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STRUCTURES AND THEIR EVOLUTION IN TURBULENT SHEAR LAYER

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1. Introduction

In the experiments based on the turbulent flow visualization methods the coherent structures has been observed. These structures look like regions of relatively ordered motion whose arrangement in space and the emergence and disappearance in time are random. The main goal of a variety of theoretical and experimental studies on turbulence is to determine the coherent structures and their space and time distributions. The reader is refered to Townsend (1976) and to Fiedler (1978). By now the more complete information available concerns the structures in free turbulent flows: in wakes, jets and mixing layers. In the latter case in the shadow pictures the structures look like the two-dimensional vortices, that merge pairwise thus thickening the mixing layer (Winant & Browand (1974), Brown & Roshko (1974)).

Theoretical investigations usually deal with a simple problem concerning the time evolution of a shear layer produced after the development of Helmholtz instability in a plane vortex sheet rather than with a more complicated problem concerning a space-evolving shear layer. At the nonlinear stage a row of structures of the "cat's eye" type is formed. The further evolution of the shear layer is the merging of such structures.

The possible mechanisms of vortex mergings are discussed by Lundgren & Pointin (1977), Saffman & Baker (1978), Saffman & Szeto (1980). The attempts of modelling a shear layer as a line of Rankine's vortices, that merge pairwise, have been made by Ferziger (1980), Saffman & Baker (1979). Ferziger (1980) have analysed consequences of the energy conservation for that model. The modelling of a shear layer by the row of Rankine's vortices seems attractive. What seems to be a serious disadvantage of the theory is the unjustified choice of the structures configuration. Moreover, the reason for self-similarity whose existence may be assumed from the experimental data available, remains unclear.

Roberts & Christiansen (1972), Kida (1975), Saffman & Baker (1979) suggest that coherent structures in two-dimensional turbulent flows are close to statistical equilibrium state. Aref & Siggia (1980) investigate the shear layer as a dynamical system of statistical equilibrium eddies. In numerical simulation, the vorticity field was approximated by a system of the point vortices. The successful choice of the algorithm enabled Aref & Siggia to carry out the calculations with an accuracy a few times higher than that achieved by other authors. One can see from the simulation results, that the vortex layer is a linear row of structures. However, each structure is not a compact vortex but represents a bound state of a few vortex blobs. Aref & Siggia suggested that the vorticity profile of these blobs is the statistical equilibrium one. The statistical equilibrium state of the point vortices system is found by Lundgren & Pointin (1977) and by Kida (1975).

In the simple model proposed by Aref & Siggia the shear layer is treated as a linear row of statistically equilibrium vortex blobs at each stage. The configuration of statistically equilibrium blobs is completely determined by the magnitudes of its internal energy and angular momentum (Lundgren & Pointin (1977)). In the Aref & Siggia model the conservation laws determine the structures configuration after an arbitrary number of pairwise mergings. There is a number of important factors, missed in the model of Aref & Siggia. In particular, there is no angular momentum conservation for an infinite linear row. The conclusion that the vortex scatter about the midline is of the Gaussian form is not physically substantiated.

In the present paper we reconsider a simple model of a shear-layer as a row of statistically equilibrium vortices. The energy conservation law for a linear row and the total angular momentum equation are derived. It is shown that the main parameter of a linear row of vortices, the intermittency factor, can be determined, to a good accuracy, only from the recurrent formula for the energy. The model in which the loss of angular momentum in the row of vortices is taken into account, is constructed. The more complicated shear-layer model, the row of clusters which are bound states of a few vortex blobs, is studied.

2. Modelling of structures by equilibrium vortices

As the shear-layer model, we consider the linear row of an infinite number of vortex blobs. The vortex blobs are assumed to merge pairwise giving an infinite row of larger vortices. The interaction between vortices decreases very slowly with distan-

ce. So the correct limiting transition to the infinitely long row is needed. Let us separate a subsystem from $2N \gg 1$ vortices and replace the remaining part of the infinite row by a fixed vortex sheet (see Fig.1). We denote the effective vortex core radius (see below) by ℓ and the row spacing by D. The fixed vortex sheet produces an external field for the separated part of the row and hinders its rotation as a whole. The vorticity distribution in each vortex of the row is represented by $n \gg 1$ point vortices. In the final formulas the limiting transition $N \to \infty$, $n \to \infty$ is carried out under the condition of keeping D and the vorticity distribution inside each vortex. The intermittency factor $D/2\ell$ is assumed to be considerable. One can expect that the interaction between vortices, reduces to a mutual transfer while the distance between them is large compared to their size. The vortex structure is mainly determined by the nonlinear interaction inside the vortex (Aref & Siggia 1980) and may be found from the relations of equilibrium statistical mechanics (see Appendix). The statistically equilibrium distribution of the isolated vortex blob is determined by the integrals of motion

$$\bar{R} = \frac{1}{n}\sum_i \bar{r}_i ; \quad E = -\frac{\Gamma^2}{2\pi}\sum_{i<j} \ln r_{ij} ; \quad L^2 = \frac{1}{n}\sum_i (\bar{r}_i - \bar{R})^2 \qquad (2.1)$$

Here $\bar{R}$ is the centre of vorticity, Γ is the strength of point vortices. The integrals E and L^2 are distinguished by insignificant dimensional constants from the internal energy of the vortex and its angular momentum (Batchelor 1970). In what follows they are referred simply to as the energy and angular momentum, respectively. In the shear layer, while the vortex blob are far from the others, its internal energy and angular momentum are expected to conserve approximately. It is therefore required to find out how the quantities E and L^2 change in the vortex merging processes.

