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ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ
СО АН СССР

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GENERAL STRUCTURE OF NONLINEAR EVOLUTION EQUATIONS IN 1+2 DIMENSIONS INTEGRABLE BY THE TWO-DIMENSIONAL GELFAND-DIKIJ SPECTRAL PROBLEM AND THEIR TRANSFORMATION PROPERTIES

ПРЕПРИНТ 82 - 20

Новосибирск

Abstract

The general form of nonlinear evolution equations connected with the matrix two-dimensional Gelfand-Dikij spectral problem is found. Infinitedimensional abelian group of general Backlund transformations and infinitedimensional abelian symmetry group for these equations are constructed.

GENERAL STRUCTURE OF NONLINEAR EVOLUTION EQUATIONS IN 1+2 DIMENSIONS INTEGRABLE BY THE TWO-DIMENSIONAL GELFAND-DIKIJ SPECTRAL PROBLEM AND THEIR TRANSFORMATION PROPERTIES

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I. Introduction

One of the main problem of the inverse scattering transform (IST) method is problem of description of the equations integrable by this method (see e.g. [1,2]). All the equations to which IST method is applicable form the classes of the equations integrable by the same spectral problem. Very convenient and simple description of the partial differential equations integrable by second order problem (*) $\frac{\partial\psi}{\partial x} = \lambda A \psi + P(x,t)\psi$ has been given in AKNS paper [3]. Then this approach (AKNS approach) was generalised to the problem (*) of any order [4-10] and to some other spectral problems [11,12] in particular, to one-dimensional Gelfand-Dikij spectral problem [13].

Recently the two-dimensional generalisation of AKNS-technique has been given [14]. Namely, the two-dimensional arbitrary order spectral problem $\frac{\partial\psi}{\partial x} + A\frac{\partial\psi}{\partial y} + P(x,y,t)\psi = 0$ where A is any diagonalisable constant matrix was considered and the general form of the nonlinear equations integrable by this problem, their Backlund transformations were found [14].

In the present paper we consider the two-dimensional matrix Gelfand-Dikij spectral problem

$$\frac{\partial^N \chi}{\partial x^N} + V_{N-2}(x,y,t)\frac{\partial^{N-2}\chi}{\partial x^{N-2}} + \dots + V_0(x,y,t)\chi + \frac{\partial \chi}{\partial y} = 0 \qquad (1.1)$$

where N is arbitrary integer, coefficients $V_0(x,y,t),\dots,V_{N-2}(x,y,t)$ are matrices of arbitrary order M depending on two coordinates x, y and time t and $V_k(x,y,t) \xrightarrow[\sqrt{x^2+y^2}\to\infty]{} 0$ $(k=0,\dots,N-2)$. The applicability of the IST method to the problem (1.1) was discussed in Refs. [15,16] . In the case N=2 the problem (1.1) is used for integration of Kadomtsev-Petviashvili equation [15,17, 18] .

In the present paper we find the general form of nonlinear evolution equations in 1+2 dimensions (t, x, y) integrable by the problem (1.1). We construct the infinitedimensional abelian group of general Backlund transformations and infinite-dimensional abelian group of symmetry for these equations. As an example we consider the case N=2. In this case we obtain also the nonlinear superposition formulas for simplest Backlund transformation.

The paper is organised as follows. In the second section we rewrite the problem (1.1) in a matrix form, then we consider the direct scattering problem and obtain some important relations. In section 3 we calculate the recursion operators which play a main role in our constructions. The general form of the integrable equations and Backlund transformations are found in section 4.Group-theoretical properties of the integrable equations are discussed briefly in section 5. In section 6 the case N=2 is considered: the infinite family of the equations, simplest of which is Kadomtsev-Petviashvili equation, their Backlund transformations and nonlinear superposition formulas are described.

II. Direct scattering problem and some important relations

Let us note first of all that the problem (1.1) is equivalent to NM order matrix problem

$$\frac{\partial \hat{\psi}}{\partial x} + A\frac{\partial \hat{\psi}}{\partial y} + P(x,y,t)\hat{\psi} = 0 \qquad (2.1)$$

where

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \\ I_M & 0 & \dots & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & -I_M & 0 & \dots & 0 \\ 0 & 0 & -I_M & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 & -I_M \\ V_0 & V_1 & \dots & V_{N-2} & 0 \end{pmatrix} \qquad (2.2)$$

and I_M is identical M×M matrix. The adjoint spectral problem is

$$\frac{\partial \check{\psi}}{\partial x} + \frac{\partial \check{\psi}}{\partial y}A - \check{\psi}P(x,y,t) = 0. \qquad (2.3)$$

The matrix problems (2.1) and (2.3) are more convenient for our purpose than the problem (1.1) and the problem adjoint to (1.1).

Let us consider firstly the direct scattering problem for (2.1) and (2.3). We will follow to Refs. [16,14]. We will assume that $V_k(x,y,t) \to 0$ at $\sqrt{x^2+y^2} \to \infty$ so fast that all the quantities and integrals, which will appear in our constructions, will exist and that $\int_{-\infty}^{+\infty} dy \frac{\partial}{\partial y}(\dots) = 0$.

We introduce the matrices-solutions $\hat{F}^{+}_{\lambda}(x,y,t)$ and $\hat{F}^{-}_{\lambda}(x,y,t)$ of the problem (2.1) given by their asymptotic behaviour

$$\hat{F}^{+}_{\lambda}(x,y,t) \xrightarrow[x\to+\infty]{} (2\pi i)^{-\frac{1}{2}} \mathcal{D}(\lambda) \exp\{-\lambda^{N} y + \bar{A} x\},$$

$$\hat{F}^{-}_{\lambda}(x,y,t) \xrightarrow[x\to-\infty]{} (2\pi i)^{-\frac{1}{2}} \mathcal{D}(\lambda) \exp\{-\lambda^{N} y + \bar{A} x\} \qquad (2.4)$$

where λ is a complex number, $\bar{A}$ is a diagonal matrix: $\bar{A}_{ik} = \lambda q^{i-1} \delta_{ik}$ $\left(q = \exp\frac{2\pi i}{N}, \delta_{ik} = \begin{cases} 1, & i=k \\ 0, & i\neq k \end{cases}\right)$ and $\mathcal{D}_{ik} = \frac{1}{\sqrt{N}} \left(\lambda q^{(k-1)}\right)^{i-1}$ $(i,k = 1,\dots,N)^{*}$. The quantities λq^{i-1} are eigenvalues of the matrix $\hat{A} = \lambda^{N} A + P_{\infty}$ where $P_{\infty} \overset{def}{=} -\lim_{\sqrt{x^2+y^2}\to\infty} P(x,y,t)$ and $\hat{A} = \mathcal{D} \bar{A} \mathcal{D}^{-1}$.

The scattering matrix $\hat{S}(\tilde{\lambda},\lambda,t)$ for problem (2.1) is defined as follows

$$\hat{F}^{+}_{\lambda}(x,y,t) = \int_{-\infty}^{+\infty} d\tilde{\lambda}\, \hat{F}^{-}_{\tilde{\lambda}}(x,y,t)\, \hat{S}(\tilde{\lambda},\lambda,t). \qquad (2.5)$$

Correspondingly for the adjoint problem (2.3) we introduce matrices-solutions $\check{F}^{+}_{\lambda}(x,y,t)$ and $\check{F}^{-}_{\lambda}(x,y,t)$:

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*) Here and below latin indices take the values 1,2,...,N (or N - 1) and numerate the block elements of matrices of the order NM which are themselvs the matrices M × M. Greek indices mark the usual matrix elements of NM × NM matrices and take the values 1,2,...,NM.

$$\check{F}^{\pm}_{\lambda}(x,y,t) \xrightarrow[x\to\pm\infty]{} (2\pi i)^{-\frac{1}{2}} \exp\{\lambda^{N} y - \bar{A} x\} \cdot \mathcal{D}^{-1}(\lambda) \qquad (2.6)$$

and scattering matrix $\check{S}(\tilde{\lambda},\lambda,t)$:

$$\check{F}^{+}_{\lambda}(x,y,t) = \int_{-\infty}^{+\infty} d\tilde{\lambda}\, \check{S}(\lambda,\tilde{\lambda},t)\, \check{F}^{-}_{\tilde{\lambda}}(x,y,t). \qquad (2.7)$$

