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THE GENERAL FORM OF NONLINEAR EVOLUTION EQUATIONS INTEGRABLE BY MATRIX GELFAND-DIKIJ SPECTRAL PROBLEM AND THEIR GROUP-THEORETICAL AND HAMILTONIAN STRUCTURES

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A b s t r a c t

The Gelfand-Dikij spectral problem is considered in the AKNS-technique framework. The general form of nonlinear evolution equations connected with matrix Gelfand-Dikij spectral problem is found. The infinitedimensional abelian group of general Backlund transformations is constructed. The infinite family of Hamiltonian structures connected with the nonlinear equations under consideration is found.

I. INTRODUCTION

The spectral problem

$$\frac{\partial^N \chi}{\partial x^N} + V_{N-2}(x,t)\frac{\partial^{N-2}\chi}{\partial x^{N-2}} + \ldots + V_1(x,t)\frac{\partial \chi}{\partial x} + V_0(x,t)\chi = \lambda^N \chi \quad (1.1)$$

and nonlinear differential equations connected with it are the objects of intensive study starting from the Gelfand-Dikij papers [1,2]. Then the algebraic and Hamiltonian structures of nonlinear equations connected with (1.1), some special Backlund transformations, the factorization of the operator in the left-hand side of (1.1) and some other important properties have been investigated in Refs. [3-10].

In the present paper we consider the spectral problem (1.1) where $V_0(x,t)$, $V_1(x,t), \ldots, V_{N-2}(x,t)$ are matrices of arbitrary order M such that $V_k(x,t) \xrightarrow[|x|\to\infty]{} 0$ ($k = 0,1,\ldots,N-2$) and N is arbitrary one in the framework of AKNS-technique. This technique has been formulated in Refs. [11,12] for the second order linear spectral problem $\frac{\partial \psi}{\partial x} = \lambda A \psi + P(x,t)\psi$. Then AKNS-technique has been generalized to the case of linear spectral problem of the arbitrary matrix order [13-20]. The AKNS- technique in the form which we will use in the present paper (generalized AKNS-technique) has been developed by one of the authors (B.G.K.) [16-19]. The advantage of AKNS- technique consists in the following: it allows

1) to find the general form of nonlinear equations connected with given spectral problem in simple and convenient form,
2) to calculate the infinitedimensional group of general Backlund transformations for these equations and 3) to investigate the Hamiltonian structure simultaneously of the whole class of the equations integrable by given spectral problem.

The main result of the present paper is a construction of the infinitedimensional abelian group of transformations (BC-group) connected with the spectral problem (1.1). This group acts on the manifold of the scattering matrices $\{S(\lambda,t)\}$ for the problem (1.1) in the following simple linear manner:

$$S(\lambda,t) \rightarrow S'(\lambda,t) = (\bar{B}(\lambda,t))^{-1} S(\lambda,t)\, \bar{C}(\lambda,t)$$ where

$\bar{B}(\lambda,t)$ and $\bar{C}(\lambda,t)$ are arbitrary diagonal matrices with elements $B_k(\lambda,t)$ which are arbitrary functions entire on λ. The action: $V\to V'$ of this BC-group on the manifold of potentials $\{V(x,t)\}$ where $V(x,t)$ is a column with N-1 matrix components $V_0(x,t), V_1(x,t), \ldots, V_{N-2}(x,t)$ is given by the formula

$$\sum_{k=0}^{N-1} B_k(\Lambda^+,t)(\mathcal{K}_k V'-\mathcal{M}_k V)=0 \tag{1.2}$$

where $B_k(\Lambda^+,t)$ are arbitrary functions entire on Λ^+ and $\Lambda^+, \mathcal{K}_k, \mathcal{M}_k$ are certain matrix integro-differential operators which depends only on V and V'. The explicit form of these operators is presented in section 4.

The infinitedimensional abelian group of transformations (1.2) plays a fundamental role in the analysis of nonlinear systems connected with spectral problem (1.1) and their group-theoretical properties. As we shall see, the nonlinear evolution equations integrable by (1.1) are the infinitesimal form of transformations (1.2) generated by time displacement. The general form of the integrable equations is

$$\frac{\partial V(x,t)}{\partial t}-\sum_{k=1}^{N-1}\Omega_k(L^+,t)\mathcal{L}_k V=0 \tag{1.3}$$

where $\Omega_k(L^+,t)$ are arbitrary functions meromorphic on L^+, and $L^+ \overset{def}{=} \Lambda^+(V'=V)$, $\mathcal{L}_k \overset{def}{=} \mathcal{K}_k(V'=V)-\mathcal{M}_k(V'=V)$.

The infinitedimensional group of transformations (1.2) is the group of general Backlund transformations for the equations (1.3). At $\frac{\partial B_k}{\partial t}=0$ $(k=0,1,\ldots,N-1)$ the transformations (1.2) are auto Backlund transformations for the equations (1.3): they convert the solutions of definite equation of the form (1.3) into the solutions of the same equation. If $\frac{\partial B_k}{\partial t}\neq 0$ then the transformations (1.2) are generalized Backlund transformations for the class of equations (1.3): such transformations convert the solutions of given equation (1.3) into the solutions of other equations of the form (1.3).

BC-group of transformations (1.2) contains also the infinitedimensional abelian symmetry group of the equations (1.3) as a subgroup. In the infinitesimal form these symmetry transformations are $(V\to V'=V+\delta V)$

$$\delta V(x,t)=\sum_{k=1}^{N-1} f_k(L^+)\mathcal{L}_k V \tag{1.4}$$

where $f_k(L^+)$ are arbitrary entire functions. Symmetry transformations (1.4) are connected with the infinite set of integrals of motion of the equations (1.3). Symmetry group (1.4) as well the group of auto Backlund-transformations is a universal one, i.e. any equation of the form (1.3) (with any functions $\Omega_k(L^+,t)$) is invariant under transformations (1.4).

We consider also the Hamiltonian structure of the equations (1.3). Hamiltonian character of the equations connected with problem (1.1) was proved in the first papers [2]. Here we show with the use of AKNS- technique that the infinite family of Hamiltonian structures is connected with the equations (1.3), namely the following infinite family of Poisson brackets $\{\ ,\ \}_n$ $(n=0,\pm1,\pm2,\ldots)$:

$$\{\mathcal{F},\mathcal{H}\}_n=\int_{-\infty}^{+\infty}dx\, tr\Big(\frac{\delta\mathcal{F}}{\delta V^{\tau\tau}}(L^+)^n \mathcal{J}\frac{\delta\mathcal{H}}{\delta V^{\tau}}\Big) \tag{1.5}$$

where $\mathcal{J}$ is certain matrix differential operator (Gelfand-Dikij operator). Bracket $\{\ ,\ \}_0$ is a well-known Gelfand-Dikij bracket [2]. Bracket $\{\ ,\ \}_1$ corresponds to second Hamiltonian structure which was considered in Refs. [4,21,22,8,9].

The paper is organized as follows. In the second section we consider the direct scattering problem for (1.1) which is rewritten in Frobenius form and obtain some important relations. In section 3 the recursion operators which play a main role in all our constructions are calculated. The infinitedimensional group of transformations (1.2) is constructed in section 4. The general form of the integrable equations is found in section 5. In section 6 the general group-theoretical properties (namely, group of Backlund transformations and symmetry group) of the equations (1.3) is discussed. Hamiltonian structure of the equations (1.3) is considered in section 7. In section 8 the explicit forms of recursion operators, integrable equations and Backlund transformations are presented in the cases N = 2,3,4. In the conclusion we briefly consider some other approaches to Gelfand-Dikij problem (1.1).

II. Some preliminary relations

First of all let us rewrite spectral problem (1.1) in the matrix form (in the well-known Frobenius form (see e.g. [23]))

$$\frac{\partial \psi}{\partial x} = A\psi + P(x,t)\psi \tag{2.1}$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \mathbb{1} & 0 & \dots & 0 \\ 0 & 0 & \mathbb{1} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \mathbb{1} \\ \lambda^N \mathbb{1} & 0 & 0 & \dots & 0 \end{pmatrix}, P = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 & 0 \\ -V_0, & -V_1, & \dots, & -V_{N-2} & 0 \end{pmatrix} \tag{2.2}$$

and $\mathbb{1}$ is identical matrix M x M. The element ψ_i of column ψ and matrix elements of A and P are matrices M x M.

The Frobenius form (2.1) of Gelfand-Dikij spectral problem (1.1) is more convenient for our purpose.

We will assume that $V_i(x,t) \to 0$ at $|x| \to \infty$ so fast that all integrals which will appear in our calculations exist.

Now we proceed to construction of transformations (1.2). Let us introduce, according to standart procedure (see e.g. [24]), the fundamental matrices-solutions $F^+(x,t,\lambda)$ and $F^-(x,t,\lambda)$ of the problem (2.1) given by their asymptotic behaviour

$$F^+(x,t,\lambda) \underset{x\to+\infty}{\longrightarrow} E(x,\lambda), \; F^-(x,t,\lambda) \underset{x\to-\infty}{\longrightarrow} E(x,\lambda) \tag{2.3}$$

where $E(x,\lambda)$ is fundamental matric-solution of matrix equation $\frac{\partial E}{\partial x} = AE$. This system of equations have, as it is well known (see e.g. [23]), the infinite set of fundamental solutions. We will consider the solution of this system, i.e. the asymptotic E of problem (2.1) of the form

$$E(x,\lambda) = \mathcal{D}(\lambda) e^{\bar{A}x} \tag{2.4}$$

where Ā is diagonal matrix: $\bar{A}_{ik} = \lambda q^{i-1} \delta_{ik} \mathbb{1}$,
$\mathcal{D}_{ik} = \frac{1}{\sqrt{N}} (\lambda q^{(k-1)})^{i-1} \mathbb{1}$ (i,k = 1,...N) and $q = exp(\frac{2\pi i}{N})$.
Here and below δ_{ik} is Kronecer symbol ($\delta_{ik} = \begin{cases} 1, & i=k \\ 0, & i\neq k \end{cases}$).
Let us note that $\lambda q^{(l-1)}$ $(l=1,\dots,N)$ are eigenvalues of matrix A, by definition $\lambda > 0$ and $A = \mathcal{D}\bar{A}\mathcal{D}^{-1}$.

In standart manner we introduce the scattering matrix $S(\lambda,t)$:

$$F^+(x,t,\lambda) = F^-(x,t,\lambda) S(\lambda,t) \tag{2.5}$$

or $S(\lambda,t) = (F^-(x,t,\lambda))^{-1} F^+(x,t,\lambda)$. Transition from one choice of the asymptotic of the problem (1.1) to the other ($E_1 \to E_2 = E_1 K$ where K is some nondegenerate matrix) leads only to a trivial redefinition of the scattering matrix ($S_1 \to S_2 = K^{-1} S_1 K$)*.

Let us take now two arbitrary potentials $P(x,t)$ and $P'(x,t)$ ($P(x,t) \underset{|x|\to\infty}{\longrightarrow} 0$, $P'(x,t) \underset{|x|\to\infty}{\longrightarrow} 0$) and two corresponding solutions $\psi(x,t,\lambda)$ and $\psi'(x,t,\lambda)$ of the problem (2.1). With the use of the equation (2.1) and the equation for ψ^{-1} $\left(-\frac{\partial \psi^{-1}}{\partial x} = \psi^{-1}(A+P)\right)$ one can to show that

$$\psi'(x,t,\lambda) - \psi(x,t,\lambda) = -\psi(x,t,\lambda) \int_x^{+\infty} dy\, \psi^{-1}(y,t,\lambda)(P'(y,t) - P(y,t))\psi'(y,t,\lambda). \tag{2.6}$$

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* One of the simplest asymptotic of (2.1) is $E_d = \exp A x$. It connected to asymptotic E by formula $E_d = ED^{-1}$.

Putting $\psi = F^{+}$ in (2.6) and going to the limit $x \to -\infty$ we obtain

$$S'(\lambda,t) - S(\lambda,t) = -S(\lambda,t)\int_{-\infty}^{+\infty} dx\, \psi'^{-1}(x,t,\lambda)(P'(x,t) - P(x,t))\psi'(x,t,\lambda). \quad (2.7)$$

Formula (2.7) which relates a change of potential $P(x,t)$ to a change of the scattering matrix $S(\lambda,t)$ plays a fundamental role in our further constructions.

The mapping $P(x,t) \longrightarrow S(\lambda,t)$ given by the spectral problem (2.1) establish a correspondence between the transformations $P \to P'$ on the manifold of potentials $\{ P(x,t),\ P(x,t) \xrightarrow[|x| \to \infty]{} 0 \}$ and the transformations $S \to S'$ on the manifold of the scattering matrices $\{ S(\lambda,t) \}$. This follows from the commutative diagramm

$$\begin{tikzcd} P \arrow[r, "(2.1)"] \arrow[d, "B"'] & S \arrow[d, "B"] \\ P' \arrow[r, "(2.1)"] & S' \end{tikzcd}$$

We will consider only such transformations B that

$$S(\lambda,t) \longrightarrow S'(\lambda,t) = \bar{B}^{-1}(\lambda,t)S(\lambda,t)\bar{C}(\lambda,t) \quad (2.8)$$

where $\bar{B}(\lambda,t)$ and $\bar{C}(\lambda,t)$ are arbitrary block-diagonal matrices (i.e. $B_{ik} = b_i(\lambda,t)\delta_{ik}\mathbb{1}$, $C_{ik} = C_i(\lambda,t)\delta_{ik}\mathbb{1}$, i,k = 1,...N). We confine ourselves by the transformations of the form (2.8) by two reasons: 1) the linearity of the transformation law (and, therefore, its readily integrability) of the scattering matrix is a main idea of the inverse scattering transform method (see e.g. [24]) and 2) the generalized AKNS--technique allows us to construct in an explicit form the transformations of the potential $P \to P'$ which correspond to the transformations of the scattering matrix of the form (2.8).

Let us rewritte the transformation law (2.8) in the form $S' - S = (1-\bar{B})S - S(1-C)$. From the comparison of it with (2.7) we find

$$(S^{-1}(1-B)S')_F = -\int_{-\infty}^{+\infty} dx ((F^{+})^{-1}(P'-P)(F^{+})')_F \quad (2.9)$$

where for arbitrary matrix φ we denote by φ_F the off-diagonal part of matrix φ : $(\varphi_F)_{\alpha\beta} = \varphi_{\alpha\beta} - \varphi_{\alpha\alpha}\delta_{\alpha\beta}$; $(\alpha,\beta = 1,\dots,NM)$. Here and below latin indices take values 1,2,...,N (or N-1) and numerate block matrix elements of the matrix NM x NM. Greek indices take values 1,2,..., NM and numerate usual matrix elements of the matrices of the order NM.

