account. An excellent review on dissipative stochasticity can be found, for example in /14/.

Returning to Hamiltonian systems, we note that in certain simplest cases a complete and rigorous investigation of stochastic motion is possible on the basis of contemporary ergodic theory. It is, for example, the already mentioned Anosov systems. However, such a class of dynamical systems turns out to be rather limited from the application point of view. We mention that recently some advances have been made toward broadening the class of dynamical systems permitting a rigorous mathematical analysis /15/.

The main contents of the present survey are some recent results of the physical theory of stochasticity, which is based on models, various approximations and estimates, and supported by numerical simulation. We shall consider both stochasticity in classical mechanics (Sec. 2) and the behaviour of quantum systems that are stochastic in the classical limit (/33/). The importance of the latter problem from the physical point of view is that quantum mechanics gives a more exact description of real systems. So the question arises: to

what extent the unusual properties of the stochastic motion of a classical system persist in quantum theory? In principle, the answer to this question has been known for a long time (cf., for example, /34/, although it may appear somewhat unexpected to somebody: the stochasticity is not possible in quantum dynamics at all; more precisely, the time evolution of the wave function (or the density matrix) of a closed quantum system, bounded in phase space, is always almost-periodic, i.e., its frequency spectrum is discrete. In classical mechanics such a motion is regarded as just the opposite of stochasticity, for example, the motion of a completely integrable system.

In the paper /33/ we attempt to resolve this apparent contradiction by using the conception of transient, or temporal stochasticity. This approach is based on the introduction of different time scales, so that the different statistical properties of classical dynamics manifest themselves over certain finite time intervals of the quantum motion.

If we now apply this approach back to classical mechanics, we arrive at the curious conclusion that here also, an almost-periodic motion, for example, the motion of a completely integrable system, may, under definite conditions, imitate a stochastic process over a finite time interval. This sort of imitation has actually been known for a long time, and moreover is the basic method for studying the statistical properties of macrosystems (in particular, for deriving kinetic equations) in statistical physics. So far as we are aware, such an approach was first taken by Bogolyubov /16/, who investigated the statistical properties of a large number $N \to \infty$ of uncoupled linear oscillators, and was completed, in a sense, in paper /25/, where the Bernoulli property (randomness) of the classical ideal gas was rigorouly proved, also for $N \to \infty$.

A new formulation of the problem, arising naturally in classical mechanics from the analysis of quantum dynamics, is the following: for a fixed (and not necessarily large) number

of degrees of freedom of a classical completely integrable system, to find the conditions and the time scales for which imitation of stochastic motion occurs. In particular, such a problem arises in numerical simulation of classical stochasticity. Because of the discrete representation of numbers in a computer, all the trajectories of any dynamical system are then simply periodic. Bearing in mind the wide use of numerical simulation for investigating various dynamical systems, estimates of the accuracy and the limits of such simulation are extremely important. This circle of questions will be discussed in /33/.

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2. Nontrivial Problems of Classical Hamiltonian Dynamics

Nontriviality of a problem is here understood in two senses. On the one hand we have in mind the problems of stochastic motion, which still remains a quite unusual regime of classical dynamics. On the other hand, we refer to those (more complicated) problems which still defy rigorous mathematical analysis. Their solution (partial, of course) is based on a qualitative picture of the mechanism of stochastic motion, approximate estimates and numerical simulation. For Hamiltonian systems the most effective method for such semiempirical investigations is the analysis of nonlinear resonances and their interaction. The corresponding technique is described in detail, for example, in the surveys /17, 18/.

In this section we consider several specific problems related to the stochastic behaviour of simple systems in classical mechanics.

2.1. Border of Stochasticity

The determination of the border of stochastic instability of nonlinear oscillations is one of the fundamental problems in the theory of stochasticity. Here we must distinguish between the stochasticity border in the space of system parameters, i.e., the critical values of parameters such as the perturbations strength, and the border of stochasticity in the phase space of the system, including the critical value of the energy for a closed system. The latter problem is especially complicated. In fact, the structure of the phase space of nonlinear oscillations is usually characterized by an extremely complex interchange of stable and stochastic components of the motion. An example of such a structure is shown in Fig. 1 and will be discussed in more detail later. That structure has been called a divided (into stable and stochastic components) phase space.

The complexity of the structure of a divided phase space is determined mainly by two factors. First, this structure is hierarchic, i.e., the interchange of stable and stochastic regions takes place over smaller and smaller spatial scales. Secondly, the border between the stable and stochastic components is itself a stochastic surface. All this makes a rigorous analysis of such dynamical systems extremely difficult. Moreover, even the formulation of the problem on the stochasticity border raises some difficulties.

However, there are certain special systems for which one can at least formulate the problem clearly. An example is the so-called standard mapping

$$\bar{p} = p + K \cdot \sin x \qquad (2.1.1)$$
$$\bar{x} = x + \bar{p} = x + p + K \cdot \sin x$$

Many specific problems in the theory of nonlinear oscillations with two degrees of freedom reduce to this mapping (cf., for example, /18/). Here, however, this mapping is of interest

to us from a different point of view. The important thing for us is that the phase space of this model (x,p) is periodic not only in x, but also in p (with period 2π). Thus there exists a critical value of the single parameter of this mapping $K=K_{cr}$, above which the motion becomes unbounded*. The trivial case of $K_{cr}=0$ is excluded by the KAM theory (Kolmogorov-Arnold-Moser), cf., for example, /19/).

Numerical experiments with the standard mapping gave the result /18/:

$$K_{cr} \approx 0.989 \qquad (2.1.2)$$

In order of magnitude this result could be gotten from a simple criterion of nonlinear resonance overlap (cf., for example, /18/). For this purpose the mappings (2.1.1) should be replaced by the equivalent continuous system with the Hamiltonian

$$H(x,p,t)=\frac{p^2}{2}+K\sum_{n=-\infty}^{\infty}\cos(x-2\pi nt) \qquad (2.1.3)$$

If we take account only of the resonances in first approximation $\dot{x}\approx p = 2\pi n$, we have

$$K_{cr}^{(1)}=\frac{\pi^2}{4}\approx 2.5 \qquad (2.1.4)$$

More complicated calculations involving resonances of higher approximations, and also the stochastic layer of resonances, enable one /18/ to improve the agreement with the numerical result (2.1.2): $K_{cr}\approx 1.1$.