It is not difficult to show with the use of (2.1)-(2.7) that the following relations hold:

$$\int_{-\infty}^{+\infty} dy\, \check{F}^{\pm}_{\tilde{\lambda}}(x,y,t)\, \hat{F}^{\pm}_{\lambda}(x,y,t) = \delta(\tilde{\lambda}-\lambda),$$

$$\int_{-\infty}^{+\infty} d\lambda\, \hat{F}^{\pm}_{\lambda}(x,y,t)\, \check{F}^{\pm}_{\lambda}(x,y',t) = \delta(y'-y), \qquad (2.8)$$

$$\int_{-\infty}^{+\infty} d\mu\, \check{S}(\tilde{\lambda},\mu,t)\, \hat{S}(\mu,\lambda,t) = \delta(\tilde{\lambda}-\lambda)$$

where $\delta(\lambda)$ is Dirac delta-function. Hence the scattering matrices can be represented as

$$\hat{S}(\tilde{\lambda},\lambda,t) = \int_{-\infty}^{+\infty} dy\, \check{F}^{-}_{\tilde{\lambda}}(x,y,t)\, \hat{F}^{+}_{\lambda}(x,y,t),$$

$$\check{S}(\tilde{\lambda},\lambda,t) = \int_{-\infty}^{+\infty} dy\, \check{F}^{+}_{\tilde{\lambda}}(x,y,t)\, \hat{F}^{-}_{\lambda}(x,y,t). \qquad (2.9)$$

Let now P and P' are two different potentials and $\hat{F}^{+}, \check{F}^{+}, \hat{F}^{+\prime}, \hat{S}, \hat{S}'$ are corresponding solutions and scattering matrices for the problems (2.1) and (2.3). One can prove (analogously to Ref. [14]) the following important relation

$$\hat S'(\tilde\lambda,\lambda,t) - \hat S(\tilde\lambda,\lambda,t) =$$

$$= -\int_{-\infty}^{+\infty} d\mu\, \hat S(\tilde\lambda,\mu,t) \int_{-\infty}^{+\infty} dx\,dy\, \check F^{+}_{\mu}(x,y,t)\big(P'(x,y,t)-P(x,y,t)\big)\hat F^{+\prime}_{\lambda}(x,y,t). \qquad (2.10)$$

The mapping $P(x,y,t)\to \hat S(\tilde\lambda,\lambda,t)$ given by the spectral problem (2.1) establish correspondence between the transformations $B_P : P\to P'$ on the manifold of the potentials $\{P(x,y,t)\}$ and the transformations $B_S : \hat S\to \hat S'$ on the manifold of the scattering matrices $\{\hat S(\tilde\lambda,\lambda,t)\}$.

We will consider only such transformations B that

$$\hat S(\tilde\lambda,\lambda,t) \xrightarrow{B} \hat S'(\tilde\lambda,\lambda,t) = B^{-1}(\tilde\lambda,t)\,\hat S(\tilde\lambda,\lambda,t)\,C(\lambda,t) \qquad (2.11)$$

where $B(\tilde\lambda,t)$ and $C(\lambda,t)$ are arbitrary block diagonal matrices, i.e. $B_{ik}= B_i(\lambda,t)\delta_{ik}I_M$, $C_{ik}=C_i(\lambda,t)\delta_{ik}I_M$. This "restricted" class of the transformations , as we shall see, is wide enough.

Further, it is not difficult to show that the following identity holds

$$-\int_{-\infty}^{+\infty} d\mu\, \check S(\tilde\lambda,\mu,t)(1-B(\mu,t))\hat S'(\mu,\lambda,t) + (1-B(\lambda,t))\delta(\tilde\lambda-\lambda) =$$

$$= \int_{-\infty}^{+\infty} dy \left\{ \check F^{+}_{\tilde\lambda}(x,y,t)(1-\tilde B(-\partial_y,t))\,\hat F^{+\prime}_{\lambda}(x,y,t)\right\}\Bigg|_{x=-\infty}^{x=+\infty} = \qquad (2.12)$$

$$= -\int_{-\infty}^{+\infty} dx\,dy\, \check F^{+}_{\tilde\lambda}(x,y,t)\Big\{ \tilde P(x,y,t)(1-\tilde B(-\partial_y,t))\,\hat F^{+\prime}_{\lambda}(x,y,t) -$$

$$-(1-\tilde B(-\partial_y,t))\,\tilde P'(x,y,t)\,\hat F^{+\prime}_{\lambda}(x,y,t)\Big\}$$

where $\tilde P \stackrel{def}{=} P+P_\infty$ and $\tilde B(\mu,t)=\mathcal{D}(\mu)B(\mu,t)\mathcal{D}^{-1}(\mu)$. Here and below $\partial_x \equiv \frac{\partial}{\partial x}$, $\partial_y\equiv \frac{\partial}{\partial y}$.

Combining the relations (2.10) and (2.11) and taking into account the identity (2.12) we find *

$$\int_{-\infty}^{+\infty} dx\,dy \Big\{ \check F^{+}_{\tilde\lambda}(x,y,t)\Big(\tilde B(-\partial_y,t)\,\tilde P'(x,y,t)\,\hat F^{+\prime}_{\lambda}(x,y,t) - \tilde P(x,y,t)\,\tilde B(-\partial_y,t)\,\hat F^{+\prime}_{\lambda}(x,y,t)\Big)\Big\}_F = 0 \qquad (2.13)$$

where $(\varphi_F)_{\alpha\beta} \stackrel{def}{=} \varphi_{\alpha\beta} - \delta_{\alpha\beta}\varphi_{\alpha\alpha}$ $(\alpha,\beta = 1,\dots,NM)$ for an arbitrary NM x NM matrix φ .

Let us represent block diagonal matrix B in the form

$$B(\lambda,t) = \sum_{k=0}^{N-1} B_k(\lambda^N,t)\,\bar A^{k}$$

where $B_k(\lambda^N,t)$ are scalar functions and $\bar A^{0} \stackrel{def}{=} I_{NM}$. Correspondingly for $\tilde B(-\partial_y,t)$ we have

$$\tilde B(-\partial_y,t) = \sum_{k=0}^{N-1} B_k(-\partial_y,t)\,(-A\,\partial_y + P_\infty)^k . \qquad (2.14)$$

It is easy to see that

$$(-A\partial_y + P_\infty)^k = -(P_\infty^z)^{N-k}\,\partial_y + P_\infty^{\,k} \qquad (2.15)$$

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*) We will omit some intermediate calculations which are typical for generalised AKNS-technique (see e.g. [8,13,14]).

where symbol τ denote a transposition of $M\times M$ blocks in $NM\times NM$ matrices.

We will consider only functions $B_K(-\partial_y,t)$ entire on the first argument, i.e. $B_K(-\partial_y,t)=\sum_{n=0}^{\infty}b_{Kn}(t)(-\partial_y)^n$ where $b_{Kn}(t)$ are arbitrary functions. In virtue of (2.14) and (2.15), for such functions $B_K(-\partial_y,t)$ the equality (2.13) is equivalent to the equality

$$\int_{-\infty}^{+\infty}dx\,dy\sum_{K=0}^{N-1}\sum_{n=0}^{\infty}b_{Kn}(t)\,tr\Big\{(P_\infty^\tau)^{N-K}\tilde{P}'(x,y,t)\;\check{\Phi}^{(\alpha\beta)}_{(n+1)}(x,y,t)+$$

$$+P_\infty^K\,\tilde{P}'(x,y,t)\;\check{\Phi}^{(\alpha\beta)}_{(n)}(x,y,t)-$$

$$-\tilde{P}(x,y,t)\,(P_\infty^\tau)^{N-K}(-1)^{n+1}\;\hat{\Phi}^{(\alpha\beta)}_{(n+1)}(x,y,t)- \tag{2.16}$$

$$-\hat{P}(x,y,t)\,P_\infty^K(-1)^n\;\hat{\Phi}^{(\alpha\beta)}_{(n)}(x,y,t)\Big\}=0 \qquad (\alpha\neq\beta)$$

where tr denote a usual matrix trace and

$$\Big(\check{\Phi}^{(\alpha\beta)}_{(n)}(x,y,t)\Big)_{\gamma\delta}\overset{def}{=}\Big(\hat{F}'^{+}_{\lambda}(x,y,t)\Big)_{\gamma\beta}\frac{\partial^n\big(\check{F}^{+}_{\tilde{\lambda}}(x,y,t)\big)_{\alpha\delta}}{\partial y^n},$$

$$\Big(\hat{\Phi}^{(\alpha\beta)}_{(n)}(x,y,t)\Big)_{\gamma\delta}\overset{def}{=}\frac{\partial^n\big(\hat{F}'^{+}_{\lambda}(x,y,t)\big)_{\gamma\beta}}{\partial y^n}\Big(\check{F}^{+}_{\tilde{\lambda}}(x,y,t)\Big)_{\alpha\delta} \tag{2.17}$$

$$(\alpha,\beta,\gamma,\delta=1,\dots,NM;\; n=0,1,2,\dots).$$

III. Recursion operators

For the further transformation of the equality (2.16) one must establish the relations between the quantities $\check{\Phi}_{(n)}$ and $\hat{\Phi}_{(n)}$ with different n, i.e. one must calculate the recursion operators.