Further, it is not difficult to justified that the following identity holds

$$\begin{aligned} &(S^{-1}(\lambda,t)(\mathbb{1} - \bar{B}(\lambda,t))S(\lambda,t))_F = \\ &= ((F^{+}(x,t,\lambda))^{-1}(\mathbb{1} - B(\lambda'',t))(F^{+}(x,t,\lambda))')_F \Big|_{-\infty}^{+\infty} = \\ &= \int_{-\infty}^{+\infty} dx \{ ((F^{+}(x,t,\lambda))^{-1}(P(x,t)(\mathbb{1} - B(\lambda'',t)) - \\ &\qquad - (\mathbb{1} - B(\lambda'',t))P'(x,t))(F^{+}(x,t,\lambda))' \}_F \end{aligned} \quad (2.10)$$

where $B(\lambda'',t) = \mathcal{D}\bar{B}(\lambda,t)\mathcal{D}^{-1}$. Equalizing the left-hand and right-hand sides of the equalities (2.9) and (2.10) we obtain

$$\int_{-\infty}^{+\infty} dx \left\{ \left(F^{+}(x,t,\lambda)\right)^{-1} \left(B(\lambda^{N},t)P'(x,t) - P(x,t)B(\lambda^{N},t)\right)\left(F^{+}(x,t,\lambda)\right)' \right\}_{F} = 0 \,. \quad (2.11)$$

Rewritting (2.11) by components and introducing the quantity $(\overset{\pm\pm}{\varphi}{}^{(\alpha\beta)})_{\gamma\delta} \overset{def}{=\!=} (F^{+})'_{\gamma\beta} (F^{+})^{-1}_{\alpha\delta}$ $(\alpha, \beta, \gamma, \delta = 1, \ldots, NM)$ (tensor product) we have

$$\int_{-\infty}^{+\infty} dx \, tr\left(\left(B(\lambda^{N},t)P'(x,t) - P(x,t)B(\lambda^{N},t)\right)\overset{\pm\pm}{\varphi}{}^{(F)}(x,t,\lambda)\right) = 0 \quad (2.12)$$

where tr denote the full matrix trace.

Let us represent how the matrix $\bar{B}(\lambda,t)$ in the form

$$\bar{B}(\lambda,t) = \sum_{k=0}^{N-1} B_{k}(\lambda^{N},t)\bar{A}^{k} \quad (2.13)$$

where $B_{k}(\lambda^{N},t)$ are scalar functions and $\bar{A}^{0} \overset{def}{=\!=}$ identical matrix NM x NM. Since all block-elements of matrix A are different, any block-diagonal matrix B can be represented in the form (2.13) (see e.g. [25] , chapter VIII).

Correspondingly for $B(\lambda^{N},t)$ we have

$$B(\lambda^{N},t) = \sum_{k=0}^{N-1} B_{k}(\lambda^{N},t)A^{k} \,. \quad (2.14)$$

In virtue of (2.14) the equality (2.12) is equivalent to the following one

$$\left\langle \sum_{k=0}^{N-1} \left((A^{k}(\lambda^{N})P' - PA^{k}(\lambda^{N})) B_{k}(\lambda^{N},t)\overset{\pm\pm}{\varphi}{}^{(F)} \right\rangle = 0 \quad (2.15)$$

where $\langle \varphi \rangle \overset{def}{=\!=} \int_{-\infty}^{+\infty} dx \, tr(\varphi(x))$.

The equality (2.15) is the relation between $P(x,t)$, $P'(x,t)$ and $F^{+}(x,t,\lambda)$, $F^{+\prime}(x,t,\lambda)$ under transformations of the form (2.8). This equality contains the quantities $A(\lambda^{N})$, $B_{k}(\lambda^{N},t)$ $(k=0,\ldots,N-1)$ which explicitly depends on spectral parameter λ^{N}. Next step (which is standart for AKNS-technique) consists in the converting of the relation (2.15) into the form which does not contain explicit dependence on λ^{N}. In order to do this one must calculate so-called recursion operators.

III. Recursion operators

Not all of the elements of the quantity $\overset{\pm\pm}{\varphi}{}^{(F)}$ are independent one. Indeed, with the use of the equation (2.1) and equation $\frac{\partial F^{-1}}{\partial x} = -F^{-1}(A+P)$ one can to show that $\overset{\pm\pm}{\varphi}$ satisfies the equation

$$\frac{\partial \overset{\pm\pm}{\varphi}}{\partial x} = [A, \overset{\pm\pm}{\varphi}] + P'\overset{\pm\pm}{\varphi} - \overset{\pm\pm}{\varphi}P \,. \quad (3.1)$$

Since matrices A and P are of the special form (2.2) then matrix elements of $\overset{\pm\pm}{\varphi}$ can be expressed due to (3.1) through N-1 independent one. One can choose various N-1 elements of $\overset{\pm\pm}{\varphi}$ (or their superpositions) as the independent elements. The choice is determined by conveniency reasons. The equality (2.15) give us certain information about what elements of $\overset{\pm\pm}{\varphi}$ should be choose as independent one. Indeed, let us consider in (2.15) the term which corresponds to $k = 0$. Since $tr\left((P'-P)\overset{\pm\pm}{\varphi}{}^{(F)}\right)_{k=0} = -\sum_{i=2}^{N-1} tr\left((V'_{i-1} - V_{i-1})\overset{\pm\pm}{\varphi}{}^{(F)}_{iN}\right.$ then only matrix elements $\overset{\pm\pm}{\varphi}_{1N}$, $\overset{\pm\pm}{\varphi}_{2N}$, ..., $\overset{\pm\pm}{\varphi}_{N-1\,N}$ give a contribution into the term with $k = 0$. These N-1 elements of $\overset{\pm\pm}{\varphi}{}^{(F)}$, as we shall see, can be choose as the independ one. This choice of independent quantities is the most convenient for our purpose.

Let us introduce the operation Δ_{k} of projection onto k-nd column of matrix: $(\varphi_{\Delta_{k}})_{i\ell} \overset{def}{=\!=} \delta_{\ell k}\varphi_{i\ell}$ $(i,k,\ell = 1,\ldots,N)$. Appling the operation Δ_{k} to the equation (3.1) and taking into account (2.2) we obtain

$$\mathcal{P}\overset{++}{\Phi}_{\Delta_1} = \lambda^N \overset{++}{\Phi}_{\Delta_N} A^{1-N} + (\overset{++}{\Phi}_{\Delta_N} \cdot P)_{\Delta_1}, \qquad (3.2)$$

$$\overset{++}{\Phi}_{\Delta_k} = \mathcal{P}\overset{++}{\Phi}_{\Delta_{k+1}} A^{-1} - (\overset{++}{\Phi}_{\Delta_N} \cdot P)_{\Delta_{k+1}} \cdot A^{-1} \qquad (3.3)$$

$$(k = 1, \ldots, N-1)$$

where $\mathcal{P} \overset{def}{=} A - \partial + P'(x,t) \quad (\partial = \frac{\partial}{\partial x})$.

From the recursion relations (3.3) we find

$$\overset{++}{\Phi}_{\Delta_k} = \{\mathcal{P}^{N-k}\overset{++}{\Phi}_{\Delta_N} - \sum_{m=0}^{N-k-1} \mathcal{P}^{N-k-m}(\overset{++}{\Phi}_{\Delta_N} \cdot P)_{\Delta_{N-m}} A^m\} A^{k-N}. \qquad (3.4)$$

By introducing of the operators $\mathcal{P}_{(k)}$:

$$\mathcal{P}_{(k)} \overline{\Phi}_{\Delta_N} \overset{def}{=} \mathcal{P}^k \overline{\Phi}_{\Delta_N} - \sum_{m=0}^{k-1} \mathcal{P}^{k-m-1}(\overline{\Phi}_{\Delta_N} \cdot P)_{\Delta_{N-m}} \cdot A^m \qquad (3.5)$$

$$(k = 1, 2, \ldots, N-1)$$

where $\mathcal{P}_{(0)} = \mathcal{P}^0 = 1_{NM}$ one can rewrite (3.4) in the form

$$\overset{++}{\Phi}_{\Delta_k} = \mathcal{P}_{(N-k)} \overset{++}{\Phi}_{\Delta_N} \cdot A^{k-N} \quad (k = 1, 2, \ldots, N-1) \qquad (3.6)$$

Taking into account (2.2) and identity

$$(\Phi_{\Delta_N} P)_{\Delta_{N-m}} A^m = -\Phi_{\Delta_N} \circ V_{N-m-1} \qquad (3.7)$$

where $(\Phi \circ V_k)_{i\ell} \overset{def}{=} \varphi_{i\ell} \circ V_k \quad (i, k, \ell = 1, \ldots, N-1)$
one can represent operators $\mathcal{P}_{(k)}$ more compactly

$$\mathcal{P}_{(k)} \overset{++}{\Phi}_{\Delta_N} = \sum_{m=0}^{k} \mathcal{P}^{k-m}(\overset{++}{\Phi}_{\Delta_N} \circ V_{N-m}), \qquad (3.8)$$

$$(k = 1, 2, \ldots, N-1).$$

In particular, from (3.6) we have

$$\overset{++}{\Phi}_{\Delta_1} = \mathcal{P}_{(N-1)} \overset{++}{\Phi}_{\Delta_N} A^{1-N}. \qquad (3.9)$$

Substituting (3.9) into (3.3) we obtain

$$\sum_{m=0}^{N-1} \mathcal{P}^{N-m}(\overset{++}{\Phi}_{\Delta_N} \circ V_{N-m}) A^{1-N} + (\overset{++}{\Phi}_{\Delta_N} \circ V_0) A^{1-N} = \lambda^N \overset{++}{\Phi}_{\Delta_N} A^{1-N}$$

or

$$\sum_{m=0}^{N} \mathcal{P}^m(\overset{++}{\Phi}_{\Delta_N} \circ V_m) = \lambda^N \overset{++}{\Phi}_{\Delta_N}. \qquad (3.10)$$

Operators $\mathcal{P}^m$ in the left-hand side of (3.10) contain an explicit dependence on λ^N. Let us single out this dependence.

Since $P^2 = 0$ the operator $\mathcal{P}^k$ can be represented in the form

$$\mathcal{P}^k = (A-\partial)^k +$$
$$+ \sum_{m=1}^{k-1} \sum_{n_1+n_2+\ldots+n_{m+1}=k-m} (A-\partial)^{n_1} P'(A-\partial)^{n_2} \ldots P'(A-\partial)^{n_{m+1}} \qquad (3.11)$$

where total number of factors in each term in the sum is equal to k.

The operator $(A-\partial)^k$ is

$$(A-\partial)^k = \sum_{m=0}^{k} C_k^m (-\partial)^{k-m} A^m \qquad (3.12)$$

where $C_k^m = \frac{k!}{m!\,k!}$ and matrix A^m can be represented in the form

$$A^m = \lambda^N (R^T)^{N-m} + R^m \tag{3.13}$$

where $R_{ik} \overset{def}{=} \delta_{k,i+1}\, \mathbb{1} \quad (i,k=1,\dots,N)$ and symbol T denote block transposition.

Taking into account (2.2) and using (3.11)-(3.13) it is not difficult to show that $\mathcal{P}^k$ is linear function on λ^N.

$$\mathcal{P}^k = \lambda^N r_k + S_k \quad (k=1,\dots,N) \quad . \tag{3.14}$$

The matrices r_k and S_k are calculated by formulas

$$\begin{aligned} S_k = \mathcal{P}^k(\lambda^N=0) = (R-\partial)^k + \\ + \sum_{m=1}^{k} \sum_{n_1+\dots+n_{m+1}} (R-\partial)^{n_1} P'(R-\partial)^{n_2} \dots P'(R-\partial)^{n_{m+1}} \end{aligned} \tag{3.15}$$

and

$$r_k = \frac{d\mathcal{P}^k}{d(\lambda^N)}\bigg|_{\lambda^N=0} = \sum_{m=0}^{k-1} S_{k-1-m} \begin{pmatrix} 0 0 \dots 0 \\ \dots\dots \\ 0 0 \dots 0 \\ \mathbb{1} 0 \dots 0 \end{pmatrix} S_m \quad . \tag{3.16}$$

Using the simple properties of matrices A and R one can show that

$$r_k = \begin{array}{c} \\ \\ \\ k\Bigg\{ \\ \\ \\ \end{array} \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ \mathbb{1} & 0 & 0 & \dots & \dots & 0 \\ -C_k^1\partial & \mathbb{1} & 0 & \dots & \dots & 0 \\ * & -C_k^1\partial & \mathbb{1} & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \underbrace{* \quad \dots \quad *}_{k} & & -C_k^1\partial & \mathbb{1} & 0 \dots & 0 \end{pmatrix} \quad (k=1,\dots,N) \tag{3.17}$$

and

$$\begin{aligned} (S_k)_{1\ell} &= C_k^{\ell-1}(-\partial)^{k+1-\ell}, \quad \ell=1,\dots,k+1 \\ (S_k)_{1\ell} &= 0, \quad \ell=k+2,\dots,N \\ (S_N)_{1\ell} &= C_N^{\ell-1}(-\partial)^{N+1-\ell} - V'_{\ell-1}, \quad \ell=1,\dots,N \quad . \end{aligned} \tag{3.18}$$

The rest elements of r_k and S_k are more complicated.

Let us now substitute (3.14) into (3.10). As a result we have

$$\lambda^N \Big\{ \sum_{m=0}^{N} r_m \big(\overset{++}{\tilde{\varphi}}_{\Delta_N} \circ V_m\big) - \overset{++}{\tilde{\varphi}}_{\Delta_N} \Big\} = -\sum_{m=0}^{N} S_m \big(\overset{++}{\tilde{\varphi}}_{\Delta_N} \circ V_m\big) \quad . \tag{3.19}$$

The first from N nontrivial equations (3.19) permit us to express $\overset{++}{\tilde{\varphi}}_{NN}$ through $\overset{++}{\tilde{\varphi}}_{1N}, \overset{++}{\tilde{\varphi}}_{2N}, \dots, \overset{++}{\tilde{\varphi}}_{N-1N}$:

$$\overset{++}{\tilde{\varphi}}_{NN}(x,t,\lambda) = \sum_{k=1}^{N-1} \ell_k \overset{++}{\tilde{\varphi}}_{kN}(x,t\lambda) + \overset{++}{\tilde{\varphi}}_{NN}(x=+\infty) \tag{3.20}$$

where

$$\begin{aligned} \ell_k \cdot = -\frac{1}{N}\partial^{-1}\big((V'_{k-1}\cdot) - (\cdot V_{k-1})\big) - \\ - \frac{1}{N}(-\partial)^{N-k} C_N^{k-1} - \frac{1}{N}\sum_{m=0}^{N-k-1} (-\partial)^{N-m-k-1} C_{N-m-1}^{k-1} (\cdot \circ V_{N-m-1}) \quad . \end{aligned} \tag{3.21}$$

Here and below $(\partial^{-1} f)(x) \overset{def}{=} -\int_x^\infty dy\, f(y)$.

The relation (3.20) contains inhomogeneous term $\overset{++}{\tilde{\varphi}}_{NN}(x=+\infty)$. Similar terms, namely $\overset{++}{\tilde{\varphi}}^{(in)}(x=+\infty)$, will appear after integrations in the further calculations too. Taking into account (2.4) one can easy to show that

$$\big(\overset{++}{\tilde{\varphi}}^{(in)}(x=+\infty)\big)_{ke} = \mathcal{D}_{kn} (\mathcal{D}^{-1})_{ie} \lim_{x\to+\infty} e^{\lambda(q^{n-1}-q^{i-1})x} \quad . \tag{3.22}$$

Let us consider those $\overset{++}{\tilde{\varphi}}^{(in)}$ for which $Re(q^{n-1}-q^{i-1})<0$, i.e.

and

$$\cos\frac{2\pi(i-1)}{N} > \cos\frac{2\pi(n-1)}{N} \quad . \tag{3.23}$$

Since $\lambda > 0$ then for indices n and i which satisfy (3.23) we have $\lim\limits_{x\to+\infty} \exp(q^{n-1}-q^{i-1})\lambda x = 0$ and therefore $\overset{++}{\Phi}{}^{(in)}(x=+\infty) = 0$. The inequality (3.23) is satisfied for example for following values of n and i : $i, n < \frac{N}{4}+1,\ i<n$.