An entirely different approach to the solution of the problem on the stochasticity border was recently developed by Greene /20/. His approach is based on the study of trajectory stability at the center of resonances. Generally, this method always over-estimates K_{cr} , since the unbounded motion is

* For $K>K_{cr}$ the motion is stochastic /18/. Actually, even for $K<K_{cr}$ there are stochastic regions of complex structure bounded in p. Therefore the condition $K=K_{cr}$ should be understood as the border of global, or connected, stochasticity.

possible over a set of touching (intersecting) separatrices, while the resonance centers are still stable. Thus, for example, the fixed points of the mapping (2.1.1) $x=\pi$; $p=2\pi n$, i.e. the trajectories with period $T=1$ at the resonance centers, remain stable up to $K=K_1=4$, which exceeds even the value (2.1.4). However, already $K_2=2$ ($T=2$); $K_3=1.52$; $K_4=1.24$ etc. /21/. Thus Greene's hypothesis that, as $T\to\infty$, $K_T\to K_{cr}$ seems to be plausible. His result for the standard mapping /20/:

$$K_{cr}=0.971635 \qquad (2.1.5)$$

is noticeably less than the value (2.1.2). In connection with this discrepancy, we have carried out a new, more careful processing of the numerical data given in /18/. But the new result $K_{cr}\approx 1.00$ increased the discrepancy even more.

One of the reasons for this discrepancy may be the following. The investigation of the stability of periodic trajectories for large T can be carried out only numerically. One can therefore not run through all the periodic trajectories, and so one has to assume the additional hypothesis that the most stable periodic trajectories at resonance centers are those with the unperturbed frequency ($K=0$)

$$\frac{\omega}{2\pi}=\frac{M}{T}\longrightarrow g=\frac{1}{1+\frac{1}{1+\dots}}=\frac{\sqrt{5}-1}{2}\approx 0.618 \qquad (2.1.6)$$

where M, T are integers, T being the full period of the trajectory before it closes. Greene calls the number g "the most irrational number" on the grounds that it is the most poorly approximated by rationals, as one sees from the representation of g by a continued fraction (2.1.6). From the view point resonance overlap, however, this last hypothesis of Greene does not seem convincing, since for the overlap not only the spacing between resonances but also their width, which depends on the denominator of the rational $\frac{\omega}{2\pi}=M/T$, is of importance.

For a more detailed investigation of this question we

chose a somewhat different model, given by the mapping /22/ :

$$\bar{p} = p + \frac{K \cdot \sin x}{(1 - a\cos x)^2} \qquad (2.1.7)$$

$$\bar{x} = x + \bar{p}$$

Unlike the standard mapping, the perturbation in this system already contains many harmonics in phase x in first approximation (for $a \longrightarrow 1$):

$$\frac{\sin x}{(1 - a\cos x)^2} \approx \frac{2}{\varepsilon} \sum_{m=1}^{\infty} e^{-\varepsilon m} \cos mx \qquad (2.1.8)$$

where $\varepsilon \approx \sqrt{1-a^2}$. Thus, already in the first approximation there "operate" not only integer resonances $p = 2\pi \cdot n$, but also fractional ones, $p = 2\pi \cdot n/m$ (with $m \leq 1/\varepsilon$), as one sees from the Hamiltonian of the equivalent continuous system /22/:

$$H(x,p,t) \approx \frac{p^2}{2} + \frac{2K}{\varepsilon} \sum_{m=1}^{\infty} e^{-\varepsilon m} \sum_{n=-\infty}^{\infty} \cos(mx - 2\pi n t) \qquad (2.1.9)$$

First, let us consider each of the resonances individually in the "pendulum" approximation /18/. The halfwidth of the resonance separatrix is equal to

$$(\Delta p)_m = 2\sqrt{\frac{2K}{\varepsilon} \cdot e^{-\varepsilon m/2}} \qquad (2.1.10)$$

while the simple sum over all resonances in the interval

$0 \leq \frac{\omega}{2\pi} \leq 1$ is

$$S(K,\varepsilon) = 2\sum_{m=1}^{\infty}\sum_{n=1}^{m} (\Delta p)_m = 4\sqrt{\frac{2K}{\varepsilon}} \cdot \sum_{m=1}^{\infty} m e^{-\frac{\varepsilon m}{2}} \qquad (2.1.11)$$

From the condition $S(K,\varepsilon) = 2\pi$ one can obtain a rough estimate of K_{cr} in the form

$$K_{cr} \approx \frac{\pi^2}{128} \varepsilon^5 \qquad (2.1.12)$$

This estimate can be improved as follows /22/. We should take the sum in (2.1.11) not over all m, n, but only over those m_0, n_0 that form irreducible fractions n_0/m_0. The number of these irreducible resonances is on the average 2/3 of all the resonances. The remaining resonances split up into groups, each corresponding to one irreducible resonance: $m = m_0 l$; $n = n_0 l$, $l = 1,2,\ldots$. For $\varepsilon \ll 1$, the motion in the neighborhood of each irreducible resonance is determined by all the perturbation terms in the Hamiltonian (2.1.9) belonging to this group, i.e., by the sum over l for given m_0, n_0. As a result, the pendulum approximation is violated, and the resonance width for low harmonics ($\varepsilon m_0 \lesssim 1$) increases significantly.

However, it is most important to take account of nonuniformity in resonance spacing. In fact, the total number of resonances with $m_0 < m_1$, per unit interval of the quantity $\omega/2\pi$, is $(2/3)m_0^2$, so that the average spacing between resonances is $\langle \Delta\omega \rangle \approx 3\pi/m_0^2$. But near the integer resonances (for example, $n_0/m_0 = 1$) the gaps are formed equal to $(\Delta\omega)_1 = 2\pi/m_0$. So it is clear that K_{cr} is actually determined not by the average overlap of resonances (2.1.11), but by the "covering" of these gaps. This significantly increases the critical perturbation for this model /22/:

$$K_{cr} \approx \frac{\pi^2 \varepsilon^4}{\left(L + 4/L\right)^2} \; ; \qquad L = \ln \frac{128}{9\varepsilon} \qquad (2.1.13)$$

Some results of a numerical simulation of the system

(2.1.7) were described in /23/. The simulation was done as follows. Suppose that we are interested in the dynamics of the system

$$\bar{p} = p + \mathcal{K} \cdot f(x)$$
$$\bar{x} = x + \bar{p} \qquad (2.1.14)$$

where f(x) is a periodic function of period 2π. We consider the auxiliary mapping

$$\bar{y} = y + \frac{f(x)}{\lambda}$$
$$\bar{x} = x + \lambda \cdot F(\bar{y}) \qquad (2.1.15)$$

where λ is some constant; f(x) is the same function as in (2.1.14), while F(y) is a new function that we will choose later.