Let us consider firstly the quantity $\hat{\Phi}^{(\alpha\beta)}_{(n)}$. From the equations (2.1) and (2.3) we obtain

$$\partial_x\hat{\Phi}^{(\alpha\beta)}_{(n)}+\partial_y\hat{\Phi}^{(\alpha\beta)}_{(n)}A=-\big[A,\hat{\Phi}^{(\alpha\beta)}_{(n+1)}\big]+$$

$$+\hat{\Phi}^{(\alpha\beta)}_{(n)}P(x,y)-\sum_{m=0}^{n}C_m^n\,P'_{(n-m)}\,\hat{\Phi}^{(\alpha\beta)}_{(m)} \qquad (n=0,1,2,\dots) \tag{3.1}$$

where $C_m^n=\frac{n!}{m!(n-m)!}$ and $P'_{(K)}\overset{def}{=}\frac{\partial^K P(x,y,t)}{\partial y^K}$.

The relations (3.1) allow us to express all matrix elements of the quantity $\hat{\Phi}^{(\alpha\beta)}_{(n)}$ through $N-1$ independent one.

Let us introduce the projection operation Δ_K: $(\Phi_{\Delta_K})_{i\ell}\overset{def}{=}\delta_{\ell K}\,\Phi_{iK}$. Applying the operations Δ_K to the equations (3.1) and taking into account the properties of the matrices $A,P_\infty,\hat{P}$ one obtain

$$\partial_x\hat{\Phi}^{(\alpha\beta)}_{(n)\Delta_1}+\partial_y\hat{\Phi}^{(\alpha\beta)}_{(n)\Delta_N}A=-A\,\hat{\Phi}^{(\alpha\beta)}_{(n+1)\Delta_1}+$$

$$+\hat{\Phi}^{(\alpha\beta)}_{(n+1)\Delta_N}A+\Big(\hat{\Phi}^{(\alpha\beta)}_{(n)\Delta_N}\hat{P}\Big)_{\Delta_1}-\sum_{m=0}^{n}C_m^n\,P'_{(n-m)}\,\hat{\Phi}^{(\alpha\beta)}_{(m)\Delta_1}, \tag{3.2}$$

$$\partial_x \hat{\varphi}^{(\alpha\beta)}_{(n)\Delta_k} = -A\,\hat{\varphi}^{(\alpha\beta)}_{(n+1)\Delta_k} + \left(\hat{\varphi}^{(\alpha\beta)}_{(n)\Delta_N}\tilde{P}\right)_{\Delta_k} -$$

$$- \hat{\varphi}^{(\alpha\beta)}_{(n)\Delta_{k-1}} P_\infty - \sum_{m=0}^{n} C^n_m P'_{(n-m)}\,\hat{\varphi}^{(\alpha\beta)}_{(m)\Delta_k}\,. \quad (k=2,3,\dots,N) \tag{3.3}$$

The relations (3.2) and (3.3) can be rewritten in a more compact form. Let us introduce the matrix infinite order triangular operators $\mathcal{P}$ and T with matrix elements

$$\mathcal{P}_{(n,m)} \cdot = -\delta_{nm}\partial_x - C^n_m P'_{(n-m)}\ , \quad n \geq m$$

$$\mathcal{P}_{(n,m)} = 0 \quad , \ m \geq n+1 \quad (n,m=0,1,2,\dots) \tag{3.4}$$

and

$$T_{(n,m)} = \delta_{m,n+1}\, I_{NM}\,. \quad (n,m=0,1,2,\dots) \tag{3.5}$$

Matrix operators $\mathcal{P}$ and T act on the infinite-component column $\Psi \overset{def}{=} (\hat{\varphi}_{(0)}, \hat{\varphi}_{(1)}, \hat{\varphi}_{(2)}, \dots\ )$ by the usual rules. For example,

$$(\mathcal{P}^2\Psi)_{(n)} = \sum_{m_1=0}^{\infty}\mathcal{P}_{(n,m_1)}\sum_{m_2=0}^{\infty}\mathcal{P}_{(m_1,m_2)}\varphi_{(m_2)} =$$

$$= \sum_{m_1=0}^{n}\sum_{m_2=0}^{m_1}\mathcal{P}_{(n,m_1)}\mathcal{P}_{(m_1,m_2)}\varphi_{(m_2)}\,; \quad (T\Psi)_{(n)} = \varphi_{(n+1)}\,.$$

With the use of the operators $\mathcal{P}$ and T the relations (3.2) and (3.3) can be represented in the form

$$(\mathcal{P} - AT)\Psi^{(\alpha\beta)}_{\Delta_1} = \partial_y \Psi^{(\alpha\beta)}_{\Delta_N} A + T\Psi^{(\alpha\beta)}_{\Delta_N} A -$$

$$- \left(\Psi^{(\alpha\beta)}_{\Delta_N}\tilde{P}\right)_{\Delta_1}\ , \tag{3.6}$$

$$\Psi^{(\alpha\beta)}_{\Delta_k} = (\mathcal{P} - AT)\Psi^{(\alpha\beta)}_{\Delta_{k+1}}(A^\tau + P^\tau_\infty) +$$

$$+ \left(\Psi^{(\alpha\beta)}_{\Delta_N}\tilde{P}\right)_{\Delta_{k+1}}(A^\tau + P^\tau_\infty)\,. \quad (k=1,2,\dots,N-1) \tag{3.7}$$

From the recurrence relations (3.7) we find

$$\Psi^{(\alpha\beta)}_{\Delta_k} = (\mathcal{P} - AT)^{N-k}\Psi^{(\alpha\beta)}_{\Delta_N}(A^\tau + P^\tau_\infty)^{N-k} +$$

$$+ \sum_{\ell=0}^{N-k-1}(\mathcal{P} - AT)^\ell \left(\Psi^{(\alpha\beta)}_{\Delta_N}\tilde{P}\right)_{\Delta_{k+\ell+1}}(A^\tau + P^\tau_\infty)^{\ell+1}\,. \quad (k=1,2,\dots,N-1) \tag{3.8}$$

In virtue of (3.8) one can express the quantity $\Psi^{(\alpha\beta)}_{\Delta_1}$ through $\Psi^{(\alpha\beta)}_{\Delta_N}$. Substituting this expresion for $\Psi^{(\alpha\beta)}_{\Delta_1}$ into (3.6) and taking into account the identity

$$\left(\varphi_{\Delta_N}\tilde{P}\right)_{\Delta_\ell}(A + P_\infty)^\ell = -\varphi_{\Delta_N}\circ V_{\ell-1}$$

where $(\varphi_{(n)}\circ V_k)_{i\ell} \overset{def}{=} (\varphi_{(n)})_{i\ell}\, V_k$ we obtain

$$\sum_{\ell=0}^{N}(\mathcal{P} - AT)^\ell\,\Psi^{(\alpha\beta)}_{\Delta_N}\circ V_\ell = T\Psi^{(\alpha\beta)}_{\Delta_N} - \partial_y\Psi^{(\alpha\beta)}_{\Delta_N}\,. \tag{3.9}$$

Let us note now that in virtue of the properties of the matrices A, P_∞ and $\tilde{P}$ (for example, $A^2 = A\tilde{P} = \tilde{P}A = AP_\infty\tilde{P} = \tilde{P}P_\infty A = 0$) the operators $(\mathcal{P} - AT)^\ell$ are linear on operator T, i.e.

$$(\mathcal{P} - AT)^\ell = -\sum_{k_1+k_2=\ell-1}\mathcal{P}^{k_1}AT\mathcal{P}^{k_2} = \Gamma_\ell T + \mathcal{P}^\ell \quad (\ell=1,2,3,\dots) \tag{3.10}$$

where

$$\Gamma_e = -\sum_{k_1+k_2=e-1} \tilde{\mathcal{P}}^{k_1} A\, \tilde{\mathcal{P}}^{k_2}$$

and $\hat{\mathcal{P}} \overset{def}{=} \mathcal{P}_{nn} = -\partial_x - P'$.