Let us denote by $\overset{++}{\Phi}{}^{(*)}$ the subspace of quantities $\overset{++}{\Phi}{}^{(in)}(x,t,\lambda)$ for which $\overset{++}{\Phi}{}^{(in)}(x=+\infty)=0$. Then in all relations which contain $\overset{++}{\Phi}{}^{(*)}$ the inhomogeneous terms $\overset{++}{\Phi}{}^{(*)}(x=+\infty)$ will be absent. In particular, instead of (3.20) we have

$$\overset{++}{\Phi}{}^{(*)}_{NN}(x,t,\lambda) = \sum_{k=1}^{N-1} \ell_k \overset{++}{\Phi}{}^{(*)}_{kN} \quad . \tag{3.24}$$

With the use of (3.24) we obtain

$$\overset{++}{\Phi}{}^{(*)}_{\Delta_N} = M \overset{++}{\Phi}{}^{(*)}_{\Delta} \tag{3.25}$$

where

$$M = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & \ldots & 1 & 0 \\ \ell_1 & \ell_2 & \ldots & \ell_{N-1} & 0 \end{pmatrix}, \quad \Phi_\Delta \overset{def}{=} \begin{pmatrix} 0 & \ldots & 0 & \Phi_{1N} \\ 0 & \ldots & 0 & \Phi_{2N} \\ . & . & . & . \\ 0 & \ldots & 0 & \Phi_{N-1N} \\ 0 & \ldots & 0 & 0 \end{pmatrix} \tag{3.26}$$

Substitution of (3.25) into (3.19) and use of the identities

$$r_k M = r_k \quad (k=1,\ldots,N-1)\ , \quad (r_N - 1_{NN})M = r_N \tag{3.27}$$

give

$$\lambda^N G \overset{++}{\Phi}{}^{(*)}_{\Delta} = \mathcal{F} \overset{++}{\Phi}{}^{(*)}_{\Delta} \tag{3.28}$$

where

$$G\cdot = \sum_{m=0}^{N} r_m(\cdot\circ V_m) - 1_{NN} \tag{3.29}$$

$$\mathcal{F} = -\sum_{m=0}^{N} s_m(M\cdot)\circ V_m \quad . \tag{3.29}$$

The relation (3.28) alrerdy containg only independent quantities $\overset{++}{\Phi}{}^{(*)}_{\Delta}$. However the matrix $\overset{++}{\Phi}{}^{(*)}_{\Delta}$ has many zero elements and have in mind further Hamiltonian treatment (section 7) it is convenient to introduce N-1 - component quantity $\chi^{(*)} = (\overset{++}{\Phi}{}^{(*)}_{1N}, \overset{++}{\Phi}{}^{(*)}_{2N}, \ldots, \overset{++}{\Phi}{}^{(*)}_{N-1N})^T$. The relation (3.28) is now of the form

$$\lambda^N \tilde{G} \chi^{(*)} = \tilde{\mathcal{F}} \chi^{(*)} \quad . \tag{3.30}$$

Matrices $\tilde{G}$ and $\tilde{\mathcal{F}}$ are of the order N-1 and equal

$$(\tilde{G})_{i\ell} = \sum_{m=0}^{N} (\tilde{r}_m)_{i\ell}(\cdot\circ V_m) - R_{i\ell} \tag{3.31}$$

$$(\tilde{\mathcal{F}})_{i\ell} = -\sum_{m=0}^{N} (\tilde{S}_m)_{i\ell}(\cdot\circ V_m) - \sum_{m=0}^{N} (S_m)_{iN}(\ell_\ell\,\cdot)\circ V_m \quad (i,\ell=1,\ldots,N-1)$$

where $R_{ik} = \delta_{k\,i+1} 1 \quad (i,k=1,\ldots,N-1)$ and

$$(\tilde{r}_k)_{i\ell} = (r_k)_{i+1,\ell}\,,$$
$$(\tilde{S}_k)_{i\ell} = (S_k)_{i+1,\ell}\,, \quad (i,\ell = 1,\ldots,N-1) \tag{3.32}$$

and matrices r_k and S_k are given by formulas (3.15) and (3.16).

It follows from (3.31),(3.32) and (3.15),(3.16) that

$$\tilde{G} = \begin{pmatrix} -1N\partial & 0 & 0 & \ldots & 0 & 0 \\ G_{21} & -1N\partial & 0 & \ldots & 0 & 0 \\ . & . & . & . & . & . \\ G_{N-11} & G_{N-12} & \ldots & & G_{N-1N-2} & -1N\partial \end{pmatrix} \tag{3.33}$$

Matrix $\widetilde{G}^{-1}$ is easily calculated. It is a lowertriangular one $\left((\widetilde{G}^{-1})_{ik} = 0, \quad i<k\right)$ and

$$(\widetilde{G})^{-1}_{ii} = -\frac{1}{N}\partial^{-1},$$

$$(\widetilde{G}^{-1})_{ik} = \sum_{l,\dots,m,n} (\widetilde{G})^{-1}_{ii}(-\widetilde{G}_{il})(\widetilde{G})^{-1}_{ll}\dots(-\widetilde{G}_{mn})(\widetilde{G})^{-1}_{nn}(-\widetilde{G}_{nk})(\widetilde{G})^{-1}_{kk} \quad (i>k) \tag{3.34}$$

where summation in (3.34) is performed over all possible dividing of integers from k to i into pairs. For example,

$$(\widetilde{G})^{-1}_{21} = -(\widetilde{G})^{-1}_{22}\widetilde{G}_{21}(\widetilde{G})^{-1}_{11}, \quad (\widetilde{G})^{-1}_{32} = -(\widetilde{G})^{-1}_{33}\widetilde{G}_{32}(\widetilde{G})^{-1}_{22},$$

$$(\widetilde{G})^{-1}_{31} = -(\widetilde{G})^{-1}_{33}\widetilde{G}_{31}(\widetilde{G})^{-1}_{11} + (\widetilde{G})^{-1}_{33}\widetilde{G}_{32}(\widetilde{G})^{-1}_{22}\widetilde{G}_{21}(\widetilde{G})^{-1}_{11}.$$

One can verify that matrix operator $\widetilde{G}^{-1}$ has no nontrivial kernel. Thus, from (3.30) we have

$$\Lambda\chi^{(*)}(x,t,\lambda) = \lambda^N\chi^{(*)}(x,t,\lambda) \tag{3.35}$$

where

$$\Lambda = \widetilde{G}^{-1}\widetilde{\mathcal{F}}. \tag{3.36}$$

The operator Λ is just a recursion operator which we are interested in. It plays a fundamental role in AKNS-technique. The explicit form of Λ can be found by formulas (3.36), (3.34), (3.31), (3.32), (3.15) and (3.16).

In our further constructions we will need the operator Λ^+ adjoint to operator Λ with respect to bilinear form $\langle\langle\chi'\chi\rangle\rangle \overset{def}{=} \sum_{i=1}^{N-1}\int_{-\infty}^{+\infty}dx\, tr\,(\chi'_i(x)\,\chi_i(x))$. The operator Λ^+ is calculated by standart rule $\left(\langle\langle\chi'\Lambda\chi\rangle\rangle = \langle\langle(\Lambda^+\chi')\chi\rangle\rangle\right)$ and it is

$$\Lambda^+ = \widetilde{\mathcal{F}}^+(\widetilde{G}^+)^{-1}. \tag{3.37}$$

Operators $\widetilde{\mathcal{F}}^+$ and $\widetilde{G}^+$ are

$$(\widetilde{\mathcal{F}}^+)_{ik} = -\sum_{m=0}^{N} V_m(S^+_m)_{i,k+1} - \sum_{m=0}^{N} l^+_i V_m(S^+_m)_{Nk},$$

$$(\widetilde{G}^+)_{ik} = \sum_{m=0}^{N} V_m(r^+_m)_{i,k+1} - \delta_{i,k+1}\mathbb{1}, \tag{3.38}$$

$$(i,k=1,\dots,N-1)$$

where

$$l^+_k\cdot = \frac{1}{N}\{(\partial^{-1}\cdot)V'_{k-1} - V_{k-1}(\partial^{-1}\cdot)\} - \mathbb{1}\frac{1}{N}\partial^{N-k}C^{k-1}_N -$$

$$- \frac{1}{N}\sum_{m=0}^{N-k-1} V_{N-m-1}\,C^{k-1}_{N-m-1}\,\partial^{N-m-k-1}. \tag{3.39}$$

$$(k=1,2,\dots,N-1)$$

In the operators l^+_k (3.39) and in all adjoint operators (denoted by symbol $+$) which will further appear $(\partial^{-1}f)(x) \overset{def}{=} \int_{-\infty}^{x} dy f(y)$.

The operators r^+_k and S^+_k are block matrices of the order N and defined analogously to r_k and S_k. Namely

$$\mathcal{P}^{+k} = \lambda^N r^+_k + S^+_k \tag{3.40}$$

where

$$\mathcal{P}^+ = \begin{pmatrix} \mathbb{1}\partial & 0 & \dots & 0 & \lambda^N-(\cdot V'_0) \\ \mathbb{1} & \mathbb{1}\partial & \dots & 0 & -(\cdot V'_1) \\ 0 & \mathbb{1} & \dots & 0 & -(\cdot V'_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \mathbb{1} & \mathbb{1}\partial \end{pmatrix}. \tag{3.41}$$

The explicit forms of S^+_k and r^+_k are calculated by formulas

$$S_k^+ = \mathcal{P}^{+k}(\lambda^N = 0) \qquad (3.42)$$

and

$$r_k^+ = \frac{d\mathcal{P}^{+k}}{d(\lambda^N)}\Big|_{\lambda^N=0} = \sum_{m=0}^{k-1} S_{k-1-m}^+ \begin{pmatrix} 0\dots 0\ 1 \\ 0\dots 0\ 0 \\ \vdots \\ 0\dots 0\ 0 \end{pmatrix} S_m^+ .$$

From (3.38) it follows that $\tilde{G}^+$ is upper-triangular matrix: $(\tilde{G})_{ik} = 0$, $i > k$ and $(\tilde{G}^+)_{ii} = N\partial$ $(i = 1, \dots, N-1)$. Matrix elements of the inverse matrix $(\tilde{G}^+)^{-1}$ are

$$(\tilde{G}^+)^{-1}_{ik} = 0, \quad i > k,$$

$$(\tilde{G}^+)^{-1}_{ii} = \frac{1}{N}\partial^{-1}, \qquad (3.43)$$

$$(\tilde{G}^+)^{-1}_{ik} = \sum_{\ell,\dots,m,n} (\tilde{G}^+)^{-1}_{ii}(-\tilde{G}^+_{i\ell})(\tilde{G}^+)^{-1}_{\ell\ell}\dots(\tilde{G}^+)^{-1}_{nn}(-\tilde{G}^+_{nk})(\tilde{G}^+)^{-1}_{kk}$$

where summation is performed as well as in (3.34) over all possible dividing of integers from i to k into pairs.

Formulas (3.37)-(3.43) give us somewhat cumbersome but direct procedure for calculation of the operator Λ^+. Explicit form of the operator Λ^+ for some concrete examples (N = 2,3,4) will be given in section 8.

IV. Construction of the transformations (1.2)

In the previous section it was shown that matrix elements of $\overset{\pm}{\Phi}^{(*)}$ can be expressed through the quantity $\chi^{(*)}$ (formulas (3.2),(3.6),(3.25)). Let us transform therefore the equality (2.15) into the form which contains only independent quantity $\chi^{(*)}$.

Using the properties of bilinear form $\langle\ \rangle$ and equalities (3.6),(3.25) we obtain

$$\langle A^k P' \overset{\pm\pm}{\Phi}^{(*)} - A^k \overset{\pm\pm}{\Phi}^{(*)} P \rangle = \langle P' \mathcal{P}_{(k)} \overset{\pm\pm}{\Phi}^{(*)}_{\Delta_N} - P A^k \overset{\pm\pm}{\Phi}^{(*)}_{\Delta_N} \rangle = \qquad (4.1)$$
$$= \langle (P'\mathcal{P}_{(k)} M - P A^k M) \overset{\pm\pm}{\Phi}^{(*)}_{\Delta} \rangle .$$

Let us single out the explicit dependence on λ^N in the operators $\mathcal{P}_{(k)}M$ and $A^k M$. In virtue of (3.6),(3.13),(3.14), (3,17) and (3.27) we have

$$\mathcal{P}_{(k)} M = \lambda^N G_{(k)} + \mathcal{F}_{(k)} , \qquad (4.2)$$

$$A^k M = \lambda^N (R^T)^{N-k} + R^k M \qquad (4.3)$$

where

$$G_{(k)} \overset{def}{=} \sum_{m=0}^{k} r_{k-m}(\cdot \circ V_{N-m}) , \qquad (4.4)$$

$$\mathcal{F}_{(k)} \overset{def}{=} \sum_{m=0}^{k} S_{k-m}((M\cdot)\circ V_{N-m}) , \qquad (4.5)$$

and matrix operators r_k and S_k are given by formulas (3.15), (3.16).

Substitution of (4.2) and (4.3) into (4.1) give

$$\langle A^k P' \overset{\pm\pm}{\Phi}^{(*)} - A^k \overset{\pm\pm}{\Phi}^{(*)} P \rangle =$$
$$= \langle (\lambda^N P' G_{(k)} + P'\mathcal{F}_{(k)} - \lambda^N P (R^T)^{N-k} - P R^k M) \overset{\pm\pm}{\Phi}^{(*)}_{\Delta} \rangle . \qquad (4.6)$$

If one introduce N-1-component column $V(x,t) \overset{def}{=} (V_0(x,t), V_1(x,t), \dots, V_{N-2}(x,t))^T$ and proceed from $\overset{\pm\pm}{\Phi}^{(*)}$ to $\chi^{(*)}$ then from (4.6) one obtain

$$\langle A^k P' \overset{\pm\pm}{\Phi}^{(*)} - A^k \overset{\pm\pm}{\Phi}^{(*)} P \rangle =$$
$$= \langle\langle (-\lambda^N V' \tilde{G}_{(k)} - V' \tilde{\mathcal{F}}_{(k)} + \lambda^N V \tilde{K}_{(k)} + V \tilde{N}_{(k)}) \chi^{(*)} \rangle\rangle \qquad (4.7)$$

where

$$(\tilde{G}_{(k)})_{ie} = \sum_{m=0}^{k} (r_{k-m})_{ie}(\cdot\circ V_{N-m}),$$

$$(\tilde{\mathcal{F}}_{(k)})_{ie} = \sum_{m=0}^{k} (S_{k-m})_{ie}(\cdot\circ V_{N-m}) + \sum_{m=0}^{k} (S_{k-m})_{iN}(\ell_e\cdot)\circ V_{N-m},$$

$$(\tilde{K}_{(k)})_{ie} = \delta_{i,e-N-k}\mathbb{1}, \tag{4.8}$$

$$(\tilde{N}_{(k)})_{ie} = \delta_{i,e-k}\mathbb{1} + \delta_{i,N-k}\ell_e,$$

$$(i,k,\ell = 1,\ldots,N-1).$$

Further in virtue of (3.35) for arbitrary entire function $B_k(\Lambda^N)$ we have

$$B_k(\Lambda^N)\chi^{(*)}(x,t,\lambda) = B_k(\Lambda)\chi^{(*)}(x,t,\lambda). \tag{4.9}$$

With the use of the equalities (4.7) and (4.9) one can rewrite (2.15) in the form

$$\ll \sum_{k=0}^{N-1}\left(-V'\tilde{G}_{(k)}\Lambda - V'\tilde{\mathcal{F}}_{(k)} + V\tilde{K}_{(k)}\Lambda + V\tilde{N}_{(k)}\right)B_k(\Lambda,t)\chi^{(*)}(\lambda)\gg = 0 \tag{4.10}$$

The equality (4.10) is the form of the equality (2.15) in which the explicit dependence on λ^N is eliminated. This elimination become possible due to existence of the recursion operators.