We linearize the second equation of this last mapping around one of the resonant values $y = y_r$, where

$$\lambda \cdot F(y_r) = 2\pi r \; ; \quad r - \text{integer} \qquad (2.1.16)$$

Setting $p = \lambda \cdot F'(y_r)\,(y - y_r)$, we get back the original mapping (2.1.14) with the parameter

$$\mathcal{K} = F'(y_r) \qquad (2.1.17)$$

Now suppose that the function $|F'(y)|$ falls off monotonically

with $|y|$; then the condition $|\mathcal{K}| > \mathcal{K}_{cr}$ determines the stochastic layer $|y| < y_\ell$ on the phase plane of the auxiliary mapping (2.1.15). Also, conversely, if by computation we find the maximum value $y_m \simeq y_\ell$ $(|y| < y_m)$ which diffusion in the system (2.1.15) reaches, then from (2.1.17) we can approximately estimate the critical perturbation for the mapping (2.1.14): $\mathcal{K}_{cr} \simeq |F'(y_m)|$. More precisely, we can assert that

$$\mathcal{K}_{cr} \le |F'(y_\ell)| \qquad (2.1.18)$$

where y_ℓ corresponds to the last integer resonance crossed by the trajectory $(\lambda F(y_\ell) = 2\pi r)$. If the original mapping (2.1.14) is symmetrical in respect to $p = \pi$, i.e., in respect to the half-integral resonance, as is the case, for example, in the standard mapping (f(x) = sin x), and also in the mapping (2.1.7) under consideration, then y_ℓ in (2.1.18) corresponds to the last half-integer resonance $(\lambda F(y_\ell) = \pi r)$.

As the auxiliary function we chose $F(y) = \ln|y|$. We note that for f(x) = sin x the mapping (2.1.15) describes the motion in the stochastic layer of the nonlinear resonance in the pendulum approximation /18/. Numerical experiments with the mapping (2.1.15) and f(x) from (2.1.7), for a number of iterations $t \sim 10^6$, showed a good agreement of $\mathcal{K}_{cr}$, determined by this method, with the estimate (2.1.13) in the range $0.03 < \sigma < 0.5$ /23/. However, as the motion time increases further, the critical value $\mathcal{K}_{cr}$ slowly "creeps" downward. Thus, for $\sigma = 0.142$ and $t = 2 \times 10^8$, the upper limit of $\mathcal{K}_{cr}$ (2.1.18) dropped to 4.3×10^{-5}, whereas the estimate (2.1.13) gives 1.3×10^{-4}. The fact that this last value, obtained from the overlap of the first approximation resonances, is too high, would be rather natural (cf. the critical value (2.1.4) for the standard mapping). The interesting point is another one in that the ratio (R) of the theoretical and numerical values of $\mathcal{K}_{cr}$ ($R \simeq 3$) differs only slightly from the corresponding ratio for the standard mapping ($R \simeq 2.5$, cf. (2.1.4)). We may add that in the case of just two overlapping resonances $R \simeq 2.3$ (from the data of /18/) and $R \simeq 2$ (from the data of /24/).

In the case considered here this ratio can be interpreted differently, as a reduction in the effective value $\sigma \to \sigma^*$ due to the effects of higher approximations. From the value $R \approx 3$ above and the estimate (2.1.13) we have $\sigma^*/\sigma \approx 0.76$.

An indirect confirmation of the decisive role of the gaps near the integer resonances, on which (2.1.13) is based, is the histogram in Fig. 2. Here the fractional part $r = \{\lambda F(y_m)/2\pi\}$ characterizes the location of the system within the period of the resonance structure corresponding to the maximum value $|y| = y_m$ reached at the given time. The ordinate gives the number of iterations T during which y_m remained within each of 20 intervals covering a period of the resonance structure. The horizontal arrow in the upper part of the Figure shows the direction of diffusion. It is clear that the most difficult section to get through is that immediately before the integer resonance $r = 1$. The two vertical arrows show the "most irrational" (à la Greene) values $r = g_n V$, (2.1.6) and $r = 1-g_n$ in the neighborhood of which no significant delay of the diffusion is observed.

Although the agreement obtained between the last computational results and the estimate (2.1.13), based on the relatively simple overlap criterion appears satisfactory, this question requires further study, in particular the performance of further numerical experiments over a much longer time of the motion. The latter is connected with a very slow diffusion in the region of a gap. This may lead to a further reduction in the numerical value of K_{cr}. In addition it is not excluded that the quite high accuracy of numerical computation (56 binary digits in the mantissa), may still be insufficient, again because of the low diffusion rate. Then, the "cycling" may occur, i.e., the change to a periodic trajectory because of the discrete representation of numbers in the computer (cf. /33/).

2.2. The Modulation Diffusion

As mentioned earlier, the structure of the stochastic components in phase space is usually very complicated. In this

section we consider a fairly simple and little-known case, where the stochastic components have the form of relatively narrow layers, along which the system can diffuse over considerable distances. Such a structure arises, for example, as a result of weak low-frequency modulation in the system.