Substitution of (3.10) into (3.9) give

$$\sum_{e=0}^{N} \Gamma_e \left(T \Psi^{(\alpha\beta)}_{\Delta_N} \circ V_e \right) - T \Psi^{(\alpha\beta)}_{\Delta_N} = -\sum_{e=0}^{N} \mathcal{P}^e \left(\Psi^{(\alpha\beta)}_{\Delta_N} \circ V_e \right) - \partial_y \Psi^{(\alpha\beta)}_{\Delta_N} \quad (3.11)$$

The equality (3.11) is the relations between N quantities $\Psi^{(\alpha\beta)}_{1N}, \Psi^{(\alpha\beta)}_{2N}, \ldots, \Psi^{(\alpha\beta)}_{NN}$. The first nontrivial equation from (3.11) allow us to express $\Psi^{(\alpha\beta)}_{NN}$ through $\Psi^{(\alpha\beta)}_{1N}, \ldots, \Psi^{(\alpha\beta)}_{N-1,N}$:

$$\Psi^{(\alpha\beta)}_{NN}(x,y,t) = \sum_{k=1}^{N-1} e_k \Psi^{(\alpha\beta)}_{kN} + \Psi^{(\alpha\beta)}_{NN}(x=+\infty, y, t) \quad (3.12)$$

where

$$e_k \cdot = \frac{1}{N} \sum_{e=1}^{N} \partial_x^{-1} \left((\mathcal{P}^e)_{1k} (\cdot \circ V_e) \right) + \frac{1}{N} \partial_x^{-1} \partial_y \delta_{k1} \cdot \quad (3.13)$$

and $(\partial_x^{-1} f)(x,y) \overset{def}{=} -\int_x^{\infty} dx' f(x',y)$.

Formula (3.12) contains the inhomogeneous term $\Psi^{(\alpha\beta)}_{NN}(x=+\infty, y, t)$. Similar inhomogeneous terms (namely $\left(\Psi^{(\alpha\beta)}_{(x=+\infty,y,t)}\right)_{ke}$) will appear after integration in further calculations too. Taking into account (2.5) and (2.6) one can show that ($\lambda > 0$)

$$\left(\hat{\Phi}^{(in)}_{(m)}(x=+\infty, y, t) \right)_{ke} = -\lambda^{Nm} D_{kn} D^{-1}_{ie} \lim_{x\to+\infty} \exp\{\lambda (q^{n-1} - q^{i-1}) x\} .$$

Let us mark by $\Psi^{(*)}$ the subspace of quantities $\Psi^{(in)}$ for which $Re(q^{n-1} - q^{i-1}) = \cos\left(\frac{2\pi}{N}(n-1)\right) - \cos\left(\frac{2\pi}{N}(i-1)\right) < 0$. For indices n and i which satisfy to this inequality one have $\lim_{x\to+\infty} \exp\{\lambda(q^{n-1} - q^{i-1})x\} = 0$. Therefore in the relations which contain the quantity $\Psi^{(*)}$ the inhomogeneous terms will be absent. In particular, instead of (3.12) we have

$$\Psi^{(*)}_{NN} = \sum_{k=1}^{N-1} e_k \Psi^{(*)}_{kN} . \quad (3.14)$$

In virtue of (3.14) one find

$$\Psi^{(*)}_{\Delta_N} = M \Psi^{(*)}_{\Delta} \quad (3.15)$$

where

$$M = \begin{pmatrix} I_M & 0 & \cdots & & 0 \\ 0 & I_M & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & I_M & 0 \\ e_1 & e_2 & \cdots & e_{N-1} & 0 \end{pmatrix}, \quad \Psi_{\Delta} = \begin{pmatrix} 0 & \cdots & 0 & \Psi_{1N} \\ 0 & \cdots & 0 & \Psi_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & \Psi_{N-1,N} \\ 0 & \cdots & 0 & 0 \end{pmatrix} . \quad (3.16)$$

Substituting (3.15) into the relation (3.11) we obtain

$$\sum_{\ell=0}^{N} r_\ell\, T(M\Psi_\Delta^{(*)} \circ V_\ell) - TM\Psi_\Delta^{(*)} =$$

$$= -\sum_{\ell=0}^{N} \mathcal{P}^\ell (M\Psi_\Delta^{(*)} \circ V_\ell) - \partial_y M\Psi_\Delta^{(*)} . \qquad (3.17)$$

Let us now rewrite the equality (3.17) directly in the components $\hat{\varphi}_{(n)\Delta}^{(*)}$ of the infinite-component column $\Psi_\Delta^{(*)}$. Taking into account the explicit forms of the operators $\mathcal{P}$ and T (see (3.4) and (3.5)) one find

$$G\,\hat{\varphi}_{(n+1)\Delta}^{(*)} = \mathcal{F}\,\hat{\varphi}_{(n)\Delta}^{(*)} + \sum_{m=0}^{n-1} \mathcal{F}_{(n,m)}\, \hat{\varphi}_{(m)\Delta}^{(*)} \qquad (3.18)$$

$$(n = 0,1,2,\ldots,)$$

where

$$G\cdot \stackrel{def}{=} \sum_{\ell=0}^{N} r_\ell (\tilde{M}\cdot \circ V_\ell) - \tilde{M}\cdot ,$$

$$\mathcal{F}\cdot \stackrel{def}{=} -\sum_{\ell=0}^{N} \tilde{\mathcal{P}}^\ell (\tilde{M}\cdot \circ V_\ell) - \partial_y \tilde{M}\cdot , \qquad (3.19)$$

$$\mathcal{F}_{(n,m)} \stackrel{def}{=} -\sum_{\ell=0}^{N}\sum_{m'=0}^{n} (\mathcal{P}^\ell)_{(n,m')} (M_{(m',m)}\cdot \circ V_\ell) .$$

Operator $\tilde{M}$ is of the form (3.16) where instead of ℓ_k one must put the operators $\tilde{\ell}_k$ which are

$$\tilde{\ell}_k \cdot = \frac{1}{N}\sum_{\ell=1}^{N} \partial_x^{-1} (\tilde{\mathcal{P}}^\ell)_{1k} (\cdot \circ V_\ell) + \frac{1}{N}\delta_{1k}\partial_x^{-1}\partial_y . \qquad (3.20)$$

The relations (3.18) contain only independent quantities

$\hat{\varphi}_\Delta^{(*)}$. Let us introduce the N - 1-component column $\hat{X}_{(n)} = (\hat{\varphi}_{(n)1N}^{(*)}, \hat{\varphi}_{(n)2N}^{(*)}, \ldots, \hat{\varphi}_{(n)N-1,N}^{(*)})^T$. In the terms of these quantities the relation (3.18) is

$$\tilde{G}\hat{X}_{(n+1)} = \tilde{\mathcal{F}}\hat{X}_{(n)} + \sum_{m=0}^{n-1} \tilde{\mathcal{F}}_{(n,m)} \hat{X}_{(m)} \qquad (3.21)$$

where operators $\tilde{G}$, $\tilde{\mathcal{F}}$, $\tilde{\mathcal{F}}_{(n,m)}$ are block matrices of the order N - 1 . Their matrix elements are

$$\tilde{G}_{ik}\cdot = \sum_{\ell=0}^{N} (r_\ell)_{i+1,k} (\cdot \circ V_\ell) - \delta_{k,i+1} I_M ,$$

$$\tilde{\mathcal{F}}_{ik}\cdot = -\sum_{\ell=0}^{N} \{ (\tilde{\mathcal{P}}^\ell)_{i+1,k} (\cdot \circ V_\ell) + (\tilde{\mathcal{P}}^\ell)_{iN} (\tilde{\ell}_k \cdot)\circ V_\ell \} -$$