At last, the equality (4.10) is equivalent to the following one

$$\ll \chi^{(*)}(\lambda)\sum_{k=0}^{N-1} B_k(\Lambda^+,t)\left((\Lambda^+\tilde{G}^+_{(k)} + \tilde{\mathcal{F}}^+_{(k)})V' - (\Lambda^+\tilde{K}^+_{(k)} + \tilde{N}^+_{(k)})V\right)\gg = 0 \tag{4.11}$$

where Λ^+, $\tilde{G}^+_{(k)}$, $\tilde{\mathcal{F}}^+_{(k)}$, $\tilde{K}^+_{(k)}$, $\tilde{N}^+_{(k)}$ are operators adjoint to operators Λ, $\tilde{G}_{(k)}$, $\tilde{\mathcal{F}}_{(k)}$, $\tilde{K}_{(k)}$, $\tilde{N}_{(k)}$ with respect to bilinear form $\ll\chi'\chi\gg = \int_{-\infty}^{+\infty}\sum_{i=1}^{N-1} tr\,(\chi'_i(x)\chi_i(x))$. Operator Λ^+ is given by formulas (3.37)-(3.43) and

$$(\tilde{G}^+_{(k)})_{ie} = \sum_{m=0}^{k} V_{N-m}(r^+_{k-m})_{ie},$$

$$(\tilde{\mathcal{F}}^+_{(k)})_{ie} = \sum_{m=0}^{k} V_{N-m}(S^+_{k-m})_{ie} + \sum_{m=0}^{k} \ell^+_i V_{N-m}(S^+_{k-m})_{Ne},$$

$$(\tilde{K}^+_{(k)})_{ie} = \delta_{i,e-N+k}\mathbb{1}, \tag{4.12}$$

$$(\tilde{N}^+_{(k)})_{ie} = \delta_{i,e+k}\mathbb{1} + \delta_{e,N-k}\ell^+_i,$$

$$(i,k,\ell = 1,\ldots,N-1),$$

where operators r^+_k, S^+_k and ℓ^+_k are calculated by formulas (3.42),(3.39).

The equality (4.11) is just the relation between V, V' and $\chi^{(*)}$ under the transformations of the scattering matrix of the form (2.8). The equality (4.11) is satisfied if

$$\sum_{k=0}^{N-1} B_k(\Lambda^+,t)(\mathcal{K}_k V' - \mathcal{M}_k V) = 0 \tag{4.13}$$

where

$$\mathcal{K}_k \stackrel{def}{=} \Lambda^+\tilde{G}^+_{(k)} + \tilde{\mathcal{F}}^+_{(k)},$$

$$\mathcal{M}_k \stackrel{def}{=} \Lambda^+\tilde{K}^+_{(k)} + \tilde{N}^+_{(k)}. \tag{4.14}$$

If quantities $\chi^{(*)}(x,t,\lambda)$ form a complete set (as in the case N = 2) then the equation (4.13) is also necessary condition of fulfilment of (4.11).

Thus, we find the transformations of potential $V(x,t) \to V'(x,t)$ which correspond to the transformations (2.8). These transformations are given by the relation (4.13) where $B_k(\Lambda^+, t)$ are arbitrary entire functions on Λ^+.

It is important that the relation (4.13) contains only the potential V and transformed potential V'. We restricted overselvs by the transformations law of scattering matrix of the form (2.8) just in order that it will be possible to convert the transformation law of scattering matrix into the explicit transformation law of potential which contains only V and V'. It is remarkable that these "restricted" transformations (2.8),(4.13) are quite wide and, as we shall see, contain all the transformations typical for equations integrable by the spectral problem (2.1) and these integrable equations themselves.

It easy to see that the transformations (2.8),(4.13) form a group. Indeed, let we have two transformations of the type (2.8),(4.13): $S \to S_1 = \bar{B}_1^{-1} S \bar{C}_1$, $S_1 \to S_2 = \bar{B}_2^{-1} S_1 \bar{C}_2$. Since matrices $\bar{B}_1, \bar{B}_2, \bar{C}_1, \bar{C}_2$ are diagonal one, then

$$S \to S_2 = \bar{B}_2^{-1} S_1 \bar{C}_2 = \bar{B}_2^{-1} \bar{B}_1^{-1} S \bar{C}_1 \bar{C}_2 = (\bar{B}_2 \bar{B}_1)^{-1} S \bar{C}_2 \bar{C}_1 ,$$

i.e. the product of transformations of the type (2.8) is the transformation of the same type. In virtue of commutativity of diagonal matrices the group of transformations (2.8),(4.13) is abelian group, more exactly, abelian infinitedimensional group. The transformations from this group are indicated by N functions $B_k(\lambda^N, t)$ $(k = 0, 1, \ldots, N-1)$ entire on λ^N, which can be arbitrary one. The group of transformations (2.8),(4.13) can be considered as infinite-parametrical Lie group with parameters $b_{kn}(t)$ which are coefficients of the expansion of $\ln B_k(\lambda^N, t)$ on λ^N $\left(\ln B_k(\lambda^N, t) \overset{def}{=} \sum_{n=0}^{\infty} b_{kn}(t) (\lambda^N)^n\right)$.

We will refer the infinitedimensional abelian group of transformations (2.8),(4.13) as BC-group (Backlund-Calogero group)*.

BC-group which acts on the manifold of potential $\{V(x,t)\}$ by the formula (4.13) and on the manifold of the scattering

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* Backlund was the first who considered concrete transformation of the type (4.13) (see e.g. [26]) and Calogero constructed the general transformations of the form (4.13) (in the case N = 2, N = 1) for the first time [27].

matrices $\{S(\lambda, t)\}$ by the formula (2.8) plays a fundamental role in the analysis of nonlinear systems connected with problem (2.1) and their group-theoretical properties.

For the first time the importance of the transformations of the type (4.13) was emphasized by Calogero and Degasperis (see e.g. [27-29]). They used the technique of generalized Wronskians and constructed, in particular, the transformations (4.13) in the case N = 2 and arbitrary M [29].

A structure of BC-group is determined by the form of spectral problem. In the AKNS-technique framework BC-group has been constructed for general linear spectral problem $\frac{\partial \psi}{\partial x} = (\lambda A + P)\psi$ where A is constant matrix, for general polynomial bundle and some other spectral problems in Refs. [16,17,19].

V. General form of the integrable equations

BC-group constructed in the previous section contains the transformations of various types. Let us consider its one-parameter subgroup given by matrices

$$\bar{B}(\lambda, t) = \bar{C}(\lambda, t) = \sum_{k=0}^{N-1} e^{-\int_t^{t'} ds\, \Omega_k(\lambda^N, s)} \cdot \bar{A}^k \tag{5.1}$$

where $\Omega_k(\lambda^N, s)$ are some (in general, arbitrary) functions entire on λ^N. It is easy to show that the transformation (2.8) with matrices $\bar{B}$ and $\bar{C}$ of the form (5.1) is a displacement in time t:

$$S(\lambda, t) \to S'(\lambda, t) = \sum_{k=0}^{N-1} e^{\int_t^{t'} ds\, \Omega_k(\lambda^N, s)} (\bar{A}^{-1})^k \times$$
$$\times S(\lambda, t) \sum_{\ell=0}^{N-1} e^{-\int_t^{t'} ds\, \Omega_\ell(\lambda^N, s)} \bar{A}^\ell = S(\lambda, t') . \tag{5.2}$$

The corresponding transformation of potential is $V(x,t) \to V(x,t')$ and is given by formula

$$\sum_{k=0}^{N-1} e^{-\int_t^{t'} ds\, \Omega_k(\Lambda^+, s)} \left(\mathcal{K}_k V(t') - \mathcal{M}_k V(t)\right) = 0 \tag{5.3}$$

where in the operators Λ^+, $\mathcal{K}_k$ and $\mathcal{M}_k$ one must put $V'(x,t) = V(x,s)$. For the first time the transformations of the type (5.3) were considered in Ref. [27], (for problem (1.1) at N = 2, M = 1). See also Refs. [28,29,16,17,19].

At fixed functions $\Omega_k(\lambda^N, t)$ one-parameter group of the transformations (5.3) determine a flow Y_Ω: $V(x,t) \to V(x,t')$, in other words, an evolution system. This evolution system can be also described by certain nonlinear evolution equation.

Indeed, let us consider the infinitesimal displacement in time: $t \to t' = t + \varepsilon$ where $\varepsilon \to 0$. In this case

$$V(x,t') = V(x,t) + \varepsilon \frac{\partial V(x,t)}{\partial t},$$

$$B_k(\lambda^N, t) = \delta_{k0} - \varepsilon \Omega_k(\lambda^N, t), \tag{5.4}$$

$$k = 0, 1, \ldots, N-1)$$

Substituting (5.9) into (5.3), taking into account that $\mathcal{K}_0 = \mathcal{M}_0 = 1_{NM}$ and keeping the terms of the first order on ε we obtain

$$\frac{\partial V(x,t)}{\partial t} - \sum_{k=1}^{N-1} \Omega_k(L^+, t) \mathcal{L}_k V = 0 \tag{5.5}$$

where $L^+ \overset{def}{=} \Lambda^+|_{V'=V}$, $\mathcal{L}_k \overset{def}{=} \mathcal{K}_k|_{V'=V} - \mathcal{M}_k|_{V'=V}$. Operators L^+, $\mathcal{K}_k|_{V'=V}$, $\mathcal{M}_k|_{V'=V}$ are calculated by formulas (3.37)-(3.43), (4.14), (4.12) at $V' = V$.

It is not difficult to show that operators $\mathcal{L}_k$ can be also represented in the form

$$\begin{aligned}\mathcal{L}_k = {} & 1_{(N-1)M}\partial + 2L^+ \sum_{m=0}^{k-1} \partial^{k-m} R^{N-m} + \\ & + 2\sum_{m=1}^{k-1} \widetilde{M}(R^T_{(N-1)})^{m-1} \partial^{k-m} - \\ & - 1_{(N-1)M} \sum_{m=0}^{k-1} \left((\partial^{k-m-1} V_{N-m-1}) \cdot - V_{N-m-1} \partial^{k-m-1} \cdot \right)\end{aligned} \tag{5.6}$$

where $(R_{(N-1)})_{ik} = \delta_{k,i+1} \mathbb{1}$ $(i, k = 1, \ldots, N-1)$ and

$$\widetilde{M} = \begin{pmatrix} 0 & 0 & \ldots & 0 & \ell_1^+ \\ 1 & 0 & \ldots & 0 & \ell_2^+ \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & \ell_{N-2}^+ \\ 0 & 0 & \ldots & 1 & \ell_{N-1}^+ \end{pmatrix}$$

In particular, for arbitrary N $\mathcal{L}_1 = 1_{(N-1)M}\partial$.

For the scattering matrix under transformation (5.4) $S(\lambda, t') = S(\lambda, t) + \varepsilon \frac{\partial S(\lambda,t)}{\partial t}$ and from (5.2) we obtain linear evolution equation

$$\frac{dS(\lambda,t)}{dt} = [Y(\lambda,t), S(\lambda,t)] \tag{5.7}$$

where

$$Y(\lambda, t) = \sum_{k=1}^{N-1} \Omega_k(\lambda^N, t) \bar{A}^k.$$

Thus the consideration of the infinitesimal transformation (5.3) leads to nonlinear evolution equation (5.5). So evolution equation (5.5) give the flow Y_Ω: $V(x,t) \to V(x,t')$ in the infinitesimal form. The relation (5.3) which does not contain the derivative $\partial V/\partial t$ is an "integrated" form of the evolution equation (5.5).

Class of the equations (5.5) is characterized by integers N, M, by recursion operator L^+ and N-1 arbitrary functions $\Omega_1(\lambda^N, t), \Omega_2(\lambda^N, t), \ldots, \Omega_{N-1}(\lambda^N, t)$ entire on λ^N. The choice of concrete N , M and functions $\Omega_k(\lambda^N, t)$ give us concrete equation of the form (5.5). A few examples will be given in section 8.

The nonlinear evolution partial differential equations (5.5) are just the equations integrable by the inverse scattering transform method with the help of spectral problem (1.1). Using the equations of the inverse scattering problem for (1.1) (Gelfand-Levitan-Marchenko type equations) one can find, in principle, a broad class of exact solutions of the equations (5.5). At N = 2 and arbitrary M the inverse scattering problem was considered in Ref. [30]. At N = 3, M = 1 it was discussed in Refs. [31,32]. See also Ref. [24].

In the form (5.5) one can represent a broader class in the integrable equations. These are the equations

$$\sum_{m=1}^{p} f_m(L^+, t_1, \ldots, t_p) \frac{\partial V(x, t_1, \ldots, t_p)}{\partial t_m} - \sum_{k=1}^{N-1} \widetilde{\Omega}_k(L^+, t_1, \ldots, t_p) \mathcal{I}_k V = 0 \qquad (5.8)$$

where $f_m(\mu, t_1, \ldots, t_p)$ and $\widetilde{\Omega}_k(\mu, t_1, \ldots, t_p)$ are arbitrary functions entire on μ. In this case the scattering matrix $S(\lambda, t_1, \ldots, t_p)$ satisfies an equation

$$\sum_{m=1}^{p} f_m(\lambda^N, t_1, \ldots, t_p) \frac{\partial S(\lambda, t_1, \ldots, t_p)}{\partial t_m} = \left[\sum_{k=1}^{N-1} \widetilde{\Omega}_k(\lambda^N, t_1, \ldots, t_p) \bar{A}^k, S(\lambda, t_1, \ldots, t_p) \right].$$

At N = 2 the equations of the form (5.8) have been considered in Ref. [29].

In the case $p = 1$ the equations (5.8) are equivalent to the equations (5.5) with meromorphic functions $\Omega_k(\mu, t) = \widetilde{\Omega}_k(\mu, t) / f_1(\mu, t)$. In the present paper we will consider the

equations (5.8) (or (5.5)) only with one time-type variable t.

Let us turn the attention to the fact that in virtue of (5.7) at any functions $\Omega_k(\lambda^N, t)$ the diagonal elements $S_{\alpha\alpha}$ $(\alpha = 1, \ldots, NM)$ of the scattering matrix are time-independent

$$\frac{d S_{\alpha\alpha}}{d t} = 0 \quad (\alpha = 1, \ldots, NM).$$

So the quantities $S_{\alpha\alpha}(\lambda)$ $(\alpha = 1, \ldots, NM)$ at any value of spectral parameter λ are integrals of motion for the equations (5.5). One can extract a counting set of explicit integrals of motion from this continual set of inexplicit integrals of motion. Indeed, expanding $tr(\bar{A}^p \ln S_D(\lambda))$ (where $(S_D)_{\alpha\beta} \overset{def}{=} \delta_{\alpha\beta} S_{\alpha\alpha}$) in asymptotic series on λ^{-1}:

$$tr(\bar{A}^p \ln S_D(\lambda)) = \sum_{n=0}^{\infty} (\lambda)^{-n} C_n^{(p)}$$

we obtain a counting set of the integrals of motion $C_n^{(p)}$ $(p = 1, \ldots, N-1,\ n = 0, 1, 2, \ldots)$. With the use of standart procedure (see e.g. Refs. [24,17]) one can calculate an explicit dependence of $C_n^{(p)}$ on potential $V(x,t)$. At N = 3, M = 1 see e.g. Ref. [31].