The variation in the internal energy after vortex pairing is calculable from the total energy conservation law. Prior to merging the total energy of the linear row in an external velocity field produced by the vortex sheet equals

$$H_0 = 2N_0 E_0 - \frac{n_0^2 \Gamma^2}{4\pi}\sum_{i,j} \ln D_{ij} + n_0 \Gamma \sum_i \Psi(R_i)$$

where E_0 is the internal energy of the vortices (2.1),

$R_i = D_0(i+1/2)$ is the centre of vorticity of the i - th vortex, $i=-N_0,\ -N_0+1, \dots N_0-1$; $D_{ij}=|R_i-R_j|$; n_0 is the number of point vortices of strength Γ in each structure, Ψ is the current function of the vortex sheat

$$\Psi(x,y)=\frac{\Delta U}{4\pi}\left\{(A+x)\ln\left[\left(1+\frac{x}{A}\right)^2+\frac{y^2}{A^2}\right]+(A-x)\ln\left[\left(1-\frac{x}{A}\right)^2+\frac{y^2}{A^2}\right]\right\},$$

$A=(N_0+1)D_0$; ΔU is the velosity jump on the vortex sheet; $\Delta U=\frac{n_0\Gamma}{D_0}$. One suggest that the structures interact with each other and with an external field in the same way as the point vortices of the total strength $n_0\Gamma$. For a large degree of intermittency such an approximation works well. H_0 is convenient to represent as a sum of interaction energies of the pairs of vortices with numbers 2κ, $2\kappa+1$ and $2\kappa'$, $2\kappa'+1$, where κ, κ' run the values $-N_0/2, -N_0/2+1, \dots, N_0/2-1$

$$H_0=\sum_\kappa\Big\{2E_0-\frac{n_0^2\Gamma^2}{2\pi}\ln D_0-\frac{n_0^2\Gamma^2}{4\pi}\sum_{\kappa'}\ln\{4(\kappa'-\kappa)^2[4(\kappa'-\kappa)^2-1]\times D_0^4\}+n_0\Gamma\{\Psi[(2\kappa+1/2)D_0]+\Psi[(2\kappa+3/2)D_0]\}\Big\} \qquad (2.2)$$

The total energy after pairwise merging can be written as follows.

$$H=\sum_\kappa\Big\{E_1-\frac{n_0^2\Gamma^2}{\pi}\sum_{\kappa'}\ln(2|\kappa'-\kappa|D_0)+2n_0\Gamma\Psi[(2\kappa+1)D_0]\Big\}, \qquad (2.3)$$

where E_1 is the internal energy of each vortex after mergings. By equalizing the right-hand parts of the expressions (2.2) and (2.3), we obtain the desired relation between E_1 and E_0

$$E_1=2E_0-\frac{n_0^2\Gamma^2}{2\pi}\ln D_0-\frac{1}{N_0}\sum_\kappa\Big\{\frac{n_0^2\Gamma^2}{4\pi}\sum_{\kappa'}\ln\Big[1-\frac{1}{4(\kappa'-\kappa)^2}\Big]+n_0\Gamma\{2\Psi[(2\kappa+1)D_0]-\Psi[(2\kappa+1/2)D_0]-\Psi[(2\kappa+3/2)D_0]\}\Big\}.$$

In the limit $N_0\to\infty$ the contribution of the ends of the row and the difference in the interaction energies with an external field can be neglected and the sum over κ, κ' may be extended from $-\infty$ to ∞. Whence

$$E_1=2E_0-\frac{n_0^2\Gamma^2}{2\pi}\ln\Big[D_0\prod_{p=1}^{\infty}\Big(1-\frac{1}{4p^2}\Big)\Big]=2E_0-\frac{n_0^2\Gamma^2}{2\pi}\ln\frac{2D_0}{\pi}. \qquad (2.4)$$

The magnitude of the internal energy after m mergings is determined by the recurrent formula

$$E_m=2E_{m-1}-\frac{n_{m-1}^2\Gamma^2}{2\pi}\ln\frac{2D_{m-1}}{\pi}, \qquad (2.5)$$

where $n_m=2^m n_0$, $D_m=2^m D_0$. In order to define completely the configuration of a statistically equilibrium vortex, it is necessary to give, besides its energy, the magnitude of its internal angular momentum. The Hamiltonian of the system of vortices in Fig. 1 is not rotationally invariant and the total angular momentum of the row is not conserved.

To study the influence of angular momentum nonconservation, we consider below a simple model. However, we shall first show that the most important parameter of a vortex layer - the intermittency factor - depends slightly on the degree of angular momentum nonconservation. It can be found, to a reasonable accuracy, from the recurrent formula (2.5) in limit $m\to\infty$.