$$- \delta_{k,i+1} I_M \partial_y - \delta_{Nk}\partial_y \tilde{\ell}_k , \qquad (3.22)$$

$$(\tilde{\mathcal{F}}_{(n,m)})_{ik}\cdot = -\sum_{\ell=0}^{N} \{ (\mathcal{P}^\ell_{(n,m)})_{i+1,k} (\cdot \circ V_\ell) +$$

$$+ \sum_{\ell=0}^{N}\sum_{m'=0}^{n} (\mathcal{P}^\ell_{(n,m')})_{iN} ((\ell_k)_{(m',m)} \cdot) \circ V_\ell \} .$$

$$(i,k = 1,\ldots, N-1)$$

It is not difficult to show that the operator $\tilde{G}$ is lower-triangular one ($\tilde{G}_{ii} = -N\partial_x$, $i=1,\ldots,N-1$) and it has no nontrivial kernel. As a result from (3.21) we have

$$\hat{X}_{(n+1)} = \tilde{G}^{-1}\tilde{\mathcal{F}}\hat{X}_{(n)} + \sum_{m=0}^{n-1} \tilde{G}^{-1}\tilde{\mathcal{F}}_{(n,m)}\hat{X}_{(m)} . \qquad (3.23)$$

$$(n=0,1,2,\ldots,)$$

From the relations (3.23) it follows that there exist the recursion operators $\hat{\Lambda}_n$ such that

$$\hat{\chi}_{(n)} = \hat{\Lambda}_n \chi_{(0)} \qquad (n=1,2,3,\dots). \qquad (3.24)$$

The operators $\hat{\Lambda}_n$ are calculated by the recurrence relations

$$\hat{\Lambda}_{n+1}\cdot = \hat{\Lambda}_1 \hat{\Lambda}_n\cdot + \sum_{m=0}^{n-1} \tilde{G}^{-1} \tilde{\mathcal{F}}_{(n,m)} \hat{\Lambda}_m\cdot ,$$

$$\hat{\Lambda}_1 = \tilde{G}^{-1} \tilde{\mathcal{F}}, \qquad \hat{\Lambda}_0 \equiv I_{NM} \qquad (3.25)$$

where the operators $\tilde{G}$, $\hat{\mathcal{F}}$ and $\tilde{\mathcal{F}}_{(n,m)}$ are given by the formulas (3.22).

The operators $\hat{\Lambda}_n$ are just the recursion operators which we are interesting in.

In the analogous manner one can show that

$$\check{\chi}_{(n)} = \check{\Lambda}_n \chi_{(0)} \qquad (n=1,2,3,\dots) \qquad (3.26)$$

where recursion operators $\check{\Lambda}_n$ are calculated by the recurrence relations analogous to (3.25). It is easy also to show that

$$\check{\Lambda}_n = \sum_{k=0}^{n} (-1)^k C_k^n \, \partial_y^{\,n-k} \hat{\Lambda}_k . \qquad (3.27)$$

In the further constructions we will also need the

operators $\hat{\Lambda}^+_n$ and $\check{\Lambda}^+_n$ adjoint to the operators $\hat{\Lambda}_n$ and $\check{\Lambda}_n$ with respect to bilinear form

$$\ll \chi' \chi \gg = \int_{-\infty}^{+\infty} dx\,dy\; tr\left(\chi'^{T}(x,y)\,\chi(x,y)\right)$$

where χ and χ' are column with N - 1 components. The recurrence relations for calculation of the operators $\hat{\Lambda}^+_n$ are of the form

$$\hat{\Lambda}^+_{n+1}\cdot = \hat{\Lambda}^+_n \hat{\Lambda}^+_1\cdot + \sum_{m=0}^{n-1} \hat{\Lambda}^+_m \tilde{\mathcal{F}}^+_{(n,m)} (\tilde{G}^+)^{-1},$$

$$\hat{\Lambda}^+_1 = \hat{\mathcal{F}}^+ (\hat{G}^+)^{-1} \qquad (n=1,2,3,\dots) \qquad (3.28)$$

where matrix elements of the operators $\tilde{G}^+$, $\tilde{\mathcal{F}}^+$ and $\tilde{\mathcal{F}}^+_{(n,m)}$ are

$$\tilde{G}^+_{ik}\cdot = \sum_{l=0}^{N} V_l \left(\Gamma_l^+\right)_{i,k+1} - \delta_{i,k+1} I_M ,$$

$$\tilde{\mathcal{F}}^+_{ik}\cdot = -\sum_{l=0}^{N} V_l \left(\hat{\mathcal{P}}^+\right)^l_{i,k+1} + \delta_{kN}\, \tilde{e}^+_k \partial_y - \sum_{l=0}^{N} \tilde{e}^+_i V_l \left(\hat{\mathcal{P}}^+\right)^l_{NK} + \delta_{i,k+1} I_M \partial_y , \qquad (3.29)$$

$$\left(\tilde{\mathcal{F}}^+_{(n,m)}\right)_{ik}\cdot = -\sum_{l=0}^{N} V_l \left((\mathcal{P}^+)^l_{(n,m)} \right)_{i,k+1} - \sum_{l=0}^{N} \sum_{m'=0}^{n} (e^+_i)_{(m',m)} \left((\mathcal{P}^+)^l_{(n,m)} \right)_{NK}$$

and

$$(\ell_i^+)_{(n,m)} = \frac{1}{N}\sum_{\ell=1}^{N} V_\ell \left((\mathcal{P}^+)^\ell_{(n,m)} \right)_{K1} \partial_x^{-1} + \frac{1}{N}\delta_{K1}\delta_{(n,m)}\partial_y\partial_x^{-1}, \qquad (3.30)$$

$$\tilde{e}_i^{+} = \frac{1}{N}\sum_{\ell=1}^{N} V_\ell \left(\tilde{\mathcal{P}}^+\right)^\ell_{K1}\partial_x^{-1} + \frac{1}{N}\delta_{K1}\partial_y\partial_x^{-1}.$$

In the formulas (3.30) and in all adjoint operators (marked by +) $(\partial_x^{-1} f)(x,y) = \int_{-\infty}^{x} dx' f(x',y)$.

The operators Γ_ℓ^+ are calculated by the formulas

$$\Gamma_\ell^+ = \sum_{K_1+K_2=\ell-1} (\tilde{\mathcal{P}}^+)^{K_1} A^T (\tilde{\mathcal{P}}^+)^{K_2} \qquad (3.31)$$

where $\tilde{\mathcal{P}}^+ \overset{def}{=} \mathcal{P}^+_{nn}$ and

$$\mathcal{P}^+_{(n,m)} = \delta_{(n,m)}\partial_x - C_m^n \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdot V_0 \\ -I_M & 0 & \cdots & 0 & \cdot V_1 \\ 0 & -I_M & \cdots & 0 & \cdot V_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots 0 & -I_M & \cdot V_{N-2} \\ 0 & 0 & \cdots 0 & 0 & 0 \end{pmatrix}^{(n-m)}. \qquad (3.32)$$

Formulas (3.28)-(3.32) give somewhat cumbersome but direct method for calculation of the recursion operators $\hat{\Lambda}^+_n$. The operators $\check{\Lambda}^+_n$ can be found by the recurrence relations analogous to (3.28) or by the formula

$$\check{\Lambda}^+_n = (-1)^n \sum_{K=0}^{n} C_K^n \hat{\Lambda}^+_K \partial_y^{n-K}. \qquad (3.33)$$

IV. General form of the integrable equations

In the previous section it was shown that matrix elements of the matrices $\hat{\Phi}^{(*)}_{(n)}$, $\check{\Phi}^{(*)}_{(n)}$ can be expressed through $\hat{\chi}_{(m)}$ and $\check{\chi}_{(m)}$. And so let us transform the equality (2.16) into the form which contain only independent quantities $\hat{\chi}$ and $\check{\chi}$.