An important property of the integrals of motion $C_n^{(p)}$ is their universality: they are integrals of motion for any equation of the form (5.5). Indeed, in their construction we use only the time-independence of $S_{\alpha\alpha}(\lambda)$ and the form of spectral problem (1.1) but not the concrete equations (5.5).

VI. Group-theoretical structure of the integrable equations

The general transformation properties of the equations (5.5) are mainly analogous to those for the equations integrable by spectral problem $\frac{\partial \psi}{\partial x} = (\lambda A + P(x,t))\psi$ (see Refs. [33,34]). In view of this we shall dwell upon them briefly.

BC-group considered in section 4 plays a main role in the analysis of the general transformation properties of the equations (5.5).

Let us consider the transformations (2.8),(4.13) with time-independent matrices $\bar{B}$ and $\bar{C}$. These transformations form an infinitedimensional subgroup of BC-group which we will refer as B-group (Bäcklund-group). It is easy to see that the transformation law of the scattering matrix (5.7) is invariant under transformation from B group, i.e. at B: $S \to S'$ we have $\frac{dS'}{dt} = [Y, S']$. Therefore, in virtue of one-to one correspondence between the equations (55) and (5.7) each concrete equation of the form (5.5) (with fixed function $\Omega_k(\lambda^N, t)$) converts into itself under the transformations of B-group. Thus, B-group is the infinitedimensional abelian group of auto-Backlund transformations: these transformations convert the solutions of definite equation (5.5) into the solutions of the same equation.

The simplest auto-Backlund transformation $V \to V'$ corresponds to constants B_k $(k = 0, \dots, N-1)$ and it is of the form

$$\sum_{k=0}^{N-1} B_k (\mathcal{K}_k V' - \mathcal{M}_k V) = 0 \qquad (6.1)$$

In the recent paper [10] an auto Backlund transformation for the equations connected with problem (1.1) at arbitrary N and M = 1 was considered. This Backlund transformation is a special case of the transformation (6.1) with $M = 1$, $B_0 = 0$, and certain constants $B_1, \dots, B_{N-1}$.

The explicit form of the transformation (6.1) at N = 2,3, will be given in section 8.

B-group contains as a subgroup the infinitedimensional abelian symmetry group of the equations (5.5). Symmetry transformations are the transformations (4.13) with $B_k = C_k = \exp f_k(\lambda^N)$ where $f_k(\lambda^N)$ $(k = 1, \dots, N-1)$ are arbitrary entire functions. Really such transformations, as it follows from (2.8), does not change the diagonal elements of the scattering matrix and, therefore, the integrals of motion $C_n^{(k)}$ and as a result they does not change the Hamiltonians of the equations (5.5).

In the infinitesimal form $V \to V' = V + \delta V$ symmetry transformatios are

$$\delta V(x,t) = \sum_{k=1}^{N-1} f_k(L^+) \mathcal{L}_k V \qquad (6.2)$$

The infinitedimensional abelian symmetry group can be also considered as infinite-parameter abelian Lie group. Indeed, let us expand entire functions $f_k(L^+)$ in the power series: $f_k(L^+) = \sum_{n=0}^{\infty} f_{kn} (L^+)^n$. As a result symmetry transformation (6.2) is rewritten as

$$\delta V = \sum_{k=1}^{N-1} \sum_{n=0}^{\infty} f_{kn} (L^+)^n \mathcal{L}_k V$$

i.e. as a superposition of the infinite number of one-parameter symmetry transformations

$$\delta_{(k,n)} V = f_{kn} (L^+)^n \mathcal{L}_k V \qquad (6.9)$$

where expansion coefficients f_{kn} $(-\infty < f_{kn} < +\infty)$ play a role of the transformation parameters. Let us note that symmetry transformations (6.3) are connected with integrals of motion $C_n^{(k)}$.

If one omit the transformation parameters f_{kn} from $\delta_{(k,n)} V$ then the corresponding quantities $\tilde{\delta}_{(k,n)} V$ ($\delta_{(k,n)} V = f_{kn} \tilde{\delta}_{(k,n)} V$) are related each to other by a simple formula

$$\tilde{\delta}_{(k,n+1)} V = L^+ \tilde{\delta}_{(k,n)} V \qquad (6.4)$$

At N = 2, M = 1 an analogous properties of the operator L^+ was firstly noted by Lenart (see Ref. [35]).

Let us emphasize now that B-group of Backlund-transformations and symmetry group are universal one: B-group and symmetry group are those for any equation of the form (5.5). The universality of the symmetry group is of the same nature as the universality of the integrals of motion.

Let us, lastly, consider the transformations (4.13) with time-dependent functions $B(\lambda,t) = \bar{C}(\lambda,t)$. In virtue of (2.8) the evolution law of the scattering matrix (5.7) conserve its form under such transformations too, i.e. under $S \to S' = \bar{B}^{-1}(t)S(t)\bar{B}(t)$

$$\frac{dS'(\lambda,t)}{dt} = [\,Y'(\lambda,t), S'(\lambda,t)] \qquad (6.5)$$

where

$$Y'(\lambda,t) = Y(\lambda,t) - \frac{\partial}{\partial t}\ln \bar{B}(\lambda,t). \qquad (6.6)$$

It follows from (6.5) that the transformations (2.8), (4.13) with time-dependent functions $B_k(\lambda,t)$ convert definite equation of the form (5.5) (with given $Y(\lambda,t)$) into other equations (with $Y'(\lambda,t)$ given by (6.6) of the form (5.5). We refer such transformations, follows Refs. [27-29] , as generalized Backlund transformations. These transformations connect the solutions of different equations from the class (5.5). It follows from (6.6) that generalized Backlund transformations act in a transitive manner on the whole family of the equations (5.5) (at fixed N and M).

Thus we see that BC-group of the transformations (2.8), (4.13) contains complete information on the general group-theoretical properties of the equations integrable by (1.1) and these integrable equations themselves.

VII. Hamiltonian structure of the integrable equations

Let us consider the equations (5.5) with arbitrary entire functions $\Omega_k(L^+_\cdot, t)$, i.e.

$$\Omega_k(L^+_\cdot, t) = \sum_{n=0}^{\infty} \omega_{kn}(t)(L^+)^n \qquad (7.1)$$

where $\omega_{kn}(t)$ are arbitrary functions. We will use the method developed in Refs. [12,16-19] for proof of the Hamiltonian character of these equations.

First of all let us note that from the relation (2.7) we have

$$\delta S_{\alpha\beta}(\lambda,t) = -\langle\!\langle \delta V^\tau \bar{\chi}^{+(\alpha\beta)}(\lambda,t) \rangle\!\rangle \qquad (7.2)$$

where $\delta V(x,t)$ is arbitrary variation of potential V, $\bar{\chi}^{+(\alpha\beta)}$ is a column with N-1 components $\bar{\varphi}^{+(\alpha\beta)}_{1N}(x,t,\lambda)$, $\bar{\varphi}^{+(\alpha\beta)}_{2N}(x,t,\lambda), \ldots,$ $\bar{\varphi}^{+(\alpha\beta)}_{N-1\,N}(x,t,\lambda)$ which are matrices $M \times M$ and $(\bar{\varphi}^{+(\alpha\beta)}(x,t,\lambda))_{\gamma\delta} \stackrel{def}{=} (F^+)_{\gamma\beta}(F^-)^{-1}_{\alpha\delta}$ $(\alpha,\beta,\gamma,\delta = 1,2,\ldots,NM)$. Here and below $\langle\!\langle \chi'\chi \rangle\!\rangle \stackrel{def}{=} \int_{-\infty}^{+\infty} dx\, tr\,(\chi'^\tau(x)\chi(x)) = \sum_{i=1}^{N-1}\int_{-\infty}^{+\infty} dx\, tr(\chi'_i(x)\chi_i(x))$ where χ and χ' are columns with N-1 components.

The relation (7.2) means that

$$(\bar{\chi}^{+(\alpha\beta)}(x,t,\lambda))_l = -\frac{\delta S_{\alpha\beta}(\lambda,t)}{\delta V^\tau_l(x,t)}, \begin{pmatrix} l=1,\ldots,N-1 \\ \alpha,\beta=1,\ldots,NM \end{pmatrix} \qquad (7.3)$$

where $\frac{\delta}{\delta V}$ denote a variational derivative and V_l^τ is a transposed matrix V_l .

In the further constructions we will use the quantity $\Pi^{(k)}$

$$\Pi^{(k)}(x,t,\lambda) \stackrel{def}{=} \sum_{\alpha=1}^{NM} (\bar{A})^k_{\alpha\alpha} \frac{\bar{\chi}^{+(\alpha\alpha)}(x,t,\lambda)}{S_{\alpha\alpha}(\lambda)} \qquad (7.4)$$

or $\Pi_l^{(k)}(x,t,\lambda) = (F^+\bar{A}^k(S_D)^{-1}(F^-)^{-1})_{lN}$ $(l=1,\ldots,N-1)$.

From (7.3) we have

$$\Pi^{(k)}(x,t,\lambda) = -\frac{\delta\, tr\,(\bar{A}^k \ln S_D(\lambda))}{\delta V^\tau(x,t)} \qquad (7.5)$$

that is our basic variational equality.

The further step is to express the nonlinear terms in the equation (5.5) through the quantities $\Pi^{(k)}$. It is necessary for this purpose to find the equations to which $\Pi^{(k)}$ satisfy. Let us start from the equation for quantity $\bar{\varphi}^{+(\alpha\beta)}$. It is obtained in a manner completely analogous to that is used for derivation of the equation for $\bar{\varphi}^{\pm}$ (section 3) and is of the form

$$\lambda^N\Big(\sum_{m=0}^{N} r_m(\bar{\varphi}^{+(\alpha\beta)}_{\Delta_N}\circ V_m) - \bar{\varphi}^{+(\alpha\beta)}_{\Delta_N}\Big) = -\sum_{m=0}^{N} S_m(\bar{\varphi}^{+(\alpha\beta)}_{\Delta_N}\circ V_m) \qquad (7.6)$$

where matrix operators r_m and S_m are given by formulas (3.15) and (3.16). From the first nontrivial equation (7.6) we have

$$-N\partial\bar{\varphi}^{+(\alpha\beta)}_{NN} = \sum_{k=1}^{N-1}[V_{k-1},\bar{\varphi}^{+(\alpha\beta)}_{kN}] - \sum_{k=1}^{N-1}(-\partial)^{N-k-1}C_N^{k-1}\bar{\varphi}^{+(\alpha\beta)}_{kN} -$$

$$-\sum_{k=1}^{N-1}\sum_{m=0}^{N-k-1}(-\partial)^{N-m-k}C^{k-1}_{N-m-1}\bar{\varphi}^{+(\alpha\beta)}_{kN}V_{N-m-1}, \qquad (7.7)$$

Integration of (7.7) over x give

$$\bar{\varphi}^{+(\alpha\beta)}_{NN}(x,t,\lambda) = \bar{\varphi}^{+(\alpha\beta)}_{NN}(x=-\infty) + \sum_{k=1}^{N-1}\bar{e}_k\bar{\varphi}^{+(\alpha\beta)}_{kN} \qquad (7.8)$$

where

$$\bar{e}_k\cdot \overset{def}{=} -\frac{1}{N}\partial^{-1}[V_{k-1},\cdot] - \frac{1}{N}(-\partial)^{N-k}C_N^{k-1} -$$

$$-\frac{1}{N}\sum_{m=0}^{N-k-1}(-\partial)^{N-m-k-1}C^{k-1}_{N-m-1}(\cdot V_{N-m-1}). \qquad (7.9)$$

Here and below $(\partial^{-1}f)(x) \overset{def}{=} \int_{-\infty}^{x} dy\, f(y)$.

Taking into account (2.4) one can show that

$$(\bar{\varphi}^{+(\alpha\alpha)}(x=-\infty))_{k\ell} = \frac{1}{N}(\lambda q^{[\frac{\alpha}{M}]})^{k-\ell} S_{\alpha\alpha}\mathbb{1}, \qquad (7.10)$$

$$(\alpha=1,\dots,NM;\ k,\ell=1,\dots,N)$$

where $[\frac{\alpha}{M}]$ denotes an integer part of the number $\frac{\alpha}{M}$. In particular, $(\bar{\varphi}^{+(\alpha\alpha)}(x=-\infty))_{NN} = \frac{1}{N}S_{\alpha\alpha}\mathbb{1}$. As a result

$$\bar{\varphi}^{+(\alpha\alpha)}_{\Delta_N} = \bar{M}\bar{\varphi}^{+(\alpha\alpha)}_{\Delta} + \frac{1}{N}\begin{pmatrix}0&\dots&0&0\\ \dots&\dots&\dots&\dots\\0&\dots&0&0\\0&\dots&0&S_{\alpha\alpha}\mathbb{1}\end{pmatrix} \qquad (7.11)$$

where $(\varphi_\Delta)_{ik} \overset{def}{=} \delta_{kN}\varphi_{iN}$, $(\varphi_\Delta)_{NN}=0$ and

$$\bar{M} \overset{def}{=} \begin{pmatrix}1&0&\dots&0&0\\0&1&\dots&0&0\\ \dots&\dots&\dots&\dots&\dots\\0&0&\dots&1&0\\ \bar{e}_1&\bar{e}_2&\dots&\bar{e}_{N-1}&0\end{pmatrix}. \qquad (7.12)$$

Substituting (7.11) into (7.6) and taking into account that

$$\sum_{m=0}^{N} S_m\begin{pmatrix}0&\dots&0&0\\ \dots&\dots&\dots&\dots\\0&\dots&0&0\\0&\dots&0&S_{\alpha\alpha}\mathbb{1}\end{pmatrix} = 0$$

we obtain

$$\lambda^N\Big\{\sum_{m=0}^{N} r_m(\bar{\varphi}^{+(\alpha\alpha)}_{\Delta}\circ V_m) - \bar{\varphi}^{+(\alpha\alpha)}_{\Delta}\Big\} = -\sum_{m=0}^{N} S_m(\bar{M}\bar{\varphi}^{+(\alpha\alpha)}_{\Delta})\circ V_m. \qquad (7.13)$$

Introducing the quantity $\bar{\chi}^{+(\alpha\alpha)} = (\bar{\varphi}^{+(\alpha\alpha)}_{1N},\dots,\bar{\varphi}^{+(\alpha\alpha)}_{N-1N})^T$ from (7.13) we have

$$\lambda^N\bar{G}^{+}\bar{\chi}^{+(\alpha\alpha)}(\lambda) = \bar{\mathcal{F}}^{+}\bar{\chi}^{+(\alpha\alpha)}(\lambda) \qquad (7.14)$$

where $\bar{G}^+$ and $\bar{\mathcal{F}}^+$ are matrix $(N-1)\times(N-1)$ operators. Their matrix elements are

$$(\bar{G}^+)_{ik} = \sum_{m=0}^{N}(r_m)_{i+1,k}(\cdot\circ V_m) - \delta_{k,i+1}\mathbb{1},$$

$$(\bar{\mathcal{F}}^+)_{ik} = -\sum_{m=0}^{N}(S_m)_{i+1,k}(\cdot\circ V_m) - \sum_{m=0}^{N}(S_m)_{iN}(\bar{e}_k\cdot)\circ V_m, \qquad (7.15)$$

$$(i,k=1,\dots,N-1).$$

It is easy to see that matrix $\vec{G}^{+}$ is lowertriangular one and $(\vec{G}^{+})_{ii} = -\mathbb{1}N\partial$ $(i = 1, \dots, N-1)$. Matrix elements of the inverse operator $\vec{G}^{+-1}$ are

$$(\vec{G}^{+})^{-1}_{ik} = 0 \ , \quad i < k, \ (i,k = 1, \dots, N-1)$$

$$(\vec{G}^{+})^{-1}_{ii} = -\mathbb{1}\frac{1}{N}\partial^{-1}, \tag{7.16}$$

$$(\vec{G}^{+})^{-1}_{ik} = \sum_{n,\dots,m} (\vec{G}^{+})^{-1}_{ii}(-\vec{G}^{+}_{in})(\vec{G}^{+})^{-1}_{nn} \dots (\vec{G}^{+})^{-1}_{mm}(-\vec{G}^{+}_{mk})(\vec{G}^{+})^{-1}_{kk}$$

where summation is performed over all possible dividing of integers from k to i into pairs.