Let us consider the case of external modulation. As an example, we take a system with the Hamiltonian

$$H = \frac{p_1^2+p_2^2}{2} + \frac{x_1^4+x_2^4}{4} - \mu x_1 x_2 - \varepsilon x_1 f(t) \qquad (2.2.1)$$

This Hamiltonian describes the behaviour of two coupled nonlinear oscillators drived by the force f(t). The system (2.2.1) was studied earlier from the view point of the so-called Arnold diffusion which is a weak universal instability occurring in many-dimensional oscillator systems (cf., for example, /26, 18, 27/ and the end of Sec. 2.3). We choose f(t), as in /27/, in the form

$$f(t) = \sum_{m=1}^{\infty} f_m \cos \psi_m(t); \qquad f_m = \frac{2e^{-\sigma m}}{\sigma} \qquad (2.2.2)$$

Suppose that the phase of the driving perturbation f(t) is modulated at a frequency Ω much smaller than the unperturbed frequencies of the system: $m\sqrt{t} + \lambda \sin \Omega t$. This type of modulation often occurs in practical problems (cf., for example, /1/). For small values of the parameters $\mu, \varepsilon \ll 1$, which characterize the strengths of the coupling and the driving perturbation, it is convenient to change to I_i, θ_i the unperturbed action-angle variables ($\mu, \varepsilon = 0$).
Then the unperturbed Hamiltonian H_0 has the form:

$$H_0 = C_0 (I_1^{4/3} + I_2^{4/3}); \qquad C_0 \approx 0.87 \qquad (2.2.3)$$

In this approximation the actions I_1 and I_2 are constants, while we have for the phase angles θ_1 and θ_2 :

$$\dot{\theta}_1 = \omega_1(I_1); \quad \dot{\theta}_2 = \omega_2(I_2) \qquad (2.2.4)$$

where ω_1, ω_2 are the unperturbed frequencies of the system, depending on I_1, I_2 because of the nonlinearity of the oscillations. Despite the strong nonlinearity, the oscillations are close to harmonic: $x_1(t) \approx a_1 \cdot \cos\theta_1(t)$, while their frequencies are proportional to the amplitudes: $\omega_1 \approx 0.85\, a_1$.

The parameter λ in (2.2.4) determines the modulation factor. In the absence of modulation the system has driving resonances $\omega_1(I_1) = m\nu$ and a coupling resonance $\omega_1(I_1) = \omega_2(I_2)$ (cf. Fig. 3). If the perturbation is large enough, neighbouring nonlinear resonances overlap, and strong stochasticity appears, leading to a rapid rise of the total energy.

Here we would like to call attention to the possibility of the so-called Modulation diffusion /28/ which is similar in a certain sense to the Arnold diffusion. The cause for the appearance of this diffusion in the case of $\lambda \neq 0$ is the overlap of modulation resonances around each of driving resonances.

Let us find the condition for overlap of modulation resonances near a single driving resonance of harmonic m. To do this we write the Hamiltonian of the motion in x without coupling ($\mu = 0$) and near a particular modulation resonance ($\omega_1 = m\nu + n\Omega$)

$$H_M = C_0 I_1^{4/3} - \frac{\varepsilon}{2} f_m a_1(I_1) J_n(\lambda) \cos(\theta_1 - m\nu t - n\Omega t) \qquad (2.2.5)$$

where $J_n(\lambda)$ is a Bessel function. Estimating the resonance width $\Delta\omega_1$ and comparing it with the spacing between resonances Ω we obtain the resonance overlap parameter:

$$s = \frac{\Delta\omega_1}{\Omega} \sim \frac{4\omega_1 e^{-sm/2}}{a_1^{3/2}\Omega} \cdot \left(\frac{\varepsilon^2}{\lambda\varepsilon^2}\right)^{1/4} \qquad (2.2.6)$$

where we have used the estimate $J_n(\lambda) \sim 1/\sqrt{\lambda\lambda}$ for $\lambda \gg 1$. For $K_M \approx 2.5 \cdot s^2 > 1$, /18/, a modulation stochastic layer is formed with width $\approx 2\lambda\Omega$. We have here introduced a stability parameter K_M, analogous to the parameter K of the standard mapping (2.1.1).

Let us take $\lambda\Omega \ll \nu$, so that neighbouring modulation layers do not touch each other (cf. Fig. 3), and any significant diffusion is possible only along the layer, with increase in the energy of the other degree of freedom (I_2). The rate of change in I_2 is (cf. (2.2.1)):

$$\dot{I}_2 \approx -\mu a_1 \cos\theta_1 \cdot a_2 \cos\theta_2 \approx \qquad (2.2.7)$$
$$\approx \frac{\mu a_1 a_2}{2}\left[\sin(\theta_1 - \omega_2 t) - \sin(\theta_1 + \omega_2 t)\right]$$

where we have set, approximately, $\theta_2 \approx \omega_2 t$. For sufficiently small coupling $\mu \ll \varepsilon$ the dependence $\theta_1(t)$ differs little from the case when $\mu = 0$, and is determined by the motion of the system in the stochastic layer according to the Hamiltonian:

$$H_{x_1} = \frac{\omega_1^2 \Delta^2}{2a_1^4} - \frac{\varepsilon a_1 f_m}{2}\cos(\theta_1 - m\nu t - \lambda \sin\Omega t) \qquad (2.2.8)$$

Here the new canonical momentum $\Delta = I_1 - I_r$ describes the deviation of the action I_1 from the resonant value I_r, which satisfies the condition: $\omega_1(I_r) = m\nu$. If the system is within the modulation stochastic layer, the change in I_2, according to (2.2.7), is of diffusion nature and is determined by the correlation properties of the phase $\theta_1(t)$. The rate of this diffusion will be found in the next Section.

We note that the diffusion mechanism in I_2 can be regarded as a "pumping over the stochasticity" from one degree of

freedom to the other. Such a picture of the diffusion in the stochastic layer was proposed in /29/, where a similar problem was considered. Actually, a similar mechanism for the "transfer" of stochasticity had been discussed briefly still earlier in /30/.

2.3. Statistical Properties of Stochastic Motion.

For the solution of concrete problems related to stochastic motion of dynamical systems, in addition to the border of stochasticity or, in the more general case, the (geometrical) structure of the stochastic component, one must know the statistical properties of the motion in the stochastic region. The most important of these is the diffusion rate in the stochastic component. In some cases this problem is trivial. Thus, for example, for the standard mapping (2.1.1), with $K \gg 1$, i.e., sufficiently deep within the stochastic region, the successive values of the phase x can be regarded to a good accuracy as random and independent /18/. It then immediately follows that the rate of diffusion in p is:

$$D_p \equiv \frac{\langle (\Delta p)^2 \rangle}{t} \approx \frac{K^2}{2} \qquad (2.3.1)$$

which agrees well with the results of numerical simulation /18/.

But the situation is not always so simple. For example, in the problem of modulation diffusion (Sec. 2.2) the rate of diffusion is determined by the square of the Fourier component modulus of a certain random continuous function (2.2.7). This means that, in the general case, we must know the Fourier spectrum of the random process, or the associated correlation function of the motion. Either of these, of course, depends on the structure of the stochastic component. For the problem of modulation diffusion the latter arises because of the overlap of several "parallel" resonances (Fig. 3).