Taking into account the properties of the matrices P_∞ and $\tilde{P}$ one can show that the equality (2.16) is equivalent to the following one

$$\int_{-\infty}^{+\infty} dx\,dy \sum_{K=0}^{N-1}\sum_{n=0}^{\infty} b_{Kn}(t)\, tr\Big\{ P_\infty^K \tilde{P}'(x,y,t)\, \check{\Phi}^{(*)}_{(n)\Delta_{N-K}}(x,y,t) - \qquad (4.1)$$

$$- \tilde{P}(x,y,t)(-1)^{n+1}(P_\infty^T)^{N-K}\hat{\Phi}^{(*)}_{(n+1)\Delta_N}(x,y,t) - \tilde{P}(x,y,t)(-1)^n P_\infty^K \hat{\Phi}^{(*)}_{(n)\Delta_N}(x,y,t)\Big\} = 0.$$

From the relation analogous to (3.8) we find

$$\check{\Phi}^{(*)}_{(n)\Delta_{N-K}} = (\partial_x + A\partial_y + P' - AT)^{N-K}\check{\Phi}^{(*)}_{(n)\Delta_N}(A^T + P_\infty^T)^{N-K} -$$

$$-\sum_{\ell=0}^{N-K-1}(\partial_x + A\partial_y + P' - AT)^{\ell}\sum_{m=0}^{n} C_m^n \left(\check{\Phi}^{(*)}_{(m)\Delta_N}\tilde{P}_{(n-m)}\right)_{\Delta_{K+\ell+1}}(A^T + P_\infty^T)^{\ell+1} = \qquad (4.2)$$

$$\overset{def}{=} \sum_m \check{G}^{(K)}_{(n,m)}\, \check{\Phi}^{(*)}_{(m)\Delta_N}.$$

Passing on in the equality (4.1) from the matrices $\Phi^{(*)}_{(n)\Delta_N}$ to the columns $\chi_{(n)}$ and introducing the N - 1-component column $V(x,y,t)=\left(V_0(x,y,t),V_1(x,y,t),\dots,V_{N-1}(x,y,t)\right)^{\tau}$

we obtain

$$\sum_{k=0}^{N-1}\sum_{n=0}^{\infty} b_{kn}(t)\ll V'^{\tau}\sum_m \check{G}^{(k)}_{(n,m)}\check{\chi}_{(m)} -$$

$$-(-1)^{n+1}V^{\tau}\tilde{K}^{(k)}\hat{\chi}_{(n+1)}-(-1)^n V^{\tau}\sum_m \tilde{N}^{(k)}_{(n,m)}\hat{\chi}_{(m)}\gg = 0 \qquad (4.3)$$

where

$$\left(\tilde{G}^{(k)}_{(n,m)}\right)_{il} = \left(\check{G}^{(k)}_{(n,m)}\right)_{il}+\sum_{m'}\left(\check{G}^{(k)}_{(n,m')}\right)_{iN}\left(\check{\ell}_l\right)_{(m',m)}$$

$$\left(\tilde{K}^{(k)}\right)_{il} = \delta_{i,l+N-k}I_M, \qquad (4.4)$$

$$\left(\tilde{N}^{(k)}_{(n,m)}\right)_{il} = \delta_{i,l-k}I_M+\delta_{i,N-k}\left(\ell_l\right)_{(n,m)}.$$

$$(i,l=1,\dots,N-1)$$

Operators $\check{\ell}_k$ are calculated analogously to the operators ℓ_k.

Lastly, in virtue of (3.24) and (3.26) the equality (4.3) is equivalent to the equality

$$\ll \check{\chi}_{(0)}\cdot\sum_{k=0}^{N-1}\sum_{n=0}^{\infty} b_{kn}(t)\Big\{\sum_m \check{\Lambda}^{+}_m\tilde{G}^{(k)+}_{(n,m)}V'-$$

$$-\left((-1)^{n+1}\hat{\Lambda}^{+}_{n+1}\tilde{K}^{(k)+}+(-1)^n\sum_m\hat{\Lambda}^{+}_m\tilde{N}^{(k)+}_{(n,m)}\right)V\Big\} = 0 \qquad (4.5)$$

where operators $\hat{\Lambda}^{+}_n$ and $\check{\Lambda}^{+}_n$ are calculated by the formulas (3.28)-(3.33) and

$$\left(\tilde{G}^{(k)+}_{(n,m)}\right)_{il} = \left(\check{G}^{(k)+}_{(n,m)}\right)_{il}+\sum_{m'}\left(\check{\ell}^{+}_i\right)_{(m',m)}\left(\check{G}^{(k)+}_{(n,m')}\right)_{Nl},$$

$$\left(\tilde{K}^{(k)+}\right)_{il} = \delta_{i,l-N+k}I_M, \qquad (4.6)$$

$$\left(\tilde{N}^{(k)+}_{(n,m)}\right)_{il} = \delta_{i,l+k}I_M+\delta_{l,N-k}\left(\ell^{+}_i\right)_{(n,m)}.$$

$$(i,k,l=1,\dots,N-1)$$

The equality (4.5) is fulfilled if

$$\sum_{k=0}^{N-1}\sum_{n=0}^{\infty} b_{kn}(t)\Big\{\sum_m \check{\Lambda}^{+}_m\tilde{G}^{(k)+}_{(n,m)}V'-$$

$$-\left((-1)^{n+1}\hat{\Lambda}^{+}_{n+1}\tilde{K}^{(k)+}+(-1)^n\sum_m\hat{\Lambda}^{+}_m\left(\tilde{N}^{(k)+}\right)_{(n,m)}\right)V\Big\} = 0. \qquad (4.7)$$

Thus, we have found the transformations of the potential $V\to V'$ which correspond to the transformations of the scattering matrix $S\to S'$ of the form (2.11). These transformations $V\to V'$ are given by the relation (4.7) where $b_{kn}(t)$ are arbitrary functions.

It is not difficult to show that the transformations (2.11),(4.7) form an infinitedimensional abelian group. The transformations from this group is characterised by N functions $B_k(\lambda^N,t)$ entire on λ^N.

The infinitedimensional abelian group of the transformations (2.11),(4.7) which act on the manifold of the potentials $\{V(x,y,t)\}$ by the formula (4.7) and on the manifold of the scattering matrices $\{\hat{S}(\tilde{\lambda},\lambda,t)\}$ by the formula (2.11) plays a fundamental role in the analysis of the nonlinear systems connected with the problem (1.1) and their group-theoretical properties.

The group of the transformations (2.11),(4.7) contains the various type transformations. Let us consider the infinitesimal displacement in time t : $t \to t' = t + \varepsilon$, $\varepsilon \to 0$. In this case

$$V'(x,y,t) = V(x,y,t') = V(x,y,t) + \varepsilon \frac{\partial V(x,y,t)}{\partial t},$$

$$B_k(\lambda^N,t) = \delta_{k0} - \varepsilon \Omega_k(\lambda^N,t) = \delta_{k0} - \varepsilon \sum_{n=0}^{\infty} \omega_{kn}(t) \lambda^{Nn}.$$

Substituting these expressions into (4.7) and keeping the terms of the first order on ε we obtain

$$\frac{\partial V(x,y,t)}{\partial t} + \mathcal{L}_{\Omega}(L^+,t) V = 0 \qquad (4.8)$$

where

$$\mathcal{L}_{\Omega}(L^+,t) = \sum_{k=1}^{N-1} \sum_{n=0}^{\infty} \omega_{kn}(t) \Big\{ \sum_m^{V_+} \hat{L}^+_m \; \tilde{G}^{(k)+}_{(n,m)} \Big|_{V'=V} - \qquad (4.9)$$

$$- (-1)^{n+1} \hat{L}^+_{n+1} \tilde{K}^{(k)+} - (-1)^n \sum_m \hat{L}^+_m \big(\tilde{N}^{(k)+}\big)_{(n,m)} \Big|_{V'=V} \Big\}$$

and $L^+_n \stackrel{def}{=} \Lambda^+_n \big|_{V'=V}$. The operators $\hat{L}^+_n$ and $\check{L}^+_n$ are calculated by the formulas (3.28)-(3.33) at $V' = V$.

For the scattering matrix $\hat{S}$ under infinitesimal time displacement $\hat{S}'(\tilde{\lambda},\lambda,t) = \hat{S}(\tilde{\lambda},\lambda,t) + \varepsilon \frac{\partial \hat{S}(\tilde{\lambda},\lambda,t)}{\partial t}$ and correspondingly from (2.11) one have

$$\frac{\partial \hat{S}(\tilde{\lambda},\lambda,t)}{\partial t} = Y(\tilde{\lambda},t)\hat{S}(\tilde{\lambda},\lambda,t) - \hat{S}(\tilde{\lambda},\lambda,t) Y(\lambda,t) \qquad (4.10)$$

where $Y(\lambda,t) = \sum_{k=1}^{N-1} \Omega_k(\lambda^N,t) \bar{A}^k$.

