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THEORY AND PHENOMENOLOGY OF THE QCD VACUUM

2. SEMICLASSICAL METHODS

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Abstract

In this preprint we discuss applications of semiclassical methods, based on the topologically nontrivial configurations, the instantons. In section 2.1 we start with the simplest problem of penetration through potential barrier in one-dimensional quantum mechanical problem, using the Euclidean time formalism. Generalization of the method to gauge theories is considered in section 2.2, and the role of light quarks is discussed in section 2.3. In section 2.4 we consider instanton interactions and study applicability limits of t'Hooft formulae, while in section 2.5 we discuss various "instanton liquid" models. Finally, in section 2.6 applications of the instanton models to the problem of chiral symmetry breaking is discussed.

2. SEMICLASSICAL METHODS

Historically, discussion of semiclassical methods was the first attempt to go out of the perturbation theory domain in the field theory context. Its primary goal was evaluation of asymptotic increase of the coefficients of the perturbative series, but the real turning point was discovery by Polyakov and coworkers of the topologically nontrivial extremal configurations -- the instantons -- in four dimentional Yang-Mills theory.

For several years beautiful mathematics connected with instantons was intensely discussed in the literature by multiple theoreticians. Many unexpected phenomena were discovered during this period, such as tunneling between topologically different classical vacua. Some old problems like the U(1) problem raised by Weinberg has found their solution, at least in principle. Also, as it always happens, new insight has lead to new problems we never think of before. As an example, the problem of CP conservation in strong interactions turns out to be rather nontrivial.

However, attempts to make some quantitative and reliable calculations based on the instanton physics have so far failed. Among them the most widely known work is that by Callan, Dashen and Gross [2.13], in which some first order phase transition in the instanton gas in colour field was predicted. According to this work, hadrons are some drops of "dilute" phase, which can be more easily understood than the vacuum itself. Unfortunately, leter studies have shown that the so called dilute gas approximation used in this work can not be in fact justified. The fact that this problem is much more difficult than originally expected was rather disappointing, and most theoreticians have turn to other problems.

New impetus to the instanton physics was given by the development of the QCD sum rules. First some intriquing correlation between the dependence of vacuum field effects on particular correlator and the instanton quantum numbers were discovered, and later there appeared attepts to collect all such candidates for the instanton-induced effects using some simplified model. The most important result of these considerations is the observation, that although instantons really interact strongly, they are not at all "melted" completely, thus the semiclassical approach is not hopeless.

Recently very interesting approach to the instanton theory was suggested by Dyakonov and Petrov[2.22], which is based on the Feynman variational principle, very adequate method for the evaluation of the ground state energy of quantum complicated systems. Although the particular results obtained are somehow preliminary and many open questions are so far left, but it seems evident that this approach also point toward rather dilute "instanton liquid".

Now we make some remarks about the contents of this chapter. In order to make the formalism with imaginary time more familiar to the reader, we start with the simplest quantum mechanical problem of tunneling through some potential barrier in section 2.1, and only then proceed to Yang-Mills case (section 2.2). Rather nontrivial role of light quarks in the instanton theory is considered in section 2.3. The next section is devoted to instanton interactions, and then, in section 2.5 we come to "instanton liquid" models. In the last section 2.6 we discuss attempts to solve the SBCS problem using the instanton-induced forces.

2.1. Tunneling in quantum mechanics

We start this chapter with the problem which can be found in any textbook on quantum mechanics. In addition, the method of its solution to be discussed below is much more combersome than standard WKB method, and the result obtained will be shown to be somehow less precise. Therefore, it is reasonable to emphasize from the start why it still is more important for us: it can be generalized to problems with many degrees of freedom. Our discussion is based mainly on the work by Polyakov [2.4].