Let the initial configuration of a row be determined by parameters E_0, n_0, D_0 and by the magnitude of the internal angular momentum of vortices L_0^2. After vortex pairing the array is determined by parameters $n_1=2n_0$, $D_1=2D_0$, E_1 and L_1. The relationship of L_1 with L_0 is unknown.

The equilibrium vorticity profile is determined by dimensionless energy (Lundgren & Pointin (1977)). The dimensionless energies of vortices $\tilde{E}_0$, $\tilde{E}_1$ prior to and after mergings are defined by equalities

$$E_0=n_0^2\Gamma^2\Big(\tilde{E}_0-\frac{1}{4\pi}\ln L_0\Big),$$

$$E_1=4n_0^2\Gamma^2\Big(\tilde{E}_1-\frac{1}{4\pi}\ln L_1\Big).$$

By using (2.4), we have

$$\frac{D_1}{2L_1}\cdot\frac{\exp(4\pi\tilde{E}_1)}{\pi}=\Big[\frac{D_0}{2L_0}\cdot\frac{\exp(4\pi\tilde{E}_0)}{\pi}\Big]^{1/2}.$$

After m mergings

$$\frac{D_m}{2L_m}\cdot\frac{\exp(4\pi\tilde{E}_m)}{\pi}=\Big[\frac{D_0}{2L_0}\cdot\frac{\exp(4\pi\tilde{E}_0)}{\pi}\Big]^{1/2^m}. \qquad (2.6)$$

It is seen that the limiting value of the parameter $D/2L$ depends only on the dimensionless energy of a vortex

$$\frac{D_\infty}{2L_\infty} = \pi \exp(-4\pi \tilde{E}_\infty).$$

The quantity L is the r.m.s. radius of vortex. If the values of inverse dimensionless temperature λ (see Appendix) are not too close to -1, then L determines the size of the vortex. At $\lambda \to -1$ the distribution of vorticity is more and more peaked. The concentrated core of vorticity proves to be surrounded by the extended atmosphere from the point vortices. The quantity L determines the radius of this atmosphere at $\lambda \to -1$.

With this in mind, it is convenient to take, as the size of a vortex, the mean radius rather than the quantity L

$$\ell = \int r P(r) d^2 r$$

where $P(r)$ is the equilibrium distribution of vorticity. The parameter

$$\frac{D}{2\ell} = \Delta(\tilde{E}) \equiv \frac{\pi}{\ell_*} \exp(-4\pi\tilde{E}), \tag{2.8}$$

where $\ell_* = \ell/L$, determines the limiting intermittency of the linear row. The quantity ℓ_* is easy to calculate for particular values $\lambda = 0$, $\lambda = \infty$, $\lambda \to -1$. By using the analytical expressions for $P(r)$ (see Appendix) we obtain

$$\ell_*(\lambda = \infty) = 2\sqrt{2}/3\,; \quad \ell_*(0) = \sqrt{\pi}/2\,; \tag{2.9}$$

$$\ell_*(\lambda) = \pi/2\sqrt{e} \exp(-4\pi\tilde{E}),\ \lambda \to -1$$

The dimensionless energy values corresponding to the above values are equal to

$$\tilde{E}(\lambda = \infty) = -\frac{1}{8\pi} \ln\left(\frac{2}{\sqrt{e}}\right); \quad \tilde{E}(0) = \frac{1}{8\pi} \ln\left(\frac{e^C}{2}\right) \tag{2.10}$$

where $C = 0.577$ is the Euler constant. By substituting (2.9), (2.10) into (2.8), we have

$$\Delta(\lambda = \infty) = \frac{3\pi}{2e^{1/4}} \approx 3{,}67; \qquad \Delta(0) = \frac{2^{3/2}\pi^{1/2}}{\exp(C/2)} \approx 3{,}76\,;$$

$$\Delta(-1) = 2\sqrt{e} \approx 3{,}3.$$

The function $\Delta(\lambda)$ for the other values of λ can be calculated numerically with the use of formulas of Appendix. The relevant diagram is illustrated in Fig.2. For convenience, on the abscissa axis we plot not the quantity λ, but the associated dimensionless energy. One can see that the dependence of Δ on $\tilde{E}$ is weak and the relative difference between the maximum value of and its minimum value is only 0.14.

In order to determine the configuration of the structures in the row and to determine more accurately the value of Δ, the magnitude of dimensionless energy $\tilde{E}$ has to be calculated. According to (2.7), $\tilde{E}_m$ is determined by parameter $D_m/2L_m$ in the limit $m \to \infty$. After pairwise merging the distance D becomes two times longer. Let us deduce the recurrent formula for L_m. As noted above, the total angular momentum of the array $\Lambda^2 = \frac{1}{2nN}\sum_i (x_i^2 + y_i^2)$ is not conserved. Its time dependence may be obtained by using the equations of motion for point vortices (Batchelor (1970), Appendix)

$$\frac{d\Lambda^2}{dt} = \frac{1}{nN} \sum_i \vec{r}_i \vec{V}(\vec{r}_i),$$

where $\vec{V}(\vec{r}_i)$ is the velocity field of the vortex sheet, the current function of which is written above. If the deviations of the vortices from the plane $y = 0$ are small compared to ND, then

$$\frac{d\Lambda^2}{dt} \approx \frac{1}{nN} \sum_i y_i \vec{V}(x_i), \tag{2.11}$$

where $V_y \approx \frac{\Delta U}{2\pi} \ln \frac{A - x}{A + x}$.