Therefore we obtain nonlinear evolution equations in 1+2 dimensions (t, x, y) as the infinitesimal form of the transformations (4.7) generated by the time displacement.

The class of nonlinear equations (4.8) is characterised by arbitrary integers N and M, by recursion operators L^+_n and by arbitrary functions $\omega_{kn}(t)$ $(k=1,\dots,N-1)$. A choice of the concrete N, M and functions $\omega_{kn}(t)$ leads to the concrete equation of the form (4.8). The case N=2 will be considered in section 6.

Nonlinear evolution equations (4.8) in 1+2 dimensions are just the equations integrable by IST method with the help of the two-dimensional problem (1.1). With the use of the two-dimensional version of IST method (see [15,16]) one can find a broad class of the exact solutions of the equations (4.8).

In the conclusion of this section let us attract attention to the fact that in virtue of (4.10) the diagonal elements $\hat{S}_{\alpha\alpha}(\lambda,\lambda)$ of the scattering matrix are time-independent: $\frac{d\hat{S}_{\alpha\alpha}(\lambda,\lambda)}{dt} = 0$ at any functions $\Omega_k(\lambda^N,t)$. Therefore the quantities $\hat{S}_{\alpha\alpha}(\lambda,\lambda)$ at any λ are integrals of motion for the equations (4.8). If one expands $tr(\bar{A}^p \ln \hat{S}_D(\lambda,\lambda))$ $\big((S_D)_{\alpha\beta} \stackrel{def}{=} \delta_{\alpha\beta} S_{\alpha\alpha}\big)$ in the asymptotic series on λ^{-1} : $tr(\bar{A}^{p} \ln \hat{S}_{0}(\lambda,\lambda)) = \sum_{n=0}^{\infty} \lambda^{-n} C_n^{(p)}$ then one obtain a counting set of the integrals of motion $C_n^{(p)}$ $(p=1,\dots,N-1; n=0,1,2,\dots,)$. By standart procedure (see e.g. [1, 14]) one can find the explicit dependence of the integrals of motion $C_n^{(p)}$ on $V_0(x,y,t), V_1(x,y,t),\dots, V_{N-2}(x,y,t)$. Let us emphasize that these integrals of motion are universal one, i.e. they are integrals of motion for any equation of the form (4.8).

In the case $\frac{\partial V_k}{\partial y} = 0$ $(k=0,1,\dots,N-2)$ we have $\chi(x,y,t) = e^{-\lambda^N y}\, \tilde{\chi}(x,t,\lambda)$ and problem (1.1) is reduced to the one-dimensional Gelfand-Dikij spectral problem

$$\frac{\partial^N \tilde{\chi}}{\partial x^N} + V_{N-2}(x,t)\frac{\partial^{N-2}\tilde{\chi}}{\partial x^{N-2}} + \dots + V_0(x,t)\,\tilde{\chi} = \lambda^N \tilde{\chi}.$$

In this case the transformations (4.7) and the equations (4.8) are reduced to the corresponding transformations and equations connected with one-dimensional Gelfand-Dikij problem (see [13]).

V. Transformation properties of the integrable equations

General transformation properties of the equations (4.8) mainly are analogous to those for the equations integrable by the problem $\frac{\partial \psi}{\partial x} + A\frac{\partial \psi}{\partial y} + P(x,y,t)\psi = 0$ [14]. So we consider them briefly.

Group of the transformations (2.11),(4.7) plays a main role in the analysis of the general group-theoretical properties of the equations (4.8).

Let us firstly consider the transformations (2.11), (4.7) with time-independent matrices B≠C. These transformations form an infinitedimensional abelian group which as easy to see, does not change the evolution law (4.10) of the scattering matrix. Therefore they convert the solutions of the concrete equation of the form (4.8) into the solutions of the same equation, i.e. these transformations are auto Backlund transformations for the equations (4.8).

Group of the transformations (2.11),(4.7) contains also as subgroup the infinitedimensional abelian symmetry group of the equations (4.8). In the infinitesimal form these symmetry transformations are ($V \to V' = V + \delta V$)

$$\delta V = -\sum_{k=1}^{N-1}\sum_{n=0}^{\infty} f_{kn}\Big\{\sum_m \check{L}^{+}_m\, \tilde{G}^{(k)+}_{(n,m)}\Big|_{V'=V} - (-1)^{n+1}\hat{L}^{+}_{n+1}\,\tilde{K}^{(k)+} - (-1)^n \sum_m \hat{L}^{+}_m\, \tilde{N}^{(k)+}_{(n,m)}\Big|_{V'=V}\Big\} V \qquad (5.1)$$

where f_{kn} are arbitrary constant . The transformations (5.1) are symmetry transformations for any equation of the form (4.8).

And finally, the transformations (2.11),(4.7) with time-dependent matrices B and C are generalised Backlund transformations: they convert the solutions of certain equation (4.8) into the solutions of the other equation (with other functions $\omega_{kn}(t)$) of the form (4.8).

VI. An example N=2: nonstationary Schrödinger spectral problem

Here we illustrate the general results of the previous sections. Problem (1.1) at N=2 is nonstationary Schrodinger spectral problem $\frac{\partial \chi}{\partial y} + \frac{\partial^2 \chi}{\partial x^2} + V_0(x,y,t)\chi = 0$. In the scalar case M=1 this problem was used for the integration of Kadomtsev-Petviashvili equation [15,17,18] .

At N=2 the general equations (4.8) are of the form $(V_0 \equiv u)$

$$\frac{\partial u(x,y,t)}{\partial t} - \frac{1}{2}\sum_{n=0}^{\infty}\omega_{1n}(t)\Big\{\hat{L}^{+}_{n}\left(\partial_y\partial_x^{-1}u + \partial_x u + u\,\partial_x^{-1}u\right) + (-1)^n\check{L}^{+}_{n}\left(-\partial_y\partial_x^{-1}u + \partial_x u + (\partial_x^{-1}u)u\right) - \sum_{m=0}^{n}C^n_m\left(\hat{L}^{+}_{m}(\partial_x^{-1}u)u_{(n-m)} + (-1)^n\check{L}^{+}_{m}\,u_{(n-m)}\,\partial_x^{-1}u\right)\Big\} = 0 \qquad (6.1)$$

Recursion operators $\hat{L}^{+}_{n}$ and $\check{L}^{+}_{n}$ are calculated by the formulas (3.28)-(3.33) at N=2 ($u' = u$). For example

$$\hat{L}^{+}_{1}\cdot = -\frac{1}{4}\Big\{\left(\partial_y\partial_x^{-1}+\partial_x\right)^2 + 2\left[u(x,y),\cdot\right]_+ + \left[\partial_x u(x,y),\partial_x^{-1}\cdot\right]_+ - \left[u(x,y),\partial_y\partial_x^{-2}\cdot\right]_- + \partial_y\partial_x^{-1}\left[u,\partial_x^{-1}\cdot\right]_- + \left[u(x,y),\partial_x^{-1}\left[u,\partial_x^{-1}\cdot\right]_-\right]_-\Big\} \qquad (6.2)$$

and $\check{L}^{+}_{1} = -\hat{L}^{+}_{1} - \partial_y$ where $[A,B]_{\pm} \overset{def}{=} AB \pm BA$.

The simplest equation (6.1) corresponds to $\omega_{12} = \omega_{13} = \dots = 0$ and it is

$$\frac{\partial u(x,y,t)}{\partial t} - \omega_{10}(t)\frac{\partial u(x,y,t)}{\partial x} + \frac{\omega_{11}(t)}{4}\Big(\frac{\partial^3 u(x,y,t)}{\partial x^3} + 3\int_{-\infty}^{x}dx'\frac{\partial^2 u(x',y,t)}{\partial y^2} + 3\frac{\partial}{\partial x}\left(u^2(x,y,t)\right) + 3\Big[u(x,y,t),\int_{-\infty}^{x}dx'\frac{\partial u(x',y,t)}{\partial y}\Big]_-\Big) = 0. \qquad (6.3)$$

In the scalar case (M=1) and ω_{10} =0, ω_{11}=-4 the equation (6.3) is well known Kadomtsev-Petviashvili (KP) equation. At M>1 the equation (6.3) is matrix KP equation (see e.g. [19]). KP equation (6.3) is the lowest one (KP_1) from the infinite family (KP family) of the 1+2 dimensional equations (6.1): KP_n equation corresponds to $\omega_{11} = \dots = \omega_{1n-1} = \omega_{1n+1} = \dots = 0$, $\omega_{1n} = -2^{2n}$ (n=1,2,3,...).