First, let us remind the reader some elements of Feynman formulation of quantum mechanics based on path integrals. Let us approximate the path $x(t)$ by N+2 numbers, giving the coordinates at time moments $t_k=k\,a$, $k=0,1,\ldots N+1$. The initial and final points are also called $x_0=x_i$, $x_{N+1}=x_f$. The limit of some N-dimentional integral over $x_1\ldots x_N$ is called the path integral. For its explicit formulation one has to compute the transfer matrix of the transition during small time interval a

$$\langle out|e^{-iHa}|in\rangle = \langle out|T|in\rangle + O(a^2) \qquad (2.1)$$

Here H is the Hamiltonian, and T is the transfer matrix. If T is known, the whole amplitude is the product of N its copies. In the limit $N\to\infty$ and $a\to 0$ we obtain Dyson evolution operator.

It is instructive to define T for nonrelativistic particle without internal quantum numbers (spin or colour), moving in time-indepentent potential V(x). The Hamiltonian is

$$H = \frac{p^2}{2m} + V(x) \qquad (2.2)$$

With the expression for the transfer matrix

$$\langle x_f|T|x_i\rangle = \langle x_f|e^{-\frac{iVa}{2}}\, e^{-\frac{ip^2}{2m}a}\, e^{-\frac{iV}{2}a}|x_i\rangle \qquad (2.3)$$

one should "sandwich" in between the states with definite momenta, with the result

$$\langle x_f | T | x_i \rangle = \int \frac{dp}{2\pi} e^{ip(x_f - x_i)} e^{-iH(p, x_f, x_i)a} \qquad (2.4)$$

$$H(p, x_f, x_i) = \frac{p^2}{2m} + \frac{V(x_i)}{2} + \frac{V(x_f)}{2}$$

so that the total amplitude is equal to

$$\langle x_f | e^{-iHt} | x_i \rangle =$$

$$= \lim_{N \to \infty} \int \prod_{k=1}^{N} \frac{dp_k\, dx_k}{(2\pi)} \exp\left\{ ia \left[\sum p_k \left(\frac{x_k - x_{k-1}}{a} \right) - H(p_k, x_k, x_{k-1}) \right] \right\} \qquad (2.5)$$

In the case considered it is possible to integrate over momenta, for the relevant integrals are Gaussian. The result is the famous Feynman formula. for the nonrelativistic propagator

$$\langle x_f | e^{-iHt} | x_i \rangle = \int \mathcal{D}x(t) \exp\{ iS[x(t)] \} \qquad (2.6)$$

where $S[x]$ is the action for given path

$$S = \int_0^t dt' \left(\frac{m\dot{x}^2}{2} - V(x) \right) \qquad (2.7)$$

and

$$\mathcal{D}x(t) = \prod_{i=1}^{N} \sqrt{2\pi a}\, dx_i$$

The method of transfer matrix is rather general, and its further applications to numerical evaluation of propagators will be considered in section 4.4.

Our next step is the famous Euclidean time transformation. Let us turn t to iτ and consider the path integral (2.6) in τ representation. Even at the classical level one finds that the equations of motion become

$$m \frac{d^2 x}{d\tau^2} = + \frac{\partial V}{\partial x} \qquad (2.8)$$

and they allow the nontrivial solutions for the motion under the barrier. In quantum theory, we have new weight exp($-S^E$) where

$$S^E = \int d\tau \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right] \qquad (2.9)$$

For definiteness, let us consider the popular example of the two well nonlinear oscillator with the potential

$$V(x) = c(x^2 - f^2)^2 \qquad (2.10)$$

for which the nontrivial solutions of classical equation (2.8) is like follows

$$x_{cl}(\tau) = f\, \mathrm{th}\left[\frac{\omega}{2}(\tau - \tau_c) \right], \quad \omega \equiv \left(\frac{8cf^2}{m} \right)^{1/2} \qquad (2.11)$$

Its importance is connected with the fact, that it has the minimal action among all paths leading from one well to another:

$$S_0 = \frac{4}{3} \frac{\sqrt{2m}}{c\hbar} f^3 \qquad (2.12)$$

If this action is large $S_0 \gg 1$ (or, in fact, the Plank constant $\hbar$, taken to be unity in the present work), the tunneling probability is very small, proportional to exp($-S_0$), and only paths close to the classical one (2.11) are important.

So, in Feyman formulation the semiclassical approximation is reduced to the statement that <u>the path integral can be evaluated in Gaussian approximation near some extreme configurations</u>.