Let us first consider a simple case of so strong attraction between vortices that they move to each other not deviating practically from the plane $y = 0$. In this case, all y_i in (2.11) are small and one can put $d\Lambda^2/dt = 0$, i.e. the total angular momentum is conserved. The angular momentum Λ^2 after merging is equal to

$$n_m N_m \Lambda^2 = n_m N_m L_m^2 + n_m D_m^2 \sum_{i=1}^{N_m} (i - 1/2)^2 = n_m N_m \left[L_m^2 + D_m^2\left(\frac{N_m^2}{3} - \frac{1}{12}\right)\right].$$

Here $n_m N_m$ is the total number of the point vortices in the array, N_m is the number of structures, n_m is the number of point vortices in each structure

$$n_m = 2^m n_0, \quad N_m = 2^{-m} N_0.$$

By equalizing Λ^2 prior to and after pairwise mergings, we obtain in the recurrent formula

$$L^2_{m+1} = L^2_m + \frac{1}{4} D^2_m \qquad (2.12)$$

Whence

$$L^2_m = L^2_0 + \frac{D_0^2}{12}(2^{2m}-1)$$

$$\frac{D_m}{2L_m} = D_0 2^{m-1} / \left[L_0^2 + \frac{D_0^2}{12}(2^{2m}-1)\right]^{1/2} \qquad (2.13)$$

Within the infinite number of mergings

$$\frac{D_\infty}{2L_\infty} = \sqrt{3}. \qquad (2.14)$$

From (2.7) we get that the dimensionless energy tends to a limiting value which is equal to

$$\tilde{E}_\infty = -\frac{1}{8\pi}\ln(3/\pi^2) \approx 0,047 \qquad (2.15)$$

The inverse dimensionless temperature is calculable by means of formula (Appendix)

$$1+\lambda = \frac{\pi^2 e}{3}\exp\left(-C-\frac{\pi^2 e}{3}\right) \approx 0,00066 \qquad (2.16)$$

The intermittency factor Δ is close to its limiting value 3.3 (see Fig.2).

The recurrent formula for L^2 (2.12) coincides with that used by Aref & Siggia (1980). The dimensionless energy (2.15) is 4 times higher than that obtained by Aref & Siggia and the quantity $1+\lambda$ (2.16) is nearly by 2 orders of magnitude lower than that in Aref & Siggia's (1980) theory. These distinctions are accounted for by that Aref & Siggia (1980) takes into account only some interactions in the infinite array.

It is noteworth that the formulas (2.12)-(2.16) hold only if a line of vortex blobs keeps its linear nature at any moment of time. This assumption makes it possible to use, along with the energy conservation law, the angular momentum conservation law.

The experimental data (Winant & Browand (1974), Brown & Roshko (1974)) show that this assumption is too restrictive. The vortices are on the midline only at some moments of time between

the mergings. During the merging process the vortices deviate noticeably from the plane $y=0$. So the total angular momentum is not conserved (see (2.11)).

However, one can expect that the internal angular momentum of a vortex is conserved while it is far from its neighbours. Also, the summary angular momentum with respect to the common centre of vorticity for a pair of vortex blobs, is conserved, the distance between these vortices being small compared to the distances to the other vortices. This enables one to assume that a certain critical distance $a_m < D_m$ exists such that the change of angular momentum resulted from the merging is given by a formula similar to (2.12)

$$L^2_{m+1} = L^2_m + \frac{1}{4} a^2_m .$$

The quantity a_m depends on D_m, L_m, $\tilde{E}_m$

$$a_m = D_m \cdot f\left(\frac{L_m}{D_m}, \tilde{E}\right)$$

where f is a dimensionless function, which can be defined numerically. To reveal the degree of influence of the angular momentum nonconservation on a value of the limiting temperature, we consider the following example.

The vortex blob trajectories observed in the mixing layers are qualitatively similar to the trajectories of point vortices in the linear array after the loss of stability (Lamb 1932). The development of the most rapid instability in the linear array of point vortices leads to the pairwise relative rotation of vortices along the trajectories

$$ch\,\frac{\pi Y}{D} - \cos\frac{\pi X}{D} = 2,$$

where X, Y is the relative distances of vortices in a pair. The vortices approach pairwise to a minimum distance which equals to 0.56 of the initial and then move off. The example considered by Winant & Browand (1974) shows that the nonpoint vortices can approach to a distance shorter than 0.56 D.