To study the statistical properties of such a component

we consider the following simple model with the Hamiltonian:

$$H(x,p,t) = \frac{p^2}{2} + k\cos(x + \lambda \cdot \cos \Omega t) \qquad (2.3.2)$$

Here Ω is the modulation frequency, while λ determines the modulation factor (the effective number of lines in the multiplet); the total width of the modulation spectrum is known to be approximately $2\lambda\Omega$. The condition for formation of a stochastic layer is (cf. Sec. 2.2):

$$K_N \approx 2.5\, s^2 \sim 23 \cdot \frac{k}{\sqrt{\lambda}\,\Omega^2} > 1 \qquad (2.3.3)$$

where s is the overlap parameter for the modulation resonances. We shall assume that this condition is well satisfied, so that a quite uniform stochastic layer is formed. Then the dynamics of the system (2.3.2) is described by some random function x(t) and we must find its correlation properties.

We introduce the auxiliary dynamical variable z, whose change is determined by the equation:

$$\dot{z} = \varepsilon \cdot \sin(qx - \omega t) \qquad (2.3.4)$$

where ε, q, ω are constants. This equation has the same form as (2.2.7) (with q = 1) in the problem of modulation diffusion. Thus the latter corresponds to diffusion in z for the model under consideration. We have slightly generalized the model (2.3.2,4) by introducing a new parameter q which is relevant in the other cases of modulation diffusion.

We can draw some first preliminary qualitative conclusions about the dependence of the diffusion rate

$$D \equiv \frac{\overline{(\Delta z)^2}}{t}\ ;\quad (\Delta z)_t = \varepsilon \int_0^t \mathrm{Sin}\left(q x(t') - \omega t'\right) dt' \qquad (2.3.5)$$

on the perturbation frequency ω . In the interval $|\omega| \lesssim q\lambda\Omega$ the rate of diffusion is approximately constant and relatively high since within this interval the exact resonance $q\dot{x} = \omega$ does occur (cf. (2.3.5)) due to the diffusion in frequency $\dot{x} = p$ which fills up all the modulation stochastic layer $|p| \lesssim \lambda\Omega$. On the other hand, for $|\omega| \gg q\lambda\Omega$ the diffusion rate falls off exponentially since the function Sin(qx(t)) is analytic (cf. (2.3.2)), and the Fourier spectrum of any analytic function is known to be exponential as $|\omega| \to \infty$. Let

$$D \longrightarrow A \cdot \exp\left(-\frac{|\omega|}{\omega_i}\right);\quad \omega \gg \omega_i \qquad (2.3.6)$$

The most important problem is the determination of the scale of exponential falloff ω_i . Since we don't know any analytic method for solution of this problem, we went to a numerical simulation of the combined system (2.3.2), (2.3.4).

When $k \ll 1$ the differential equations of motion can be replaced approximately by the mapping:

$$\begin{aligned} \bar{p} &= p + k \cdot \mathrm{Sin}\left(x + \lambda \cdot \mathrm{Cos}\,\Omega t\right) \\ \bar{x} &= x + \bar{p} \\ \bar{z} &= z + \varepsilon \cdot \mathrm{Sin}\left(qx - \omega t\right) \end{aligned} \qquad (2.3.7)$$

where t now takes on only integer values and $|\omega| \ll 2\pi$. To reduce the "background" associated with the discreteness of this computational scheme, we used an averaging of $z(t)$ over the ten intervals into which the total time of computation was divided. This very efficient technique is described at length in /26, 18/.

We first consider the case of $q = 1$. Preliminary numerical experiments showed that the exponent scale is approximately proportional to the width of the stochastic layer. Figure 4 gives the dependence of $D(\omega)$ in normalized coordinates: $\log(D_R)$, $y = \omega/\Delta\omega$, where $D_R = \Delta\omega \cdot D/\varepsilon^2$ is the renormalized diffusion rate, and $\Delta\omega$ is the actual halfwidth of the layer, which usually somewhat exceeds $\lambda\Omega$. In the Figure are collected data for various values of the model parameters in the intervals:

$$10^{-5} \le k \le 3\cdot 10^{-3};\quad 0.014 \le \Delta\omega \le 0.36$$

$$10^{-3} \le \Omega \le 2\cdot 10^{-2};\quad 3.8 \le S \le 30$$

$$10 \le \lambda \le 100$$

Disregarding the quite large fluctuations, we note the following features of the function $D(\omega)$:

1. a plateau $|\omega| \lesssim \Delta\omega$ with maximum diffusion rate;
2. an exponential "tail" for $\omega > \Delta\omega$ with $\omega_i = \Delta\omega/\delta$,

where the average value of δ for all the data is $\langle\delta\rangle = 6.21 \pm 0.17 \approx 2\%$;

3. a sharp drop in the diffusion rate at $|\omega| \approx \Delta\omega$;
4. an irregular dependence of $D(\omega)$ on the "tail".

The maximum diffusion rate on the plateau can be found from the condition of normalization of the Fourier transform for the function sin x(t) (Parseval equality):

$$\int_{-\infty}^{\infty} D_e(y)\,dy = \pi \qquad (2.3.8)$$

Whence the diffusion rate on the plateau ($|\omega| < \Delta\omega$) is:

$$D^{(pl)} \approx \frac{\pi\varepsilon^2}{2\lambda\Omega} \qquad (2.3.9)$$

where we have neglected the contribution of the "tail" to the integral (2.3.8) in view of the sharp drop of $D(\omega)$ for $|\omega| > \Delta\omega$. From the data in Fig. 4, the average value is $\langle D_R^{(pl)} \rangle = 1.21 \pm 0.10$.

The diffusion rate on the "tail" is

$$D_R \approx B \cdot \exp\left(-\frac{2\pi|\omega|}{\Delta\omega}\right); \quad |\omega| > \Delta\omega \qquad (2.3.10)$$

where the average value of the quantity B over all the data of Fig. 4 is $\langle B \rangle = 16$. Because of the large fluctuations, the individual values of B differ by more than an order of magnitude. It is therefore difficult to decide from the available data whether B depends on the system parameters, although the numerical results apparently indicate such a dependence, mainly on the parameter λ. We shall return to this question later.