Let us write down the path as consisting of classical and quantum parts $x(\tau) = x_{cl}(\tau) + y(\tau)$ with subsequent expansion up to y^2 order for the action

$$S[x(\tau)] = S_0 + \frac{1}{2} \int_0^{\tau_0} d\tau \left\{ y \left(-\frac{d^2}{d\tau^2} + V'' \right) y \right\} + O(y^3)$$

General method of the calculation of such integrals is the expansion over eigenfunctions of the differential operator

$$\left[-\frac{d^2}{d\tau^2} + V'' \right] y_n(\tau) = \varepsilon_n y_n(\tau) \;, \quad y(\tau) = \sum_n C_n y_n(\tau) \qquad (2.13)$$

so that one obtains the result of the type

$$\langle x_f | e^{-H\tau_0} | x_i \rangle = const \cdot \left[\det \left(-\frac{d^2}{d\tau^2} + V'' \right) \right]^{-1/2} \exp(-S_0) \qquad (2.14)$$

$$\det \left(-\frac{d^2}{d\tau^2} + V'' \right) \equiv \prod_n \mathcal{E}_n$$

There are infinitely many modes with large n, so the determinant is in fact divergent. Its renormalization is made by the consideration of its ratio to that for the free motion, using

$$\langle 0 | e^{-H\tau_0} | 0 \rangle = \int \frac{d^3 p}{2\pi} e^{-\frac{p^2 \tau_0}{2m}} = \sqrt{\frac{m}{2\pi\tau_0}} = \qquad (2.15)$$

$$= const \cdot \prod_{n=1}^{\infty} \left(\frac{\pi^2 n^2 m^2}{\tau_0^2} \right)$$

Another general problem is the appearence of one mode with zero $\mathcal{E}_0$ (at $\tau_0 \to \infty$), so that $(\det\ldots)^{-1/2}$ in (2.14) is also infinite. It is easy to trace this <u>zero mode</u> and point out its physical meaning: solution (2.11) shifted in time by an arbitrary amount τ_c is also a solution. It is quite clear that the integral over such coordinate τ_c is not at all Gaussian. On the contrary, the action does not depend on it at all, so we have the unlimited integral over τ_c, explaining the encounted infinity. Clear, that in order to work with finite quantities one should consider <u>transfer amplitude per unit τ_c</u> $dn/d\tau_c$. Transition from the integration over coefficient c_0 to the integration over τ_c leads to some Jacobian of general type, which can be found as follows

$$\Delta y(\tau) = \Delta\tau_c \frac{dx_{cl}}{d\tau_c} = -\sqrt{S_0}\, x_{cl} \Delta\tau_c = \Delta c_0 \cdot y_0 \qquad (2.16)$$

$$dc_0 / d\tau_c = -\sqrt{S_0}$$

Our result now looks as follows

$$\langle x_f | e^{-H\tau_0} | x_i \rangle = \qquad (2.17)$$

$$= const \left[\det' \left(-\frac{d^2}{d\tau^2} + V'' \right) \right]^{-1/2} \sqrt{S_0} \exp(-S_0)\, \tau_0$$

where primed determinant reminds that it corresponds only to non-zero modes. Note, that the factor $\sqrt{S_0}$ for each zero mode is the general result, which will be used in Yang-Mills case.

In order to determine the numerical constant in (2.17) one should find all $\mathcal{E}_n$. In the problem considered at the moment, the two well oscillator (2.10), it was made in Ref. [2.5], see also rather detailed discussion in review [2.3]. The final result looks as follows

$$\langle -f | e^{-H\tau_0} | f \rangle = (d \cdot \tau_0) \left(\sqrt{\frac{\omega}{\pi}} e^{-\frac{\omega\tau_0}{2}} \right) \qquad (2.18)$$

$$d = \sqrt{\frac{6 S_0}{\pi}}\, \omega \exp(-S_0)$$

The amplitude of the transition from one well to another is proportional to time τ_0, so for its large values one should take into account multiple transitions.