Let us assume that the structures in the shear layer merge when the distance between them equals αD, where $\alpha < 1$. Then $a_m = \alpha D_m$

$$\frac{D_m}{2L_m} = \frac{\sqrt{3}}{\alpha} / \left[1+\frac{1}{2^{2m}}\left(\frac{12L_0^2}{\alpha^2 D_0^2}-1\right)\right]^{1/2} \to \sqrt{3}/\alpha . \qquad (2.17)$$

The values of $\tilde{E}$ and $1+\lambda$ for $\alpha=1$ are given above (see (2.15), (2.16)). If $\alpha\neq 1$, then from (2.7) and (2.17) we get

$$\tilde{E} = -\frac{1}{8\pi}\ln\frac{3}{\alpha^2\pi^2}. \qquad (2.18)$$

If the vortices amalgamate at closest separation (α =0.56) then

$$\tilde{E}\approx 0{,}0012; \qquad 1+\lambda\approx 0{,}3. \qquad (2.19)$$

The comparison of (2.19) with (2.15) and (2.16) shows that the numerical parameter α influences strongly the values of $\tilde{E}$, λ defining the vorticity distribution inside the equilibrium structures. However, the important parameter $D/2L$ does not depend so strongly on the value of α and the parameter Δ may be determined to an accuracy of 0.14, only from the energy formula and depends even weakly on the choice of the magnitude of α.

3. The model of a vortex layer - the row of clusters.

The idea that a free vortex layer may be modelled by a linear row of vortex structures is consistent with the experimental data. Nevertheless, the recent numerical simulation (Aref & Siggia (1980)) shows that the vortex structures are not apparently in a fully statistical equilibrium but they are the bound states of a few equilibrium vortices. There is no difficulty in understanding a possible reason for emerging such bound states. According to (2.16), if $\alpha\sim 1$, then $1+\lambda$ is very small. The velocity of statistical attraction of two vortices is determined by their effective scale $\ell(\lambda)$ which is small in this case. Therefore the emerged bound state from two vortices relaxes for a long time to the statistically equilibrium state.

The time of developing the instability on a double scale is only two times larger than that of developing the most rapid instability (Saffman & Baker (1979)). So, the instability of the next order is excited before occuring the complete relaxation inside the vortex blob pairs and the bound fours of vortex blobs are formed.

As was shown by Novikov & Sedov (1978), a relative motion of the separated four of point vortices is stochastic. One can

expect therefore that the probability of approaching and merging the vortex blobs inside the clusters from four blobs increases significantly. A compound though deterministic, motion occurs in a triple of vortices (Novikov (1976), Aref (1979)). The probability of collision is likely to increase to a larger extent as the number of vortices in a cluster increases.

It is therefore natural to assume that there will be the internal merging in the clusters of vortices and they turn into pairs of vortices. The relaxation inside the produced pair will occur very slowly, and so on.

It is noteworth that the calculations were carried out by Aref & Siggia (1980) in terms of a two-dimensional hydrodynamics. In experiments the flow deviates from the two-dimensional (Fiedler (1978)). As well known the curvature of vortex lines results in appearing the self-induced motion. This motion can hasten the relaxation inside the vortex blob pairs to the statistically equilibrium state and considerably decrease the probability of appearing the bound states from a large number of vortex blobs. The experimental data available do not allow to conclude whether a vortex structure is statistically equilibrium or it is the bound state of several equilibrium vortex blobs.

Nevertheless let us consider a simple model distinguished from the previous by that the linear row of equilibrium vortex blobs is replaced by a linear row of clusters (bound states) from statistically equilibrium vortex blobs. Let a cluster consist of q vortex blobs uniformly spaced on a circumference of diameter d. Let us first assume that the size of the cluster is much less than the distance between the clusters D.

The energy and angular momentum calculations made above are independent of the configurations of structures in this approximation. For this reason we can write down immediately the final formulas. The total dimensionless energy and the total angular momentum of a cluster in the limit of a large number of mergings are equal to (see (2.17), (2.18))

$$\tilde{E}_c = -\frac{1}{8\pi}\ln\frac{3}{\alpha^2\pi^2}, \qquad (3.1)$$

$$L_c^2 = \frac{\alpha^2 D^2}{12} = L_v^2 + \frac{1}{4}d^2, \qquad (3.2)$$

where L_V^2 is the internal angular momentum of each vortex blob in the cluster. It is seen from the second equality that the r.m.s. size of vortex blob L_V is the smaller the larger the cluster diameter d is. At $d^2=d_*^2(=\frac{\varkappa^2D^2}{3})$, L_v vanishes. The value $d/2=d_*/2$ serves as an upper limit of a possible spread of vortex blobs about the midline.

Let us find the relations between the main parameters of a row of clusters. In the evolution of the row the number of vortex blobs in the clusters changes from a certain minimum to a certain maximum. The moments of time for which the recurrent formulas are written, can always be chosen so that a number of vortex blobs in the cluster is minimum. Let us consider, to be concrete, a case when this number equals 2. The total energy of the cluster is

$$E_c = 2E_v - \frac{n^2\Gamma^2}{2\pi}\ln d, \qquad (3.3)$$

where E_V is the internal energy of a vortex. Expressing both parts of (3.3) via dimensionless energies by means of equalities

$$E_c = 4n^2\Gamma^2\left[\tilde{E}_c - \frac{1}{4\pi}\ln L_c\right],\quad E_v = n^2\Gamma^2\left[\tilde{E}_v - \frac{1}{4\pi}\ln L_v\right],$$

we obtain the relations between dimensionless energies

$$\frac{\exp(4\pi\tilde{E}_c)}{L_c} = \left[\frac{\exp(4\pi\tilde{E}_v)}{L_v d}\right]^{1/2}. \qquad (3.4)$$

The dimensionless energy of the cluster is related to the parameter $D/2L_c$ by (see (2.7))

$$D/2L_c = \pi\exp(-4\pi\tilde{E}_c).$$

With (3.4) taken into consideration, one can derive the desired relation

$$\frac{d}{2\ell} = \frac{2\Delta(\tilde{E}_v)}{\pi}\cdot\left(\frac{\pi d}{D}\right)^2 \qquad (3.5)$$

where $\ell=\xi L_v$ is the mean radius of a vortex, $\Delta(\tilde{E}_v)$ is the function plotted in Fig.2.