The simplest Backlund transformation (BT) (4.7) corresponds to constant b_{00} and b_{10} and b_{kn} =0 (k=0,1; n=1,2,...,) and it is $B_b\,(u \to u')$:

$$b(u'-u) + \frac{\partial}{\partial x}(u'+u) - \int_{-\infty}^{x}dx'\frac{\partial}{\partial y}\left(u'(x',y,t) - u(x',y,t)\right) + \int_{-\infty}^{x}dx'\left(u'(x',y,t) - u(x',y,t)\right)u'(x,y,t) - u(x,y,t)\int_{-\infty}^{x}dx'\left(u'(x',y,t) - u(x',y,t)\right) = 0 \qquad (6.4)$$

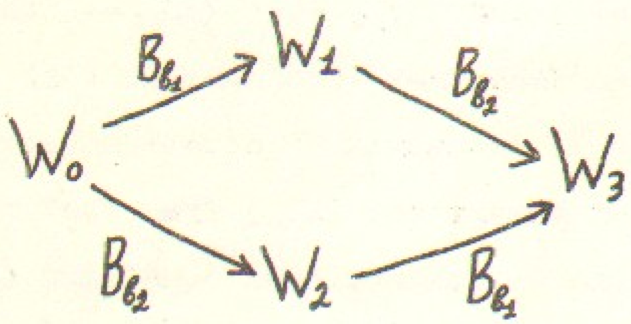
where $b = 2\frac{b_{00}}{b_{10}}$. Introducing a quantity $W(x,y,t) = \int_{-\infty}^{x}dx'\,u(x',y,t)$ we obtain a local form of BT (6.4):

$$b\frac{\partial}{\partial x}(W'-W)+\frac{\partial^2}{\partial x^2}(W'+W)-$$

$$-\frac{\partial}{\partial y}(W'-W)+(W'-W)\frac{\partial W'}{\partial x}-\frac{\partial W}{\partial x}(W'-W)=0. \qquad (6.5)$$

Let us emphasize that BT (6.4) is universal one, i.e. it is BT for any equation of the form (6.1).

BT (6.5) allow us to construct the infinite family of the solutions of the equations (6.1) by almost pure algebraic operations. Indeed, let us consider the following diagram



(6.6)

which expresses the commutativity of BTs (6.5): $B_{b_1}B_{b_2}=$ $=B_{b_2}B_{b_1}$. Here W_0, W_1, W_2, W_3 are four solutions of the concrete (but any) equation of the form (6.1). With the use of the relation (6.5) for all four solutions W_0, W_1, W_2, W_3 from (6.6) we obtain

$$W_3=(b_1-b_2+W_1-W_2)^{-1}\{(b_1-b_2)(W_1+W_2-W_0)-$$

$$-W_0(W_1+W_2)+W_1^2-W_2^2+2\frac{\partial}{\partial x}(W_1-W_2)\}. \qquad (6.7)$$

Therefore given three solutions W_0 , W_1 , W_2

one can easily calculate the fourth solution W_3 by the formula (6.7). Let us emphasize that the relation (6.7) is universal one, i.e. it is valid for all equations of the form (6.1) and in particular for any equation from KP family.

The relation (6.7) is just the nonlinear superposition principle for the equations (6.1). Some concrete nonlinear superposition principles for some concrete 1+1 dimensional equations (for example, for Korteweg- de Vries equation) are well known (see e.g. Ref. [20]).

In the scalar case (M=1) BT (6.5) and nonlinear superposition formula (6.7) are reduced to the following

$$b(W'-W)+\frac{\partial}{\partial x}(W'+W)+\frac{1}{2}(W'-W)^2-\int_{-\infty}^{x}dx'\frac{\partial}{\partial y}(W'(x',y)-W(x',y))=0 \qquad (6.8)$$

and

$$W_3=W_1+W_2-W_0+2\frac{\partial}{\partial x}\ln(b_1-b_2+W_1-W_2) \qquad (6.9)$$

which coincide at b=0 with those found earlier by another method in Ref. [21] .

Let us consider for definiteness scalar KP_2 equation (6.3). Let us start from the trivial solution W_0 =0. If one apply BT (6.8) to this solution then one obtain the solution W_1 which can be found from the equation

$$b_1\frac{\partial W_1}{\partial x}+\frac{\partial^2 W_1}{\partial x^2}-\frac{\partial W_1}{\partial y}+W_1\frac{\partial W_1}{\partial x}=0. \qquad (6.10)$$

One of the solutions of the equation (6.10) is well known soliton type solution of KP_1 equation (see e.g. [1]):

$$\mathcal{U}(x,y,t)=\frac{\left(\frac{b_1}{2}-a\right)^2}{2}\left(ch\,\frac{\left(\frac{b_1}{2}-a\right)x+\left(\left(\frac{b_1}{2}\right)^2-a^2\right)y-4\left(\left(\frac{b_1}{2}\right)^3-a^3\right)t+c}{2}\right)^{-2} \quad (6.11)$$

where a and c are arbitrary real constants and $b_1 < 0$.

Let us take now the trivial solution as W_0 and soliton type solutions (6.11) with constants b_1 and b_2 as W_1 and W_2. Then by formula (6.9) we find two-soliton solution W_3. An obvious proceeding of this procedure give any N-soliton solution of the KP_1 equation.

In the scalar case one can also obtain from (6.6) the another nonlinear superposition formula for BT (6.8). It is

$$W_3 = W_0 + \frac{(b_1+b_2)(W_2-W_1)-2\int_{-\infty}^{x}dx'\frac{\partial}{\partial y}\left(W_2(x',y)-W_1(x',y)\right)}{b_1-b_2+W_1-W_2}$$

which at $\frac{\partial W}{\partial y}=0$ is reduced to well known superposition formula for Korteweg- de Vries family of equations (see e.g. Ref. [20]).

VII. Conclusion

1. In addition to the problem (1.1) there exist another generalisation of the Gelfand-Dikij problem to the two dimensions, namely the problem

$$\frac{\partial^N \chi}{\partial x^N}+V_{N-2}(x,y,t)\frac{\partial^{N-2}\chi}{\partial x^{N-2}}+\dots+V_0(x,y,t)\chi-\frac{\partial^N \chi}{\partial y^N}=0. \quad (7.1)$$

Spectral problem (7.1) can be also obtained as a result of $\hat{Z}_N$ reduction of the general matrix problem $\frac{\partial \psi}{\partial x} + A\frac{\partial \psi}{\partial y} + P(x,y,t)\psi = 0$ [14].

For the two-dimensional problem (7.1) one can obtain all the relations analogous to those given in section 2. But the recursion operators of the type $\hat{\Lambda}_n$ and $\check{\Lambda}_n$ (with the properties (3.24) and (3.26)) do not exist for the problem (7.1). Therefore an essential modification of our constructions is needed for applicability of AKNS-technique to the problem (7.1).

2. Let us note also that in the present paper we consider direct scattering problem for (1.1) by treating variable x as time type variable, i.e. the scattering matrix S relates the asymptotics of the solutions χ of the problem(1.1) on x -infinites (at $x=+\infty$ and $x=-\infty$). In the paper [18] in the case N=2 ($\frac{\partial \chi}{\partial y}+\frac{\partial^2 \chi}{\partial x^2}+\mathcal{U}(x,y,t)\chi=0$) the standart version of the scattering problem for nonstationary Schrodinger equation was used in which a scattering matrix connects the solutions on y -infinites, i.e. on $y=+\infty$ and $y=-\infty$. The interrelation between these two approaches will be considered elsewhere.

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Работа поступила - I5 декабря I98I г.

Ответственный за выпуск - С.Г.Попов
Подписано к печати 3.II-I982г. МН 03070
Усл. I,5 печ.л., I,2 учетно-изд.л.
Тираж 290 экз. Бесплатно
Заказ № 20.

Отпечатано на ротапринте ИЯФ СО АН СССР
Novosibirsk-90, 630090, USSR