For large enough S_0 the tunneling probability is very small, so transitions can be considered as independent events, with the probability of n transition to be the Poisson distribution

$$W_n = \frac{1}{n!} (\tau_0 d)^n e^{-\tau_0 d} \qquad (2.19)$$

so that the complete result for the transition amplitude is as follows

$$\langle -f | e^{-H\tau_0} | f \rangle = \sqrt{\frac{\omega}{\pi}} e^{-\frac{\omega\tau_0}{2}} sh(\tau_0 d) \qquad (2.20)$$

These calculations naturally suggest the following terminology. Tunneling events are considered as some one-dimensional gas along the time axis, which is very dilute at large S_0 : transition from one well to another takes place durind time period ω^{-1} (2.11) being much smaller than their separation d^{-1} (2.18). Thus, expression (2.20) is said to follow from <u>dilute gas approximation</u>, and individual transitions are called <u>instantons</u> (or kinks).

Now we are going to discuss the result (2.20) further, in order to make contact to more familiar features of the two-well problem. First, we derive the ground state energy shift due to tunneling. In order to do this one has to expand the transition amplitude in stationary states. Evidently, for the n-th level the contribution is proportional to $\exp(-E_n \tau_0)$ and at $\tau_0 \to \infty$ only the ground state survives. Thus, it follows from (2.20) that

$$E_0 = \frac{\omega}{2} - d \tag{2.21}$$

or the shift is just equal to the instanton density.

The next instructive calculation is that of the energy of the first excited state. Following [2.4], let us consider the correlator of coordinates

$$K(\tau) = \frac{\langle x(\tau)\, x(0)\rangle}{\langle x^2(0)\rangle} \tag{2.22}$$

The solution (2.11) can be substituted by the step function

$$x_{cl} \simeq f\, \theta(\tau - \tau_c) \tag{2.23}$$

and, using again the dilute gas approximation, we may average over the ensemble of paths with n instantons with the result

$$K(\tau) \underset{\tau\to\infty}{\longrightarrow} \exp(-2d\tau) \tag{2.24}$$

Again, let us connect the definition of the correlator (2.22) with the familiar language of stationary wave functions. The ground state function is symmetric, so the average x in it is zero. So, first contribution to the correlator at large τ is given by the nondiagonal transition from the ground state to the first excited state

$$K(\tau) \sim |\langle 0|x|1\rangle|^2 \exp[-(E_1-E_0)\tau] \tag{2.25}$$

Comparing this relation with (2.24) one finds the energy level splitting

$$E_1 - E_0 = 2d \tag{2.26}$$

The reader may ask why we obtain so simple results by so complicated line of arguments, rather than directly from Schredinger equation. The reason is that the same method is used below for the evaluation of the ground energy shift and lowest excitations (hadronic masses) in QCD.

Still it is interesting to compare the results obtained with the standard WKB result

$$E_1 - E_0 \sim \exp(-S_{WKB}) \tag{2.27}$$

$$S_{WKB} \equiv \int_{x_1}^{x_2} \left[(2m)(V(x)-E_0)\right]^{1/2} dx$$

The leading term at large S_0 is the same, $S_{WKB} \to S_0$. The pre-exponent numerical factor depends on the particular approximations near the turning points. The standard applications of Airy functions give the result incorrect by the factor $\sqrt{\pi/e}$, while more accurate quadratic approximations for the potential around the turning point give the result in agreement with (2.20). However, S_{WKB} have also some further corrections , and, as demonstrated by Fig.1, is more accurate at S_0=2-6. It means that non-

Gaussian fluctuations are important at such S_0.

Another type of effects, also neglected in the dilute gas approximation used above, is the instanton interactions. By analytical methods it was discussed in Refs. [2.6] and by numerical path simulation in Ref. [3.45] by Zhirov and myself. We have observed many phenomena demonstrating that the instanton gas is not ideal even at large $S_0 \sim 6$, in particular close instanton-antiinstanton "molecules" and even clusters with several instantons. However, theory of such phenomena is not so far developed.

2.2. Instantons in gauge theories

In our discussion of the instantons in two-well oscillator we have not empasized the symmetry aspect of the problem. However, the classical solution (2.11) with finite action over the infinite time interval exists only due to the $(x) \leftrightarrow (-x)$ symmetry of the potential, allowing for different asymptotics at τ going to plus and minus infinity.