The relation (3.5) is obtained in the assumption that the distance between the vortex blobs d in the pair is small as compared with D. Let d be comparable with D and the angle of incli-

nation of a vortex blob pair to the midline equal χ. The radius of vortex blobs ℓ is assumed to be small compared to d, D. The interaction energy of two vortex blob pairs spaced κD apart depends on the angle χ

$$E_{ln} = -\frac{n^2\Gamma^2}{4\pi}\ln\left\{(\kappa D)^2\left[\left(1+\frac{d^2}{\kappa^2D^2}\right)^2 - \frac{4d^2}{\kappa^2D^2}\cos^2\chi\right]\right\}.$$

The simple, but a bit cumbersome calculations, quite similar to those carried in derivation of (2.6), yields a formula for the dimensionless complex

$$K = \frac{D}{2L_c}\frac{\exp(4\pi\tilde{E}_c)}{\pi}\left[\frac{\left(\frac{\pi d}{D}\right)^2}{\sin^2\left(\frac{\pi d}{D}\cos\chi\right)+\sinh^2\left(\frac{\pi d}{D}\sin\chi\right)}\right]^{1/4}. \qquad (3.6)$$

This formula has the form

$$K_m = (K_{m-1})^{1/2} = (K_0)^{1/2^m}.$$

Hence, within the infinite number of mergings the quantity K is independent of the initial conditions and equals unity. The relation between the parameters of the row takes the self-similar form

$$\frac{d}{2\ell} = \frac{2\Delta(\tilde{E}_v)}{\pi}\cdot\frac{\pi d}{D}\left[\sin^2\left(\frac{\pi d}{D}\cos\chi\right)+\sinh^2\left(\frac{\pi d}{D}\sin\chi\right)\right]^{1/2}. \qquad (3.7)$$

With $d/D \ll 1$ the formula (3.7) coincides, as should be, with (3.5).

It is of interest to reveal how the configuration of vortices in the cluster depends on the other parameters of the row. The configuration of equilibrium vortices is determined by the magnitude of their dimensionless energy. According to (3.4), $\tilde{E}_v$ is connected with the cluster's dimensionless energy $\tilde{E}_c$ by the equality

$$\tilde{E}_v = 2\tilde{E}_c + \frac{1}{8\pi}\ln\frac{d^2L_v^2}{L_c^4} \qquad (3.8)$$

By expressing here $\tilde{E}_c$ via the remaining parameters of the row by means of (3.6), we obtain the required relation. We shall study it for a simple case $d/D \ll 1$. The values of $\tilde{E}_c$ and L_c are equal to (3.1) and (3.2). The equality (3.2) shows that L_v decreases with increasing d and vanishes at $d=d_*$. Hence, at a fairly large $d<d_*$ the dimensionless energy $\tilde{E}_v$ begins to lo-

wer, too. From Lundgren & Pointin's (1977) theory follows that the distribution of vorticity tends to the tablelike.

4. Discussion

The recurrent formulas derived in the present paper for the simple shear layer models enable a simple assumption to be made that after a large number of mergings a further evolution of the shear layer is self-similar. In the self-similar regime the sizes of structures ℓ, L, d and the distance between them enlarge by a factor of 2 during each merging. The configuration of structures remains unchangeable.

The configuration of structures and the magnitude of the dimensionless parameters of a shear layer produced after rollup of the vortex sheet may differ considerably from the self-similar ones. The rate of approaching the dimensionless parameters to their self-similar values is different for different parameters. According to (2.17), in the simplest model considered above the parameter $D/2L$ takes its self-similar value equal to $\sqrt{3}/\alpha$ during 1-2 mergings and then L becomes two times larger during each merging. The relaxation of the parameter $D/2\ell$ is described by the recurrent formula (see (2.6), (2.8))

$$\frac{D_m}{2\ell_m \Delta(\tilde{E}_m)} = \left[\frac{D_{m-1}}{2\ell_{m-1}\,\Delta(\tilde{E}_{m-1})}\right]^{1/2}$$

and occurs a bit slower. This fact should bear in mind in the analysis of the experimental data because the structures in the mixing layers under observation and calculation carry out a few mergings only. For example, the calculations of Aref & Siggia (1980) cover about four mergings.

Despite the idealistic nature of the models considered above, it is of interest to compare some results obtained with the experimental data. Brown & Roshko (1974) who used not only their experimental results but the results of a number of other authors, obtained a self-similar value for the intermittency factor of the vortex structures in a mixing layer. The thickness of a vorticity layer was accepted as a typical scale of the structure

$$\delta = |\bar{\omega}_{max}|^{-1}\int_{-\infty}^{\infty}|\bar{\omega}|\,dy, \qquad (4.1)$$

where $\bar{\omega} = \partial\langle U\rangle/\partial y$ is the vorticity defined over a average velocity. The experimental value of the intermittency factor $\bar{D}/\delta$, where $\bar{D}$ is the mean interval between the vortices, calculated from the results of various authors is within the range 3÷5.