The frequency modulation in the model (2.3.2) is a regular external perturbation, and the question arises to what extent the empirical diffusion rate that we have found depends on this regularity. To clarify this question the frequency modulation was replaced by a "random" perturbation in the form of a set of harmonics with random phases. The results of these numerical experiments show that the exponent scale then changed insignificantly ($\langle d \rangle = 5.74 \pm 0.15$), while the value of B decreased substantially ($\langle B \rangle = 3.4$). Such a change in B apparently provides additional indirect evidence for a dependence of that quantity on the model parameters.

However, the regularity of the frequency modulation manifests itself in a different way - in a dependence on the system parameters of the so-called KS-entropy h (the Krylov-Kolmogorov-Sinai entropy), which determines the average rate of exponential divergence for close trajectories of the system. Namely, it turned out that the numerical values of h are very well described by the simple relation

$$h = \frac{\Omega}{2\pi} \ln \frac{\mathcal{K}_M}{2} \approx \frac{\Omega}{\pi} \ln s \qquad (2.3.11)$$

where $\mathcal{K}_M$ is given by (2.3.3). This relation coincides exactly with the expression for the KS-entropy of the standard mapping (upon the replacement of $\mathcal{K}$ by $\mathcal{K}_M$ cf., for example, /18/).

This coincidence becomes understandable if we transform the mapping (2.3.7) for our model. Let us introduce a new phase $u = x + \lambda \cdot \cos\Omega t$, then the first of the two equations in (2.3.7) may be rewritten in the form ($\Omega \ll 2\pi$):

$$\bar{p} = p + k \cdot \sin u$$
$$\bar{u} \simeq u + \bar{p} - \lambda\Omega \cdot \sin\Omega t \qquad (2.3.12)$$

This mapping models the periodic crossing the resonance $p = \lambda\Omega \sin\Omega t$, and that process can actually be described using the standard mapping, at least in a certain range of the parameters /31/.

For sufficiently slow resonance crossing ($\Omega^2 \ll k/\lambda$), within the modulation stochastic layer there is a stable domain, but not a fixed one, as for example in Fig. 1, but rather an oscillating domain. In particular, its center moves according to the law:

$$p_0 \approx \lambda\Omega \cdot \sin\Omega t\,; \quad x_0 \approx \bar{x} - \lambda \cdot \cos\Omega t \qquad (2.3.13)$$

We found such a stable region, indeed. It should be noted that its relative phase area, which is readily obtained from (2.3.12),

$$\sigma \approx \frac{4\sqrt{k}}{\pi\lambda\Omega} \approx \frac{s}{(\pi\lambda)^{3/4}} \qquad (2.3.14)$$

is usually small ($s \ll \lambda$), so that, for most initial conditions within the modulation layer, the motion is stochastic.

We note that the phenomenon of slow resonance crossing gives an interesting example of a process for which overlapping of resonances can, under certain initial conditions, lead to a regular rather than stochastic motion.

For a "random" perturbation the expression (2.3.11) no longer works. Instead, a "typical" estimate for $\underline{h}$, obtained in /31/ on the basis of the notion of renormalized resonances, holds approximately

$$h_\tau \approx 0.066 \cdot \Omega s^{4/3} \qquad (2.3.15)$$

The numerical factor in this expression is taken from /32/, where it was obtained by solving the corresponding kinetic equation in the quasilinear approximation. The average ratio

of the computed values of $\underline{h}$ to the theoretical values (2.3.15) is $\langle h/h_\tau \rangle = 0.89 \pm 0.05$.

Renormalization of resonances is carried out as follows /31/. The renormalized resonance width ($\Delta\omega'$) is determined by all the perturbation harmonics that fall within the renormalized width, where, because of the randomness of different harmonic phases, the squared amplitudes, i.e., the fourth powers of the nonrenormalized widths of the individual resonances ($\Delta\omega$), are summed up. Then

$$(\Delta\omega')^4 \sim (\Delta\omega)^4 \frac{(\Delta\omega')}{\Omega}$$

and

$$h \sim \Delta\omega' \sim \frac{(\Delta\omega)^{4/3}}{\Omega^{4/3}} \sim \Omega s^{4/3}$$

since $s \sim (\Delta\omega/\Omega)$.

The KS-entropy also may be considered as one of the important statistical properties of stochastic motion. Although it is not directly related to the diffusion rate, as is clear from the examples just considered, the local instability of motion, which it characterizes, is a decisive mechanism for the appearance of statistical properties of the dynamical motion. On the other hand, it is interesting to note that there is the inverse dependence as well: the random perturbation (with continuous spectrum) depending on coordinates does always result in an exponential local instability of the motion, as directly follows from the derivation of the relation (2.3.15) in /32/.

A general description of the correlation spectrum $D_R(y)$ can be given on the basis of the following notions. We shall

assume that the spectrum is determined by two different mechanisms:

1. a local law of stochastic motion with spectrum $d_R(y)$;

2. a shift of this spectrum by an amount $q \cdot p$ because of diffusion across the stochastic layer.

The resulting spectrum is then determined by the convolution

$$D_R(y) = \frac{1}{2q} \int_{-q}^{q} d_R(y - y_1)\, dy_1 \qquad (2.3.16)$$

The local spectrum can be fitted using the relation

$$d_R(y) = \frac{\pi}{2} \alpha \frac{\sin\beta}{\beta} \frac{1}{\cos\beta + \cosh(\alpha y)} \qquad (2.3.17)$$

where β is an additional parameter, related to the quantity B in (2.3.10). Then the total spectrum, including the plateau, the drop and the exponential "tail", may be described by

$$D_R(y) = \frac{\pi}{2qp}\left[\arctan\left(\tanh\left(\alpha \frac{q-y}{2}\right)\tan\frac{\beta}{2}\right) + \arctan\left(\tanh\left(\alpha \frac{q+y}{2}\right)\tan\frac{\beta}{2}\right)\right] \qquad (2.3.18)$$

Performing the Fourier transform, we get the correlation function of the process $\sin x(t)$ in the form

$$R(\tau) = \frac{\pi}{p} \frac{\sin(q\Delta\omega\cdot\tau)}{q\Delta\omega\cdot\tau} \frac{\sinh\left(\frac{p\Delta\omega\tau}{\alpha}\right)}{\sinh\left(\frac{\pi\Delta\omega\cdot\tau}{\alpha}\right)} \xrightarrow[\tau\to\infty]{} e^{-\frac{\pi\Delta\omega\cdot\tau}{\alpha}} \qquad (2.3.19)$$

where $\xi = \pi - \beta$. Thus asymptotically as $\tau \to \infty$ the correlation decays exponentially, although initially, when the correlation is still large, it decays only as $1/\tau$.