Looking for similar phenomena in gauge theories in four dimensions it is reasonable to start with discussion of A^a_μ at x going to infinity. Finite action implies that $G^a_{\mu\nu}$ decreases slower than $1/x^2$, but A^a_μ should not tend to zero: the pure gauge potential

$$\frac{g}{2} t^a A^a_\mu (x \to \infty) \longrightarrow i S \partial_\mu S^+ \qquad (2.28)$$

corresponds to zero field strength. For example, with $S(x)$ depending on angular variables only one has $A \sim 1/x$. Can such potentials be "gauged away" by some continious gauge transformation ?

The topological analysis made in the pioneer work [2.1] has

shown that for S with the asymptotics

$$S = \left[(x_4 + i \vec{x} \vec{\sigma}) / \sqrt{x^2} \right]^n$$

(the group is SU(2), and σ are Pauli matrixes) it can not be done. The meaning of n is the following: it shows how many times the tree-dimensional sphere is covered by the gauge group. (Note, that both manifolds are tree-dimensional). Evidently, no smooth variation can change n. It is important, that there exists some gauge invariant expression for n

$$n = \frac{g^2}{32\pi^2} \int dx \, G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} \qquad (2.29)$$

where $\tilde{G}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\sigma\lambda} G^a_{\sigma\lambda}$ is the so called <u>dual field</u>. The proof is based on the relation

$$G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} = \partial_\mu \left[2\epsilon_{\mu\nu\gamma\delta} \left(A^a_\nu \partial_\gamma A^a_\delta + \frac{2}{3} g \epsilon^{abc} A^a_\nu A^b_\gamma A^c_\delta \right) \right] \qquad (2.30)$$

and transformation of the volume integral in (2.29) to the surface one, with the use of the asymptotics of A_μ.

In the same work it was pointed out that the action can be rewritten as the following relation

$$S^E = \frac{1}{4} \int d^4x \, (G^a_{\mu\nu})^2 = \int dx \left[\frac{1}{4} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} + \frac{1}{8} (G^a_{\mu\nu} - \tilde{G}^a_{\mu\nu})^2 \right] \qquad (2.31)$$

from which it becomes evident that its extremum is given by the <u>selfdual fields</u> $G^a_{\mu\nu} = \tilde{G}^a_{\mu\nu}$, and the extreme action is completely determined by the topology. Let us give the explicit form of the solution of Yang-Mills equations, the instanton, corresponding to n=1

$$A^a_\mu = \frac{2}{g} \frac{\eta^a_{\mu\nu} (x-z)_\nu}{(x-z)^2 + \rho^2}, \quad G^a_{\mu\nu} = -\frac{4 \eta^a_{\mu\nu} \rho^2}{g[(x-z)^2 + \rho^2]^2} \qquad (2.32)$$

Here z_μ and ρ are some free parameters, the so called position and radius of the instanton. Another form of such configuration is given by the so called singular gauge, in which the topological charge is transfered to the origin $(x-z)^2 \to 0$:

$$A_\mu^a = \frac{2}{g} \frac{\bar{\eta}_{\mu\nu}^a (x-z)_\nu \rho^2}{(x-z)^2[(x-z)^2+\rho^2]} \qquad (2.33)$$

$$G_{\mu\nu}^a = -\frac{8}{g} \frac{\bar{\eta}_{\nu\rho}^a \rho^2}{[(x-z)^2+\rho^2]^2}\left[\frac{(x-z)_\mu (x-z)_\rho}{(x-z)^2} - \frac{\delta_{\mu\rho}}{4}\right] + (\mu \leftrightarrow \nu)$$