The magnitude of the parameter $\bar{D}/\delta$ can be calculated in terms of the model in subsection 2, too. The distribution of vorticity in each structure is given by a single-particle equilibrium distribution function (see Appendix). The vorticity field averaged along the shear layer may be determined in this case as an average over the period of a vortex row. For a fairly large intermittency the scale is determined by the relation

$$\delta = L/(2\int_0^{\infty}\tilde{P}(\lambda,\zeta)\,d\zeta). \qquad (4.2)$$

The intermittency factor D/δ, which can be determined by formulas (2.7), (4.2) varies within (3.1÷5.2), depending upon the parameter λ. The agreement with experimental data is well, but one should not exaggerate the importance of this aggreement since at present there is no clear understanding how close are the real mixing layers to the two-dimensional linear row of equilibrium structures.

For better understanding what model corresponds to the real mixing layers, the experimental data on the dynamics of vorticity inside the structures and on the first moments of vorticity distribution ℓ, L^2 are needed.

Appendix

Statistical mechanics of point vortices.

In what follows we mainly reproduce results of Lundgren & Pointin (1977). The difference in the formulas (A.5), (A.6) are due to the correction of the sign in the formulas (41) and (47) of the Lundgren & Pointin work.

The motion of a system of N point vortices of strength Γ in an infinite region is described by the system of Hamiltonian equations (Batchelor (1970))

$$\Gamma \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \qquad \Gamma \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}$$

where $H(\vec{r}_1, \dots \vec{r}_N) = -\frac{\Gamma^2}{2\pi}\sum_{i<j} \ln r_{ij}$ is the Hamiltonian of the system of vortices, $r_{ij} = |\vec{r}_i - \vec{r}_j|$. In the presence of an external velocity field described by the current function Ψ, the role of a Hamiltonian is played by $H_1 = H + \Gamma \sum_i \Psi(\vec{r}_i)$. If Ψ is time-independent, the Hamiltonian is the integral of motion. With $\Psi = 0$ the coordinates of the centre of vorticity $\vec{R} = \frac{1}{N}\sum_i \vec{r}_i$ and the quantity $L^2 = \frac{1}{N}\sum_i (\vec{r}_i - \vec{R})^2$ will be, in addition to H, the integral motions (Batchelor (1970)).

The microcanonical distribution corresponding to the integrals $\vec{R}$, L^2, H is written as follows

$$P_N(\vec{r}_1, \dots \vec{r}_N) = \frac{1}{Q}\delta\Big(-\frac{\Gamma^2}{2\pi}\sum_{i<j} \ln r_{ij} - E\Big)\delta\Big[\sum_i (\vec{r}_i - \vec{R})^2 - NL^2\Big]\delta\Big(\sum_i \vec{r}_i - N\vec{R}\Big),$$

where $Q(E, L^2)$ is the density of states defined from the normalization condition. In this case, the entropy $S = \kappa \ln Q$ will be a thermodynamic state function and temperature $T = \left(\kappa \frac{\partial \ln Q}{\partial E}\right)^{-1}$ will be a parameter of state. The differentiation of the microcanonical distribution over $\vec{r}_i$ and integration over other arguments give rise to the known chain of the coupled equations for reduced distribution functions P_S, $S < N$. For the purpose of its breaking, the Vlasov's approximation is used

$$P_2(\vec{r}_1, \vec{r}_2) = P_1(\vec{r}_1) P_1(\vec{r}_2). \tag{A.1}$$

Its validity may be argued as that in the theory of the Coulomb systems. Since the interaction between vortices decreases very

slowly with distance, the motion of each vortex will be determined by the collective action of all the remaining $N-1$ vortices. The motion of two definite vortices at $N \gg 1$ will be mutually independent and the twoparticle distribution function will be divided into a product of single-particle distribution functions. The case $N \to \infty$ correspinds to the hydrodynamic limit. In this limit the vorticity is connected with the single-particle distribution function by the relation

$$\omega(\vec{r}) = N\Gamma P(\vec{r}).$$

The condition (A.1) for quite large N makes it possible to the closed equation for an equilibrium single-particle distribution function

$$\frac{\partial P_1(\vec{r}_1)}{\partial \vec{r}_1} = 4\lambda \int \Big(\frac{\partial}{\partial \vec{r}_1} \ln r_{12}\Big) P_1(\vec{r}_1) P_1(\vec{r}_2)\, d\vec{r}_2 - 2(\vec{r}_1 - \vec{R})\frac{1+\lambda}{L^2} P_1(\vec{r}_1), \tag{A.2}$$

where $\lambda = \rho N \Gamma^2 / 8\pi \kappa T$ is the inverse dimensionless temperature. Here ρ is the fluid density, κ is the Boltzman's constant. Let us introduce the transformation of variables

$$\vec{\zeta} = (\vec{r}_1 - \vec{R})/L, \qquad \tilde{P}_1(\vec{\zeta}) = L^2 P_1(\vec{r}_1),$$

since $P_1(\vec{r}_1)$ is isotropic

$$\frac{d^2 \ln \tilde{P}_1}{d\zeta^2} + \frac{1}{\zeta}\frac{d \ln \tilde{P}_1}{d\zeta} = -4(1+\lambda) + 8\pi\lambda \exp(\ln \tilde{P}_1). \tag{A.3}$$