It follows from (2.3.19) that the exponent is determined by two dimensionless parameters ξ, α. Since α is a constant ($\alpha \approx 2\pi$) the dependence on the system parameters is determined by ξ alone, which is related to the drop of the spectrum, or to the parameter B:

$$B = \pi \frac{\sin\xi}{\pi - \xi} \cdot \frac{\sinh(\alpha q)}{q} \qquad (2.3.20)$$

One may assume that the rate of exponential decay of correlations in (2.3.19) is proportional to the KS-entropy h:

$$R(\tau) \sim \exp(-ch\tau), \quad \tau \to \infty \qquad (2.3.21)$$

where c is some constant independent of the system parameters. We then obtain a relation between ξ, h, c:

$$\xi = c \frac{\alpha h}{\Delta\omega} \qquad (2.3.22)$$

For modulation diffusion,

$$\xi = 2c \frac{\ln s}{\lambda} \qquad (2.3.23)$$

Our numerical experiments give the following average value for the constant: $c = 0.47 \pm 0.21$. The large error is related to big fluctuations (cf. Fig. 4).

For the "random" perturbation the expression for ξ takes the form:

$$\xi \approx 0.13 \cdot c \frac{s^{4/3}}{\lambda} \qquad (2.3.24)$$

The average measured value is $c = 0.13 \pm 0.028$.

For $q \neq 1$ the correlation spectrum $D(\omega)$ suffers the following changes:

1. as expected, the size of the plateau becomes equal to $q\Delta\omega$ (for $q > \xi/\lambda$), and the diffusion rate on the plateau is $D(\omega) \approx (\pi\varepsilon^2)/(2q\lambda\Omega)$;

2. in the exponential dependence on the "tail" only the coefficient B changes (cf. (2.3.20)).

These changes are well described by the relations (2.3.16-18). Thus, for $q < \xi/\lambda$ the size of the plateau is independent of q, in agreement with (2.3.18).

For the diffusion rate on the "tail" we finally get ($\xi \ll 1$, $q > 1/2\pi$):

$$D(\omega) \approx \frac{\varepsilon^2 \ln s}{2\lambda^2 \Omega q} \cdot \exp\left(-2\pi\left(\frac{\omega}{\Delta\omega} - q\right)\right) \qquad (2.3.25)$$

in the case of frequency modulation (2.3.2), and

$$D(\omega) \approx 8.5 \cdot 10^{-3} \frac{\varepsilon^2 s^{4/3}}{\lambda^2 \Omega q} \exp\left(-2\pi\left(\frac{\omega}{\Delta\omega} - q\right)\right) \qquad (2.3.26)$$

for the "random" perturbation.

Returning to the problem of modulation diffusion (Sec. 2.2) we can now give an explicit expression for its rate. Comparing (2.3.2) with (2.2.8), we find that $x = \theta_1 - m\nu \cdot t$. It then follows from (2.2.7) and (2.3.4) that $q = 1$ and $\omega = \omega_2 - m\nu$ for the first term in (2.2.7) and $\omega = \omega_2 + m\nu$ for the second term. If $\lambda\Omega \ll \nu$ then, since the rate of diffusion depends exponentially on ω (2.3.25), the second term in (2.2.7) can be neglected. Outside the plateau ($|\omega| > \lambda\Omega$) the diffusion rate in I_2 is:

$$D_{I_2} \approx \frac{\mu^2 a_1^2 a_2^2 \ln s}{8\lambda^2 \Omega} \cdot \exp\left\{-2\pi\left(\frac{|\omega_2 - m\nu|}{\lambda\Omega} - 1\right)\right\} \qquad (2.3.27)$$

Besides external frequency modulation some "self-modulation" is possible and always present actually due to the the phase oscillations of the unperturbed variables (including the unperturbed frequencies) under the action of a resonant perturbation. Since the frequency of phase oscillations $\Omega_\varphi \to 0$ near the separatrix of a nonlinear resonance, the corresponding modulation resonances do always overlap, forming a stochastic layer along the separatrix. The motion of the system along this layer has been called Arnold diffusion, and was discussed in detail, for example, in /18/. We note that the rate dependence of the Arnold diffusion on the perturbation

frequency:

$$D_A(\omega) \propto e^{-\frac{\pi\omega}{\Omega_\phi}}$$

is similar to expression (2.3.10) for modulation diffusion. This similarity becomes still more complete if we note that the halfwidth of the Fourier spectrum of the motion in the stochastic layer near the separatrix (the quantity analogous to $\Delta\omega$ for modulation diffusion) is equal to $2\Omega_\phi$. It then follows, in particular, that the diffusion rates in both cases are comparable in magnitude (for $\Delta\omega \sim \Omega_\phi$).

There are, however, two fundamental differences between the two processes:

1. Arnold diffusion is universal in the sense that it persists for arbitrarily weak perturbation, whereas modulatinal diffusion has a threshold, determined by the overlapping of resonances in the multiplet;

2. the width of the modulation stochastic layer generally greatly exceeds the width of the layer in which Arnold diffusion occurs.

Thus, if modulation stochastic layers are formed, diffusion along them is on the average much faster than Arnold diffusion.

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Figure Captions.

Fig. 1. Phase plane of the mapping (2.1.7):
$a = 0.95$; $\epsilon = 0.323$;
$K = 6.5 \times 10^{-3} > K_{cr} \approx 4.6 \times 10^{-3}$ (2.1.13)
Stochastic component shaded.

Fig. 2. Histogram of "passabilili" for various sections of the resonance set: $r = \{\lambda \cdot F(y_m)/2\pi\}$, T is the sojourn time (number of iterations) of y_m in the given interval of the period of the resonance structure.

Fig. 3. Pattern of the first approximation resonances for the system (2.2.1): ω_1, ω_2 are the oscillator frequencies; ν is the mean frequency of the driving force; Ω is the frequency of the modulation causing the formation of the multiplets.