Taking into account the nontrivial asymptotic of $\vec{\chi}^{+(\alpha\alpha)}$ at $x \to -\infty$ (see formula (7.10)) we obtain from (7.14) the following relation

$$\lambda^N \vec{\chi}^{+(\alpha\alpha)}(x,t,\lambda) = (\vec{G}^{+})^{-1} \vec{\mathcal{F}}^{+} \vec{\chi}^{+(\alpha\alpha)} + \lambda \widetilde{\widetilde{G}} \vec{\chi}^{+(\alpha\alpha)}(x=-\infty) \tag{7.17}$$

where $\widetilde{\widetilde{G}} = -N((\vec{G}^{+})^{-1}\partial)$.

Finally, multiplying left-and right-hand sides of (7.17) by $(\bar{A}^k)_{\alpha\alpha} \frac{1}{S_{\alpha\alpha}}$ and summing over α we have

$$\lambda^N \Pi^{(k)}(x,t,\lambda) = \tilde{L}\, \Pi^{(k)} + \lambda^N \widetilde{\widetilde{G}}\, \Pi^{(k)}(x=-\infty), \quad (k = 1, \dots, N-1) \tag{7.18}$$

where $\Pi^{(k)}$ are defined by (7.9) and

$$\tilde{L} \stackrel{def}{=} (\vec{G}^{+})^{-1} \vec{\mathcal{F}}^{+} .$$

The equations (7.18) are just the equations for $\Pi^{(k)}$ which we are needed. Inhomogeneous terms $\widetilde{\widetilde{G}}\, \Pi^{(k)}(x=-\infty)$ in (7.18) are easily calculated. Really, with the use of (7.10) we have $(\Pi^{(k)}(x=-\infty))_e = \delta_{k+e,N}\mathbb{1}$ and therefore

$$(\widetilde{\widetilde{G}}\, \Pi^{(k)}(x=-\infty))_e = -N((\vec{G}^{+})^{-1}_{e,N-k}\partial)\mathbb{1} =$$

$$= \begin{cases} 0, & e+k < N \\ \mathbb{1}, & e+k = N \\ \sum\limits_{n,\dots,m} (\vec{G}^{+})^{-1}_{ee}(-\vec{G}^{+}_{en})(\vec{G}^{+})^{-1}_{nn} \dots (\vec{G}^{+})^{-1}_{mn}(-\vec{G}^{+}_{n,N-k}) \cdot \mathbb{1}, & e+k > N \end{cases} \tag{7.19}$$

where summation is performed as in formula (7.16).

The shortcoming of the equations (7.18) is that they contain the operator $\tilde{L}$ instead of operator L^{+} which define the nonlinear part of the equations (5.5).

However, it is valid the following important

Theorem: The relation

$$\mathcal{J}\tilde{L} = L^{+}\mathcal{J} \tag{7.20}$$

holds where $\mathcal{J}$ is matrix $(N-1) \times (N-1)$ differential operator. Its matrix elements are

$$\mathcal{J}_{ik} = \sum_{e=1}^{N+1-i-k} \{ C^{i-1}_{e+i-1} V_{e+i+k-1}(-\partial)^e \cdot - C^{k-1}_{e+k-1}(-\partial)^e(\cdot V_{e+i+k-1}) \}, \tag{7.21}$$

$$\mathcal{J}_{ik} = 0 \quad at \quad i+k > N+1, \ (i,k = 1, \dots, N-1)$$

where $\quad C^k_e = \frac{e!}{k!(e-k)!}$.

This theorem is proved by direct calculations with the use of the relations (7.15), (7.16), (7.9), (3.15), (3.16), (3.21), (3.37)-(3.39), (3.42), (3.43). One can show that the operator $\mathcal{J}$ is invertable and $\mathcal{J}^{+} = -\mathcal{J}$. Let us also note that the operators $\mathcal{J}(\vec{G}^{+})^{-1}$ and $(\vec{G}^{+})^{-1}\mathcal{J}$ are pure differential one and $(\mathcal{J}(\vec{G}^{+})^{-1})^{+} = -(\vec{G}^{+})^{-1}\mathcal{J}$.

The relation (7.20) allow us to connect the terms $(L^{+})^n \mathcal{X}_k V$ with the quantities $\Pi^{(k)}$.

Let us substitute the asymptotic expansion of $\Pi^{(k)}$ on λ^{-1}, i.e.

$$\Pi^{(k)}(x,t,\lambda) = \sum_{n=0}^{\infty} \lambda^{-n}\, \Pi^{(k)}_n(x,t)$$

into the equations (7.28). As a result we obtain the following system of relations

$$\Pi^{(k)}_0(x,t) = \widetilde{\widetilde{G}}\,\Pi^{(k)}(x=-\infty), \tag{7.22}$$

$$\Pi^{(k)}_1(x,t) = \widetilde{L}\,\Pi^{(k)}_0, \tag{7.23}$$

$$\Pi^{(k)}_n(x,t) = \widetilde{L}\,\Pi^{(k)}_{n-1} \quad (n=2,3,\dots) \tag{7.24}$$

Using the equality (7.22) and the explicit form of the operators $\mathcal{J}$, $\widetilde{L}$ and $\mathcal{L}_k$ one can show that

$$\mathcal{J}\widetilde{L}\,\Pi^{(k)}_0 = \mathcal{J}\widetilde{L}\,\widetilde{\widetilde{G}}\,\Pi^{(k)}(x=-\infty) = \mathcal{L}_k V, \quad (k=1,\dots,N-1). \tag{7.25}$$

In virtue of (7.25) and (7.20) the system of the relations (7.23), (7.24) is equivalent to recurrence relations

$$\mathcal{L}_k V = \mathcal{J}\Pi^{(k)}_1,$$

$$L^{+}\mathcal{J}\Pi^{(k)}_{\ell} = \mathcal{J}\Pi^{(k)}_{\ell+1} \quad \left(\begin{matrix} k=1,\dots,N-1 \\ \ell=1,2,\dots \end{matrix}\right). \tag{7.26}$$

It immediately follows from (7.26) that $(L^{+})^{n}\mathcal{L}_k V = \mathcal{J}\Pi^{(k)}_{n+1}$ or

$$(L^{+})^{n}\mathcal{L}_k V = (L^{+})^{q}\mathcal{J}\Pi^{(k)}_{n-q+1} \tag{7.27}$$

where $n = 0,1,2,\dots$, and q is arbitrary integer.

The relations (7.27) give a possibility to rewrite the equations (5.5) with the functions $\Omega_k(L^{+},t)$ of the type (7.1) in the following form

$$\frac{\partial V(x,t)}{\partial t} = (L^{+})^{q}\mathcal{J}\sum_{k=1}^{N-1}\sum_{n=0}^{\infty}\omega_{kn}(t)\frac{1}{(n-q+1)!}\,\frac{\partial^{n-q+1}\Pi^{(k)}(\lambda)}{\partial(\lambda^{-1})^{n-q+1}}\bigg|_{\lambda=\infty} \tag{7.28}$$

At last, due to (7.5), the equation (7.28) (and therefore the equation (5.5)) is equivalent to the equation

$$\frac{\partial V(x,t)}{\partial t} = (L^{+})^{q}\mathcal{J}\frac{\delta \mathcal{H}_{-q}}{\delta V^{\tau}} \tag{7.29}$$

where

$$\mathcal{H}_{-q} = -\sum_{k=1}^{N-1}\sum_{n=0}^{\infty}\frac{\omega_{kn}(t)}{(n+q-1)!}\,\frac{\partial^{n+q-1}\operatorname{tr}(\bar{A}^{k}\ln S_{2}(\lambda))}{\partial(\lambda^{-1})^{n+q-1}}\bigg|_{\lambda=\infty} \tag{7.30}$$

and q is arbitrary integer.

It is easy to see that the equation (7.29) (i.e. the equation (5.5)) can be written down in the Hamiltonian form

$$\frac{\partial V(x,t)}{\partial t} = \{V(x,t), \mathcal{H}_{-q}\} \tag{7.31}$$

with respect to infinite set of Hamiltonians (7.30) and infinite set of Poisson brackets $\{\ ,\ \}_q$ where

$$\{\mathcal{F},\mathcal{H}\}_q \overset{def}{=} \int_{-\infty}^{+\infty} dy\, \operatorname{tr}\left(\frac{\delta\mathcal{F}}{\delta V^{T\tau}(y,t)}(L^{+})^{q}\mathcal{J}\frac{\delta\mathcal{H}}{\delta V^{\tau}}\right), \quad (q=0,\pm1,\pm2,\dots). \tag{7.32}$$

The fact that the brackets (7.32) are indeed the Poisson brackets is verified by straightforward but cumbersome, calculations.

Thus, it is shown that any equation of the form (5.5) is a Hamiltonian one with respect to the infinite set of the Hamiltonian structures. Let us emphasize that this family of the Hamiltonian structures is an universal one, i.e. all of the equations (5.5) are Hamiltonian one under the same family of Poisson brackets (7.32).

The bracket $\{\ ,\ \}_0$ is well-known Gelfand-Dikij bracket which was calculated (by another method) as far back in Ref. [2]. The bracket $\{\ ,\ \}_1$ corresponds to second Hamiltonian structure which was discussed in Refs. [4,21,22,89]. For the first time the existence of the infinite family of Hamiltonian structures for equations integrable by the inverse scattering transform method was pointed out by Magri (see Refs. [36,37]). Then the hierarchies of Hamiltonian structures for various integrable equations and their properties were discussed in Refs. [38,4,8,9,17,18,19,21,39].

Let us attract the attention to the operator $\mathcal{J}$ (Gelfand-Dikij operator). It can be calculated by various methods. In the AKNS-technique framework, with which we deal in the present paper, this operator has also a sense of the similarity operator which relates the operators L and L^+ acting in the adjoint one to another spaces (relation (7.20))*.

In the conclusion of this section three remarks. First remark: in virtue of (7.20) the equation (7.29) and brackets (7.32) can be rewritten as

$$\frac{\partial V(x,t)}{\partial t} = \mathcal{J}(L)^q \frac{\delta \mathcal{H}_{-q}}{\delta V^\tau}$$

and

* The same treatment is possible for analogs of the operator $\mathcal{J}$ for Hamiltonian structures connected with other spectral problems.

$$\{\mathcal{F},\mathcal{H}\}_q = \left\langle\!\left\langle \frac{\delta \mathcal{F}}{\delta V^{\tau\tau}} \mathcal{J}(L)^q \frac{\delta \mathcal{H}}{\delta V^\tau} \right\rangle\!\right\rangle .$$

Second remark. Since $\frac{1}{n!}\frac{\partial^n tr(\bar{A}^k \ln S_0(\lambda))}{\partial(\lambda^{-1})^n}\Big|_{\lambda=\infty} = C_n^{(k)}$ where $C_n^{(k)}$ are universal integrals of motion for equations (5.5) (section 5) then the Hamiltonians $\mathcal{H}_{-q}$ are

$$\mathcal{H}_{-q} = -\sum_{k=1}^{N-1}\sum_{n=0}^{\infty} \omega_{kn}(t)\, C_{n-q+1}^{(k)} .$$

Third remark. The relation between symmetry transformations (6.3) and integrals of motion $C_n^{(k)}$ is the following one

$$\delta_{(k,n)} V(x,t) = -f_{kn}\{V(x,t), C_{n+1}^{(k)}\}_0 = -f_{kn}\mathcal{J}\frac{\delta C_{n+1}^{(k)}}{\delta V^\tau}.$$

VIII. The examples: N = 2,3,4

Here we illustrate the results obtained in the previous sections by several examples.

The case N = 2

In this case $\ell_1^+ \cdot = \frac{1}{2}((\partial^{-1}\cdot)V_0' - V_0(\partial^{-1}\cdot)) - \frac{1}{2}\partial$ and transformations (4.13) are

$$B_0(\Lambda^+,t)(\mathcal{K}_0 V' - \mathcal{M}_0 V) + B_1(\Lambda^+,t)(\mathcal{K}_1 V' - \mathcal{M}_1 V) = 0 \quad (8.1)$$

where $\mathcal{K}_0 = \mathcal{M}_0 = 1$, $\quad \mathcal{K}_1 = \partial + \ell_1^+$, $\mathcal{M}_1 = \ell_1^+$

and operator Λ^+ acts as follows

$$\Lambda^+ \cdot = \frac{1}{2}\partial(\partial^{-1}\cdot)V_0') + \frac{1}{2}(\cdot V_0') + \ell_1^{+2} =$$

$$= \frac{1}{4}\{\partial^2 + 2V_0\cdot + 2(\cdot V_0') + (\partial V_0)\partial^{-1} + (\partial^{-1}\cdot)\partial V_0' + \quad (8.2)$$

$$+ (\partial^{-1}((\partial^{-1}\cdot)V_0' - V_0(\partial^{-1}\cdot))V_0' - V_0\partial^{-1}((\partial^{-1}\cdot)V_0' - V_0(\partial^{-1}\cdot))\}$$

The integrable equations are of the form

$$\frac{\partial V_0}{\partial t} - Q_1(L^+, t)\partial V_0 = 0 \qquad (8.3)$$

where

$$L^+ = \frac{1}{4}\{\partial^2 + 2[V_0, \cdot]_+ + [\partial V_0, \partial^{-1}\cdot]_+ + \qquad (8.4)$$
$$+ [V_0, \partial^{-1}[V_0, \partial^{-1}\cdot]_-]_-\}$$

and $[A,B]_\pm = AB \pm BA$. Operator $\widetilde{L}$ is

$$\widetilde{L}\cdot = \frac{1}{4}\{\partial^2 + 2[V_0, \cdot]_+ - \partial^{-1}[\partial V_0, \cdot]_+ + \partial^{-1}[V_0, \partial^{-1}[V_0, \cdot]_-]_-\}$$

and $\mathcal{J} = 2\partial$.

The operators Λ^+, L^+, transformations (8.1) and the equations (8.3) coincide with those obtained earlier by another method in Ref. [29].