Some results can be obtained directly from equations (A.2) and (A.3). So, for $\lambda = 0$

$$\tilde{P}_1 = \frac{1}{\pi}\exp(-\zeta^2),$$

with $\lambda \to \infty$

$$\tilde{P}_1 = 1/2\pi, \quad \zeta < \sqrt{2}$$
$$= 0, \quad \zeta > \sqrt{2}$$

For $\lambda \to -1$ the approximate solution of (A.3) is of the form

$$\tilde{P}_1 = \frac{A}{(1 - \pi A \lambda \zeta^2)^2}\exp[-(1+\lambda)\zeta^2]. \tag{A.4}$$

The constant A is defined by the normalization condition

$$\pi A - \ln \pi A = -C - \ln(1+\lambda), \quad C = 0{,}5772 \tag{A.5}$$

The energy of the system of vorticites with due regard for (A.1) can be written down

$$E = -\frac{(N^2-N)}{2} \cdot \frac{\Gamma^2}{2\pi} \int \ln r_{12} P_1(\bar{r}_1) P_1(\bar{r}_2) d\bar{r}_1 d\bar{r}_2 .$$

In the dimensionless variables

$$E = \rho N^2 \Gamma^2 \left[\tilde{E}(\lambda) - \frac{1}{4\pi} \ln L \right],$$

$$\tilde{E}(\lambda) = -\frac{1}{4\pi} \int \ln \tau_{12} \tilde{P}_1(\bar{\tau}_1) \tilde{P}_1(\bar{\tau}_2) d\bar{\tau}_1 d\bar{\tau}_2$$

For some values of the parameter λ one can find out the dimensionless energy $\tilde{E}$, using the results presented above. So,

$$\tilde{E}(0) = \frac{1}{8\pi}(C - \ln 2),$$

$$\tilde{E}(\infty) = -\frac{1}{16\pi}(2\ln 2 - 1).$$

At $\lambda \to -1$, using the relation (A.4), we have

$$\pi A = exp\,(1 + 8\pi \tilde{E})$$

and taking into account the relation (A.5) we get

$$1 + \lambda = exp\left[-C + (1 + 8\pi\tilde{E}) - exp(1 + 8\pi\tilde{E})\right], \; C = 0{,}5772. \quad (A.6)$$

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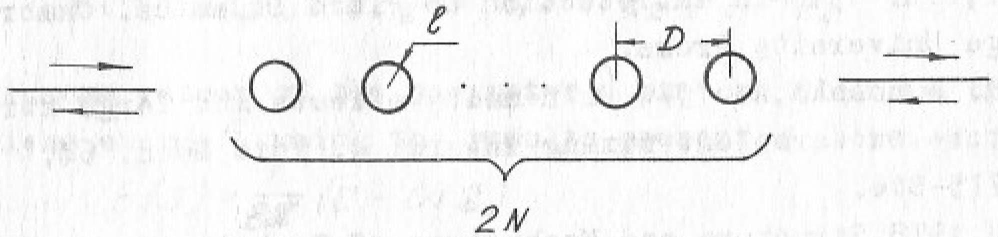

Figure 1

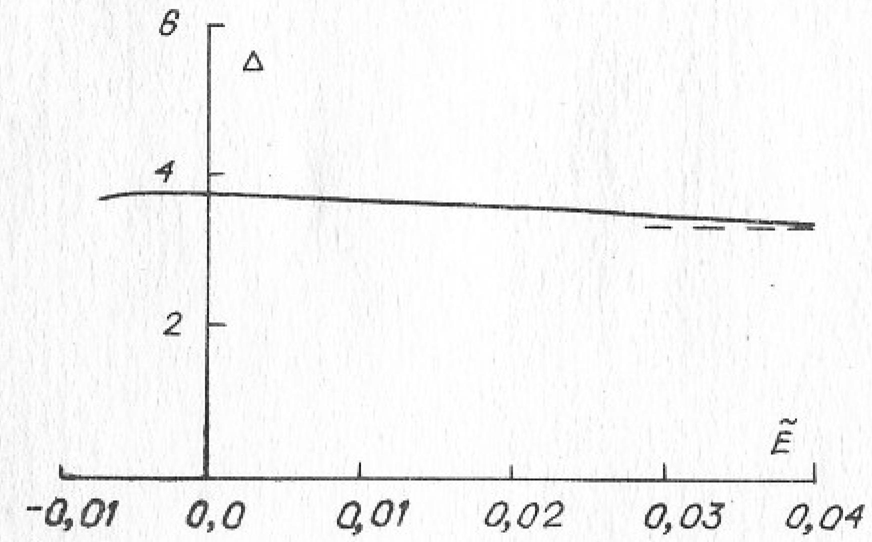

Figure 2