We have used above $\eta_{\mu\nu}^a$ - the so called t'Hooft symbol

$$\eta_{\mu\nu}^a = \begin{cases} \epsilon_{a\mu\nu} & \mu,\nu \neq 4 \\ -\delta_{a\nu} & \mu=4 \\ \delta_{a\mu} & \nu=4 \end{cases} \qquad \bar{\eta}_{\mu\nu}^a = \begin{cases} \epsilon_{a\mu\nu} \\ \delta_{a\nu} \\ -\delta_{a\mu} \end{cases} \qquad (2.34)$$

$$\eta_{\mu\nu}^a = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\eta_{\alpha\beta}^a \qquad \bar{\eta}_{\mu\nu}^a = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\bar{\eta}_{\alpha\beta}^a$$

$$\eta_{\mu\nu}^a = -\eta_{\nu\mu}^a, \quad \eta_{\mu\nu}^a \bar{\eta}_{\mu\nu}^b = 0, \quad \eta_{\mu\nu}^a \eta_{\mu\nu}^b = 4\delta^{ab}$$

$$\eta_{\mu\nu}^a \eta_{\gamma\lambda}^a = \delta_{\mu\gamma}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\gamma} + \epsilon_{\mu\nu\gamma\lambda}$$

Exchanging η and $\bar{\eta}$ one obtains the so called antiinstanton.

Now let us return from mathematics to physics. What is described by these configurations ?

After the introductory section 2.1 the reader surely knows the answer: they describe tunneling between some topologically different states with $G_{\mu\nu}^a = 0$, the so called classical vacua. Transitions are governed by the integer number n , so we have infinitely many vacua connected by tunneling. The problem is therefore similar to the well known quantum mechanical problem of the periodic potential with infinite number of wells, modelling some crystall. As in that problem, there appears some

set of states ("the zone") numerated by the phase θ which the wave function gains per one period - the so called θ-vacua [2.28]. The states with $\theta \neq 0$ have some quasimomentum with the "time arrow", so they are not T and CP invariant. Thus, as it was pointed out by t'Hooft [2.8] , the problem of CP conservation in strong interactions is rather nontrivial, and there should be some reason for the fact that $\theta = 0$ in real world.*)

There is no answer inside QCD - θ is conserved in strong interactions, so some other interaction should be responsible for it. As an electron emit phonons in solids and relaxes to the zone bottom, the vacuum also needs some degrees of freedom which can absorbe the extra energy and lead to the lowest $\theta=0$ state. The so called axion [2.29-2.31] was suggested, but its simplest version seems to be ruled out by experiment. However, there are other possibility, the "invisible" axion with much weaker interactions. For them there are only some cosmological limitations**).

Let us now return from these general remarks to the evaluation of the tunneling probability or, using the terminology explained in the preceeding section, to the instanton density in (four-dimensional) Euclidean space.

First, let us calculate the number of zero modes. There are four translations, one scale transformation (small change of ρ) and three Euler angles (in colour or coordinate space). As a

*) From experimental limits on neutron dipole moment $\theta < 10^{-9}$, see [2.32].

**) Of course, the interaction should be strong enough in order the relaxation to take place, but it is not strong condition. The limitations mentioned are such that not too much energy is stored now in stable "invisible" form, for we know its upper limit from the Universe expansion law, see [2.33] .

result, there appears the factor $(\sqrt{S_0})^8$. From dimensional reasons the following expression for the instanton density can be written:

$$\frac{dn}{dz} = const\, \frac{d\rho}{\rho^5}\, S_0^{4}\, e^{-S_0}\, , \quad S_0 = \frac{8\pi^2}{g^2(\rho)} \qquad (2.35)$$

where we have substituted the instanton action S_0 and have taken into account charge renormalization, so that g is changed to $g(\rho)$. In order to find the numerical constant one has to make very lengthy calculations, see [2.8,2.9]. For the SU(N) gauge group there are 4(N-2) extra rotations of some SU(2) subgroup, so that there are 4N zero modes. Finally, the instanton density is equal to

$$\frac{dn}{dz} = \frac{d\rho}{\rho^5}\, \frac{C_1}{(N-1)!\,(N-2)!}\, S_0^{2N}\, e^{-S_0 - C_2 N} \qquad (2.36)$$

$$C_1 = \frac{2e^{5/6}}{\pi^2} = 0.466 \qquad C_2 = 1.679$$

The numbers given depend on the particular regularization method used, namely that of Pauli-Villars. Respectively, expressing the charge via Λ we should use Λ_{PV}. Transition to other schemes is quite obvious, as far as ratios of lambdas are known.