Bakclund-transformations (6.1) is well-known soliton Backlund transformations for matrix KdV-family of the equations (8.3) (see [29]):

$$\frac{2B_0}{B_1}(V_0' - V_0) + \partial(V_0' + V_0) + (\partial^{-1}(V_0' - V_0))V_0' - \qquad (8.5)$$
$$- V_0\partial^{-1}(V_0' - V_0) = 0$$

The case N = 3

At N = 3 from (3.39) we have

$$\ell_1^+ = -\frac{1}{3}\partial^2 - \frac{1}{3}V_1 + \frac{1}{3}((\partial^{-1}\cdot)V_0' - V_0(\partial^{-1}\cdot)), \qquad (8.6)$$

$$\ell_2^+ = -\partial + \frac{1}{3}((\partial^{-1}\cdot)V_1' - V(\partial^{-1}\cdot))$$

and from (4.14),(4.12)

$$\mathcal{K}_1 = \begin{pmatrix} \partial, & \ell_1^+ \\ 1, & \partial + \ell_2^+ \end{pmatrix}, \quad \mathcal{M}_1 = \begin{pmatrix} 0 & \ell_1^+ \\ 1 & \ell_2^+ \end{pmatrix},$$

$$\mathcal{K}_2 = \begin{pmatrix} \partial^2 + V_1\cdot + \ell_1^+, & \frac{1}{3}(\partial^{-1}\cdot)\partial V_0' + 2\ell_1^+\partial + \ell_1^+\ell_2^+ \\ 2\partial + \ell_2^+, & \partial^2 + V_1\cdot + \frac{1}{3}(\partial^{-1}\cdot)\partial V_1' + \ell_1^+ + 2\ell_2^+\partial + \ell_2^{+2} \end{pmatrix}, \qquad (8.7)$$

$$\mathcal{M}_2 = \begin{pmatrix} \ell_1^+, & (\cdot V_0') + \frac{1}{3}(\partial^{-1}\cdot)\partial V_0' + \ell_1^+\ell_2^+ \\ \ell_2^+, & (\cdot V_1') + \frac{1}{3}(\partial^{-1}\cdot)\partial V_1' + \ell_1^+ + \ell_2^{+2} \end{pmatrix}.$$

Operator Λ^+ is 2 x 2 matrix operator which elements are

$$\Lambda_{11}^+ = \cdot V_0' + \ell_1^+\ell_2^+ + \frac{1}{3}(\partial^{-1}\cdot)(\partial V_0'),$$

$$\Lambda_{21}^+ = \cdot V_1' + \ell_2^{+2} + \ell_1^+ + \frac{1}{3}(\partial^{-1}\cdot)(\partial V_1'), \qquad (8.8)$$

$$\Lambda_{12}^+ = \ell_1^+\ell_2^{+2} + \ell_1^{+2} + \ell_1^+(\cdot V_1') + \cdot\partial V_0' +$$
$$+ \frac{1}{3}(\partial^{-1}\ell_2^+\cdot)\partial V_0' + \frac{1}{3}(\partial^{-1}\cdot)\partial^2 V_0' + \frac{2}{3}\ell_1^+((\partial^{-1}\cdot)\partial V_1')$$

$$\Lambda_{22}^+ = \ell_2^{+3} + [\ell_1^+, \ell_2^+]_+ + \ell_2^+(\cdot V_1') + (\cdot V_0') + (\cdot\partial V_1') +$$

$$+\frac{2}{3}(\partial^{-1}\cdot)\partial V_0' + \frac{1}{3}(\partial^{-1}\cdot)\partial^2 V_1' + \frac{1}{3}(\partial^{-1}(\ell_2^+\cdot))\partial V_1' + \frac{2}{3}\ell_2^+((\partial^{-1}\cdot)\partial V_1')$$

where ℓ_1^+ and ℓ_2^+ are given by formulas (8.6).

General transformations (4.13) are given by three arbitrary entire functions $B_0(\Lambda^+,t)$, $B_1(\Lambda^+,t)$, $B_2(\Lambda^+,t)$ where the operators Λ^+, $\mathcal{K}_i$, $\mathcal{M}_i$ are of the form (8.7), (8.8).

Simplest Backlund transformation (6.1) in the scalar case M = 1 is

$$\begin{gathered}
B_0(V_0'-V_0) + B_1\{\partial V_0' - \tfrac{1}{3}\partial^2(V_1'-V_1) - \tfrac{1}{3}V_1(V_1'-V_1) + \tfrac{1}{3}(V_0'-V_0)\partial^{-1}(V_1'-V_1)\} + \\
+B_2\{-\tfrac{1}{3}\partial^3(V_1'+V_1) + \tfrac{2}{3}\partial^2 V_0' + \tfrac{1}{3}\partial^2 V_0 + \tfrac{1}{3}(V_0'-V_0)V_1' - \tfrac{1}{3}V_1\partial(V_1'-V_1) - \\
-\tfrac{1}{3}(V_1'-V_1)\partial(V_1'-V_1) + \tfrac{1}{3}(\partial V_0)\partial^{-1}(V_1'-V_1) + \tfrac{1}{3}(V_0'-V_0)\partial^{-1}(V_0'-V_0) - \\
-\tfrac{1}{9}V_1(V_1'-V_1)\partial^{-1}(V_1'-V_1) - \tfrac{1}{9}\partial^2(V_1'-V_1)\partial^{-1}(V_1'-V_1) + \\
+\tfrac{1}{18}(V_0'-V_0)(\partial^{-1}(V_1'-V_1))^2\} = 0,
\end{gathered} \tag{8.9}$$

$$\begin{gathered}
B_0(V_1'-V_1) + B_1\{\partial V_1' + V_0'-V_0 + \tfrac{1}{3}(V_1'-V_1)\partial^{-1}(V_1'-V_1)\} + \\
+B_2\{-\tfrac{1}{3}\partial^2 V_1' - \tfrac{2}{3}\partial^2 V_1 + \partial(V_0'+V_0) + \tfrac{1}{3}V_1(V_1'-V_1) + \\
+\tfrac{1}{3}(V_1'-V_1)\partial^{-1}(V_0'-V_0) + \tfrac{1}{3}(V_0'-V_0)\partial^{-1}(V_1'-V_1) + \\
+\tfrac{1}{3}\partial V_1\partial^{-1}(V_1'-V_1) + \tfrac{1}{18}(V_1'-V_1)(\partial^{-1}(V_1'-V_1))^2\} = 0
\end{gathered}$$

where B_0, B_1, B_2 are arbitrary constants. It is easy to see that for "potentials" W_0, W_1 which are defined by formula $V_i = \partial W_i$ (i = 0,1) the transformation (8.9) have a

local form.

The general form of the integrable equations is

$$\begin{gathered}
\frac{\partial}{\partial t}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \Omega_1(L^+,t)\begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} + \\
+\Omega_2(L^+,t)\begin{pmatrix} \partial^2 + V_1, & -\frac{2}{3}\partial^3 - \frac{2}{3}V_1\partial - \frac{2}{3}[V_0,\cdot] - (\cdot V_0) \\ 2\partial, & -\partial^2 + \frac{1}{3}[V_1,\cdot] \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix}
\end{gathered} \tag{8.10}$$

where operator L^+ is calculated by formulas (8.8) at $V_0' = V_0$, $V_1' = V_1$. We present here the explicit forms of matrix elements of L^+ in the scalar case M = 1 ($\ell_1^+ = -\frac{1}{3}(\partial^2 + V_1)$, $\ell_2^+ = -\partial$):

$$\begin{gathered}
L_{11}^+ = \tfrac{1}{3}\partial^3 + V_0 + \tfrac{1}{3}V_1\partial + \tfrac{1}{3}(\partial V_0)\partial^{-1}, \\
L_{21}^+ = \tfrac{2}{3}\partial^2 + \tfrac{2}{3}V_1 + \tfrac{1}{3}(\partial V_1)\partial^{-1}, \\
L_{12}^+ = -\tfrac{2}{9}\partial^4 + \tfrac{2}{3}(\partial V_0) - \tfrac{2}{9}V_1\partial^2 - \tfrac{2}{9}\partial^2(V_1\cdot) - \\
-\tfrac{2}{9}V_1^2 + \tfrac{1}{3}(\partial^2 V_0)\partial^{-1} - \tfrac{2}{9}V_1(\partial V_1)\partial^{-1} - \tfrac{2}{9}\partial^2((\partial V_1)\partial^{-1}), \\
L_{22}^+ = -\tfrac{1}{3}\partial^3 + \tfrac{1}{3}V_0 - \tfrac{1}{3}\partial(V_1\cdot) - \tfrac{1}{3}\partial((\partial V_1)\partial^{-1}\cdot) + \tfrac{2}{3}\partial(V_0\partial^{-1}\cdot).
\end{gathered} \tag{8.11}$$

In scalar case M = 1 the equation (8.10) with linear functions $\Omega_1 = \omega_{10} + \omega_{11}L^+$, $\Omega_2 = \omega_{20} + \omega_{21}L^+$ is the following system of two equations

$$\begin{gathered}
\frac{\partial V_0}{\partial t} = \omega_{10}\partial V_0 + \omega_{11}\{-\tfrac{2}{9}\partial^5 V_1 + \tfrac{1}{3}\partial^4 V_0 + \tfrac{2}{3}\partial(V_1\partial V_0) + \tfrac{2}{3}\partial(V_0^2) - \\
-\tfrac{2}{3}V_1\partial^3 V_1 - \tfrac{4}{3}(\partial V_1)(\partial^2 V_1) - \tfrac{4}{9}V_1^2\partial V_1\} + \omega_{20}\{\partial^2 V_0 - \\
-\tfrac{2}{3}\partial^3 V_1 - \tfrac{2}{3}V_1\partial V_1\} + \omega_{21}\{-\tfrac{1}{9}\partial^5 V_0 - \tfrac{5}{9}\partial^3(V_1 V_0) - \\
-\tfrac{5}{9}\partial(V_0\partial^2 V_1) + \tfrac{5}{3}\partial(V_0\partial V_0) - \tfrac{5}{9}\partial(V_0 V_1^2)\},
\end{gathered} \tag{8.12}$$

$$\frac{\partial V_1}{\partial t} = \omega_{10}\partial V_1 + \omega_{11}\{-\tfrac{1}{3}\partial^4 V_1 + \tfrac{2}{3}\partial^3 V_0 + \tfrac{4}{3}\partial(V_0V_1) - \tfrac{2}{3}\partial(V_1\partial V_1)\} +$$

$$+ \omega_{20}\{2\partial V_0 - \partial^2 V_1\} + \omega_{21}\{-\tfrac{1}{9}\partial^5 V_1 - \tfrac{5}{9}\partial(V_1\partial^2 V_1) -$$

$$- \tfrac{5}{9}\partial(V_0\partial V_1) - \tfrac{5}{9}V_1^2\partial V_1 + \tfrac{5}{3}\partial(V_0^2)\} \quad .$$

The system of equations (8.12) contains some well-known equations as particular cases. At $\omega_{10} = \omega_{11} = \omega_{21} = 0$ system (8.12) is

$$\frac{\partial V_0}{\partial t} = \omega_{20}\left(\partial^2 V_0 - \tfrac{2}{3}\partial^3 V_1 - \tfrac{2}{3}V_1\partial V_1\right), \qquad (8.13)$$

$$\frac{\partial V_1}{\partial t} = \omega_{20}\left(2\partial V_0 - \partial^2 V_1\right).$$

It is easy to see that the system (8.13) is equivalent to the equation

$$\frac{\partial^2 V_1}{\partial t^2} = -\tfrac{2}{3}\omega_{20}^2\left(\tfrac{1}{2}\partial^4 V_1 + \partial^2(V_1^2)\right)$$

that is Bousinesq equation at $\omega_{20} = i\frac{\sqrt{3}}{2}$. The implicability of the inverse scattering transform method to Bousinesq equation was demonstrated in Ref. [40].

Another interesting special case of (8.12) is ω_{10} = ω_{11} = ω_{20} = 0, i.e. the system

$$\frac{\partial V_0}{\partial t} = \omega_{21}\left(-\tfrac{1}{9}\partial^5 V_0 - \tfrac{5}{9}\partial^3(V_1V_0) - \tfrac{5}{9}\partial(V_0\partial^2 V_1) + \tfrac{5}{3}\partial(V_0\partial V_0) - \right.$$

$$\left. - \tfrac{5}{9}\partial(V_0V_1^2)\right),$$

$$\frac{\partial V_1}{\partial t} = \omega_{21}\left(-\tfrac{1}{9}\partial^5 V_1 - \tfrac{5}{9}\partial(V_1\partial^2 V_1) - \tfrac{5}{9}\partial(V_0\partial V_1) - \right. \qquad (8.14)$$

$$\left. - \tfrac{5}{9}V_1^2\partial V_1 + \tfrac{5}{3}\partial(V_0^2)\right).$$

Under the reduction $V_0 = \frac{1}{2}\partial V_1$ and at $\omega_{21} = 3i$ the system (8.14) is reduced to the equation

$$\frac{\partial V_1}{\partial t} = \partial^5 V_1 + 5V_1\partial^3 V_1 + \tfrac{25}{2}\partial V_1\partial^2 V_1 + 5V_1^2\partial V_1$$

which was considered earlier in Refs. [8,31].

Let us note that the system of equations (8.12) does not admit the reduction V_0 = 0.