Fig. 4. Frequency dependence of the rate of modulation diffusion (for the notation, see the text; logs are base 10). The points are numerical values of the diffusion rate in $\underline{z}$ for the system (2.3.2), (2.3.4) with $q = 1$. The solid curve is drawn from formula (2.3.18) with the average empirical values $\langle\lambda\rangle = 6.21$, $\langle\beta\rangle = \pi - 0.065$.

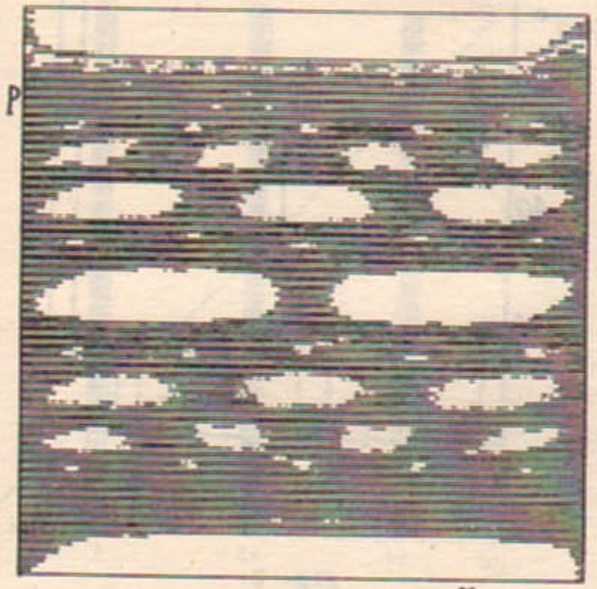

Fig. 1.

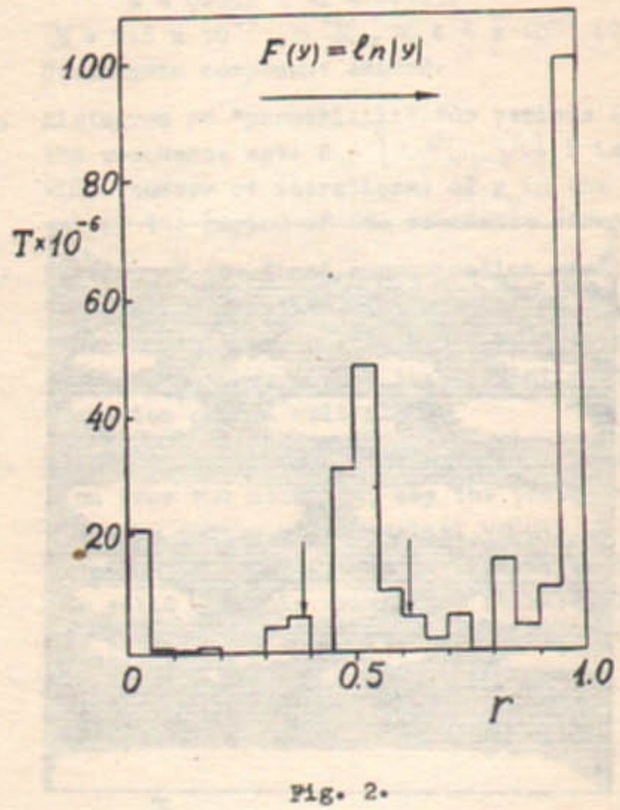

Fig. 2.

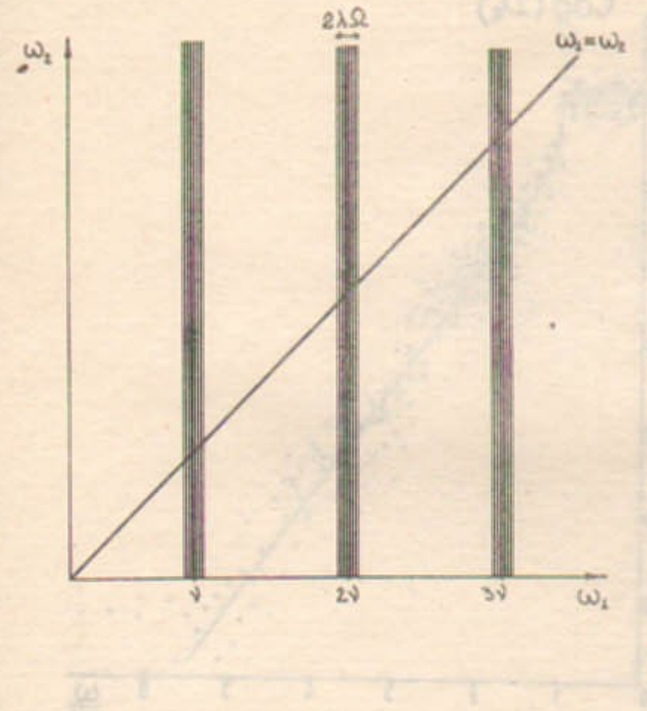

Fig. 3.

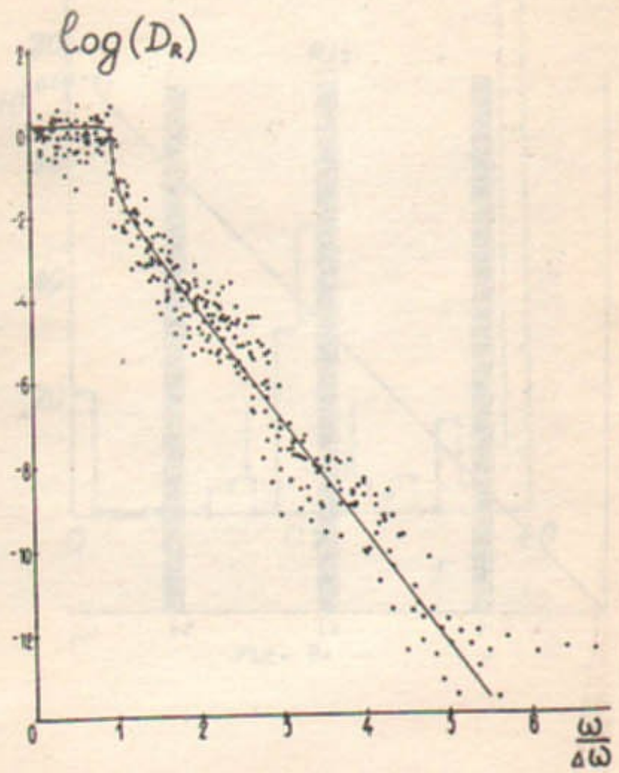

Fig. 4.