<u>The case N = 4</u>

Operators ℓ_i^+ are

$$\ell_1^+ = -\tfrac{1}{4}\partial^3 - \tfrac{1}{4}V_2\partial - \tfrac{1}{4}V_1 + \tfrac{1}{4}\left((\partial^{-1}\cdot)V_0' - V_0(\partial^{-1}\cdot)\right),$$

$$\ell_2^+ = -\partial^2 - \tfrac{1}{2}V_2 + \tfrac{1}{4}\left((\partial^{-1}\cdot)V_1' - V_1(\partial^{-1}\cdot)\right), \qquad (8.15)$$

$$\ell_3^+ = -\tfrac{3}{2}\partial + \tfrac{1}{4}\left((\partial^{-1}\cdot)V_2' - V_2(\partial^{-1}\cdot)\right).$$

For operator $\Lambda^+ = \widetilde{\mathcal{F}}^+(\widetilde{G}^+)^{-1}$ we have (i = 1,2,3)

$$\widetilde{\mathcal{F}}_{i1}^+ = 4\ell_i^+\ell_3^+\partial + 4(\partial\cdot)V_{i-1}' + \cdot\partial V_{i-1}' + \delta_{i2}4\ell_1^+\partial + \delta_{i3}4\ell_2^+\partial,$$

$$\widetilde{\mathcal{F}}_{i2}^+ = 4\ell_i^+\ell_2^+\partial - 4(\ell_3^+\partial\cdot)V_{i-1}' + 2\ell_i^+\partial(\cdot V_2') + 2\ell_i^+(\partial\cdot)V_2' +$$

$$+ 4(\partial\cdot)\partial V_{i-1}' + \cdot\partial^2 V_{i-1}' + \delta_{i2}\{4(\partial\cdot)V_0' + 2(\cdot\partial V_0')\} +$$

$$+ \delta_{i3}\{4\ell_1^+\partial + 4(\partial\cdot)V_1' + 2(\cdot\partial V_1')\},$$

$$\widetilde{\mathcal{F}}_{i3}^+ = 4\ell_i^+\ell_1\partial - 4\ell_i^+(\ell_3^+\partial\cdot)V_2' + 3\ell_i^+\partial(\cdot V_1') + \ell_i^+(\partial\cdot)V_i' +$$

$$+ 8\ell_i^+(\partial\cdot)\partial V_2' + 3\ell_i^+(\cdot\partial^2 V_2') - 4(\ell_2^+\partial\cdot)V_{i-1}' + V_2(\cdot\partial V_{i-1}') + \qquad (8.16)$$

$$+ 6(\partial^2\cdot)\partial V_{i-1}' + 4(\partial\cdot)\partial^2 V_{i-1}' + \cdot\partial^3 V_{i-1} - \partial(\cdot V_2'V_{i-1}) -$$

$$- 2(\partial(\cdot V_2'))V_{i-1}' - (\partial\cdot)V_2'V_{i-1} - \delta_{i2}\{4(\ell_3^+\partial\cdot)V_0' - 8(\partial\cdot)\partial V_0' -$$

$$- 3(\cdot\partial^2 V_0')\} - \delta_{i3}\{4(\ell_3^+\partial\cdot)V_1' - 8(\partial\cdot)\partial V_1' - 3(\cdot\partial^2 V_1') - 4(\partial\cdot)V_0' - 3(\partial V_0')\}$$

where $\delta_{ik} = \begin{cases} 1, & i=k \\ 0, & i \neq k \end{cases}$ and

$$(\tilde{G}^{+})^{-1} = \begin{pmatrix} \frac{1}{4}\partial^{-1}, & \frac{1}{4}\partial^{-1}\ell_3^{+}, & \frac{1}{4}\partial^{-1}\ell_2^{+} + \frac{1}{4}\partial^{-1}\ell_3^{+2} + \frac{3}{16}(\partial^{-1})V_2 + \frac{1}{16}\partial^{-1}(V_2) \\ 0, & \frac{1}{4}\partial^{-1}, & \frac{1}{4}\partial^{-1}\ell_3^{+} \\ 0, & 0, & \frac{1}{4}\partial^{-1} \end{pmatrix} \qquad (8.17)$$

The general equations (5.5) are characterized by three arbitrary functions, operator L^{+} is calculated by the formulas (8.16),(8.17) at $V' = V$ and (in scalar case M = 1)

$$\mathcal{L}_1 = \begin{pmatrix} \partial & 0 & 0 \\ 0 & \partial & 0 \\ 0 & 0 & \partial \end{pmatrix}, \quad \mathcal{L}_2 = \begin{pmatrix} \partial^2 + V_2, & 0, & \frac{1}{2}\partial^4 - \frac{1}{2}V_2\partial^2 - \frac{1}{2}V_1\partial - V_0 \\ 2\partial, & \partial^2 + V_2, & -2\partial^3 - V_2\partial - V_1 \\ 0, & 2\partial, & -2\partial^2 \end{pmatrix}$$

$$\mathcal{L}_3 = \begin{pmatrix} \partial^3 + V_2\partial + V_1, & -\frac{3}{4}\partial^4 - \frac{3}{4}V_2\partial^2 - \frac{3}{4}V_1\partial - V_0, & \frac{3}{8}\partial^5 + \frac{3}{8}V_2\partial^3 + \frac{3}{8}V_1\partial^2 - \frac{1}{4}\partial V_0 \\ 3\partial^2 + V_2, & -2\partial^2 - \frac{1}{2}V_2\partial, & \frac{3}{4}\partial^4 - \frac{3}{4}V_1\partial - \frac{1}{4}\partial V_1 - V_0 \\ 3\partial, & -\frac{3}{2}\partial^2, & \frac{1}{4}\partial^3 - \frac{1}{2}V_2\partial - \frac{1}{4}\partial V_2 \end{pmatrix}$$

The simplest equation (5.5) at N = 4, M = 1 corresponds to constant $\Omega_1, \Omega_2, \Omega_3$ and it is the system of equations

$$\frac{\partial V_0}{\partial t} = \Omega_1\partial V_0 + \Omega_2\left(-\frac{1}{2}\partial^4 V_2 + \partial^2 V_0 - \frac{1}{2}V_2\partial^2 V_2 - \frac{1}{2}V_1\partial V_2\right) +$$
$$+\Omega_3\left(\frac{3}{8}\partial^5 V_2 - \frac{3}{4}\partial^4 V_1 + \partial^3 V_0 + \frac{3}{8}V_2\partial^3 V_2 + \frac{3}{8}V_1\partial^2 V_2 - \frac{3}{4}V_2\partial^2 V_1 + \frac{3}{4}V_2\partial V_0 - \frac{3}{4}V_1\partial V_1\right),$$
$$\frac{\partial V_1}{\partial t} = \Omega_1\partial V_1 + \Omega_2\left(-2\partial^3 V_2 + \partial^2 V_1 + 2\partial V_0 - V_2\partial V_2\right) + \Omega_3\left(\frac{3}{4}\partial^4 V_2 -\right.$$
$$\left. -2\partial^3 V_1 + 3\partial^2 V_0 - \frac{3}{4}V_2\partial V_1 - \frac{3}{4}V_1\partial V_2\right), \qquad (8.18)$$
$$\frac{\partial V_2}{\partial t} = \Omega_1\partial V_2 + \Omega_2\left(-2\partial^2 V_2 + 2\partial V_1\right) + \Omega_3\left(\frac{1}{4}\partial^3 V_2 - \frac{3}{2}\partial^2 V_1 + 3\partial V_0 - \frac{3}{4}V_2\partial V_2\right).$$

In the particular case $\Omega_2 = 0$ the system (8.18) admit the reduction $V_1 = \partial V_2$ and it is reduced to the system

$$\frac{\partial V_0}{\partial t} = \Omega_3\left(-\frac{3}{8}\partial^5 V_2 + \partial^3 V_0 + \frac{3}{4}V_2\partial V_0 - \frac{3}{8}\partial(V_2\partial^2 V_2)\right) + \Omega_1\partial V_0,$$
$$\frac{\partial V_2}{\partial t} = \Omega_3\left(-\frac{5}{4}\partial^3 V_2 + 3\partial V_0 - \frac{3}{4}V_2\partial V_2\right) + \Omega_1\partial V_2.$$

IX. Conclusion

Here we briefly consider some other approachs to Gelfand-Dikij spectral problem and compare their with that given in the present paper.

It is considered, to our knowledge, three types of matrix spectral problems of the form

$$\frac{\partial \psi}{\partial x} = \lambda A\psi + P(x,t)\psi \qquad (9.1)$$

which are connected with Gelfand-Dikij problem (1.1) (with the exception of the problem (2.1),(2.2)).

1. The first problem is the spectral problem (9.1) with

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad P = \begin{pmatrix} P_1 & 0 & \dots & 0 \\ 0 & P_1 & \dots & 0 \\ \dots & & & \\ 0 & \dots & \dots & P_N \end{pmatrix} \qquad (9.2)$$

where $P_1, \dots, P_N$ are matrices M x M and $P_1 + P_2 + \dots + P_N = 0$ This problem is obtained as a matrix Z_N reduction [41] of the general problem (9.1) and it was considered in Refs. [41,42,8, 18]. The general form of the nonlinear equations integrable by (9.1)-(9.2) is [18]

$$\frac{\partial u_i(x,t)}{\partial t} = \sum_{k=1}^{N-1}\sum_{\ell=1}^{N-1}\left(\Omega_k(L_N^{+},t)L_k^{+}\right)_{i\ell}\left(u_{\ell+k} - u_\ell - u_k\right), \qquad (9.3)$$
$$(i = 1, \dots, N-1).$$

where $U_i \overset{def}{=} P_i + \sum_{\ell=1}^{N} P_\ell$, L_k^+ $(k=1,2,\ldots,N)$ are certain matrix integro-differential operators and $\Omega_m(\lambda,t)$ are arbitrary functions meromorphic on λ. The equations (9.3) contain the two-dimensional abelian (M = 1) and nonabelian (M > 1) Toda lattice equations [41, 42] as a particular case $\Omega_1 = \ldots = \Omega_{N-2} = 0$, $\Omega_{N-1} = (L_N^+)^{-1}$ [18].

All the equations (9.3) are Hamiltonian one with respect to the infinite set of Poisson brackets $\{\ ,\ \}_n$ $(n=0,\pm1,\pm2,\ldots)$ where [18]

$$\{\mathcal{F},\mathcal{H}\}_n = \sum_{i,k=1}^{N-1}\int_{-\infty}^{+\infty} dx\, tr\left(\frac{\delta\mathcal{F}}{\delta U_i^\tau}\left((L_N^+)^n\mathcal{D}\right)_{ik}\frac{\delta\mathcal{H}}{\delta U_k^\tau}\right) \tag{9.4}$$

and matrix operator $\mathcal{D}$ acts as follows

$$\mathcal{D}_{ik}\cdot = -\delta_{ik}\partial + \delta_{ik}[U_k(x) - \frac{1}{N}\sum_{\ell=1}^{N}U_\ell(x), \cdot]_- -$$
$$-\frac{1}{N}[U_i(x),\cdot]_- - \frac{1}{N}[U_i(x), \partial^{-1}[U_k - \frac{1}{N}\sum_{\ell=1}^{N-1}U_\ell, \cdot]_-]_-$$

At N = 3, M = 1 see also Ref. [8].

The connection between the problem (9.1)-(9.2) and Gelfand-Dikij problem (1.1) is given by Miura type transformation which is defined from the relation [7-9, 43]

$$\partial^N + V_{N-2}\partial^{N-2} + \ldots + V_1\partial + V_0 = (\partial - P_N)(\partial - P_{N-1})\ldots(\partial - P_1). \tag{9.5}$$

And what is more the spectral problems (9.1)-(9.2) and (2.1)-(2.2) are gauge equivalent [41]. So the equations integrable by the problems (9.1)-(9.2) and (2.1)-(2.2) namely the equations (9.3) and (5.5) with the same functions $\Omega_k(\lambda,t)$ are gauge equivalent. In particular, two-dimensional nonabelian Toda lattice equations are gauge equivalent to equation (5.5) at $\Omega_1 = \ldots = \Omega_{N-2} = 0$, $\Omega_{N-1} = (L^+)^{-1}$, i.e. to

$L^+\frac{\partial V}{\partial t} = \mathcal{L}_{N-1}V$. The families of Poisson brackets (9.4) and (7.32) convert one into another under Miura transformation (9.5) and what is more this transformation is a canonical one [18].

2. Second spectral problem connected with Gelfand-Dikij problem is the problem (9.1) where

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & q & 1 & 0 \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & q^{N-1} \end{pmatrix}, \quad P = \begin{pmatrix} 0 & Q_{N-2} & Q_{N-3} & \ldots & Q_2 & Q_1 & Q_0 \\ Q_0 & 0 & Q_{N-2} & \ldots & Q_3 & Q_2 & Q_1 \\ Q_1 & Q_0 & 0 & \ldots & \ldots & Q_3 & Q_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Q_{N-3} & \ldots & \ldots & \ldots & \ldots & 0 & Q_{N-2} \\ Q_{N-2} & Q_{N-3} & \ldots & \ldots & Q_1 & Q_0 & 0 \end{pmatrix} \tag{9.6}$$

where $q = exp\frac{2\pi i}{N}$ and $Q_0,\ldots,Q_{N-2}$ are matrices M x M. This spectral problem is obtained as a result of Z_N reduction too [41].

The problem (9.1),(9.6) (up to change $P \to AP$) and its relation to (1.1) have been studied in Refs. [9,44]. In the AKNS-technique framework the problem (9.1),(9.6) has been considered in Refs. [18,19]: the general form of the integrable equations and infinite family of Hamiltonian structures analogous to (9.3),(9.4) have been found.

The problems (9.1),(9.2) and (9.1),(9.6) are ultimately connected. Indeed a simple gauge transformation $\Psi \to B\Psi$, $P \to BPB^{-1}$ where $B_{ik} = \frac{1}{\sqrt{N}}q^{-(i-1)(k-1)}\mathbb{1}$ $(i,k=1,\ldots,N)$ convert the problem (9.1),(9.2) into the problem (9.1),(9.6) [41,18]. So the equations integrable by (9.1),(9.6) are gauge equivalent to the equations (9.3) integrable by (9.1),(9.2) and therefore to the equations (5.5) integrable by Gelfand-Dikij problem (1.1).

3. Third problem is (9.1) where

$$A=\begin{pmatrix} \mathbb{1} & 0 & \dots & 0 \\ 0 & q\mathbb{1} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & q^{N-1}\mathbb{1} \end{pmatrix}, \quad P_{Z_N}=\begin{pmatrix} 0 & Q_{N-2} & Q_{N-3} & \dots & Q_2 & Q_1 & Q_0 \\ \mathbb{1} & 0 & Q_{N-2} & \dots & & Q_2 & Q_1 \\ 0 & (1+q)\mathbb{1} & 0 & \dots & & \dots & Q_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & & (1+\dots+q^{N-3})\mathbb{1}, & 0, & Q_{N-2} \\ 0 & 0 & \dots & & & (1+\dots+q^{N-2})\mathbb{1}, & 0 \end{pmatrix} \qquad (9.7)$$

where $q=\exp\frac{2\pi i}{N}$. Spectral problem (9.1),(9.7) is obtained as a result of Z_N reduction of the general problem (9.1) [18,19]. One can find the general form of the equation (analogous to (9.3)) integrable by (9.1),(9.7) and infinite family of corresponding Hamiltonian structures [18,19].

The connection between (9.1),(9.7) and Gelfand-Dikij problem is due to the fact that ψ_N (N-th component of the column ψ in (9.1)) satisfies to the equation (1.1) where coefficients $V_0,\dots,V_{N-2}$ are related with $Q_0,\dots,Q_{N-2}$ by formulas [18,19]

$$\sum_{k=1}^{N}\left(V_k(x)P^{(k)}(x)\right)_{Nl}=0 \quad (l=1,\dots,N) \qquad (9.8)$$

where $P^{(k+1)}=\partial P^{(k)}+P^{(k)}P_{Z_N}$, $P^{(0)}=\mathbb{1}_{NM}$ ($k=0,1,2,\dots$). The relations (9.8) is readily solved both with respect to quantities $Q_0,\dots,Q_{N-2}$ and quantities $V_1,\dots,V_{N-2}$.

If one perform the transition $\{Q_K\}\to\{V_K\}$ in the equations integrable by (9.1),(9.7) then one obtain the equations (5.5).

4. All three problems considered above are equivalent from the point of view of Gelfand-Dikij spectral problem: any information about nonlinear equations connected with (1.1) which it is possible to obtain with the use of one of these

problems can be obtained with the use of the other problems too.

From this point of view the approach considered in the present paper seems to be preferable because it allows us to study the nonlinear equations connected with Gelfand-Dikij problem (1.1) directly in the terms of $V_0, V_1,\dots, V_{N-2}$ without introducing any auxiliary quantities and constructions.

Let us note that recursion operator (of the type L^+) for Gelfand-Dikij problem (1.1) has been also calculated by a completely another technique in Refs. [4,6]. In the explicit examples at M = 1 and N = 3,4 the recursion operators given in [4,6] look like the operator L^+ (see formulas (8.11) and (8.16)-(8.17). The absent of exact coincidence of this operators is due to different choice of basis of independent quantities.

The results of the present paper can be generalized to the Gelfand-Dikij spectral problem with $V_{N-1}\neq 0$ and to the case $\lim_{|x|\to\infty} V_K(x,t)\neq 0$ ($k=0,1,\dots,N-1$). The reduction problem can be also analysed.

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