So, using dilute gas approximation we have obtained (2.36), which imply that the instanton density grows with ρ as ρ^{6}*). Obviously, at large enough ρ this result fails. We return to this question in section 2.4, after discussion of the role of light quarks in the instanton physics.

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*) This follows from (2.36) and the asymptotic freedom formula for the charge.

2.3. Instantons and light quarks

Let us start with the explanation of what we mean by "light" quarks here: they are those with mass $m \lesssim 1/\rho$, ρ being the instanton radius. The opposite case of heavy quarks is not so interesting: they are just decoupled.

For simplicity, we start our discussion with one massless flavour. t'Hooft [2.8] has discovered very nontrivial phenomenon: well localized solution of the Dirac equation in the instanton field. It is usually called the zero mode ψ_0, because the integration over fermions lead to Mattew-Salam determinant which is computed in the standard way as the product of eigenvalues ε_n of the Dirac operator

$$(i\mathcal{D}_\mu \gamma_\mu)\, \psi_n = \varepsilon_n \psi_n$$

What is remarkable, this zero mode does not correspond to any evident symmetry (as zero modes of the gauge field determinant), so it was not predicted beforehand. The explicit form of the zero mode is

$$\psi_0(x) = \frac{\rho}{\pi}\left[(x-z)^2+\rho^2\right]^{-3/2} \left(\frac{1-\gamma_5}{2}\right)\varphi\, , \quad \varphi^\alpha_m = \frac{1}{\sqrt{2}}\,\epsilon^\alpha_m \qquad (2.37)$$

and in singular gauge (2.33) it looks as

$$\psi_0(x) = \frac{\rho}{\pi}\left[(x-z)^2+\rho^2\right]^{-3/2} \left(\frac{1-\gamma_5}{2}\right)\frac{\hat{x}}{\sqrt{x^2}}\,\varphi \qquad (2.38)$$

Note that two-component spinor φ means that spin and (SU(2)) colour of the quark are directly coupled to zero sum. Another important observation is that chirality is fixed.

Because the fermionic determinant stands in the nominator (unlike the gluonic one) its zero mode means that the amplitude is zero (and not infinite, as for ordinary zero modes). The physical meaning of this tunneling suppression is connected with direct relation between the topological and axial charges: only transitions with the simultaneous variation of both quantities have the nozero amplitude. As it was shown by t'Hooft, this is seen explicitly if one considers amplitudes with some external current j(x)able to flip chirality of a quark. So, with massless quarks the instantons can be considered as some effective vertexes with 2 N_f "legs", absorbing right-handed and emitting left-handed fermions. As far as properties are inverse for the antiinstantons, the "molecules" made of instanton-antiinstanton pairs can exists. Further discussion of relevant topics we continue in section 2.6.

The calculations with nonzero modes are rather standard, in particular, they contribute in the obvious way into charge renormalization. Apart from it, there appears some additional factor for each fermionic flavour. For nonzero quark mass the instanton density is nonzero even without external currents, and in the limits of small and large $m\rho$ the fermionic factor is equal to

$$F(m\rho) = \begin{cases} 1.34\, m\rho\left(1 + m^2\rho^2 \ln(m\rho) + \dots\right) & (m\rho \ll 1) \\ 1 - \dfrac{2}{75 m^2\rho^2} + \dots & (m\rho \gg 1) \end{cases} \qquad (2.39)$$

The former expression was found in [2.8] and [2.25] while the latter is given according to [2.3].

Now we return to the effective interaction among light quarks, generated by zero modes. For practical applications we need mainly the case in which ρ can be considered as small quantity, so that some <u>local effective Lagrangian</u> can be written for this interaction. It was derived in [2.16] by the following

simple method. In singular gauge zero mode at large x behaves as follows

$$\psi_0 \xrightarrow[x\to\infty]{} \frac{\rho}{\pi}\,\frac{\hat{x}}{x^4}\left(\frac{1+\gamma_5}{2}\right)\varphi \qquad (2.40)$$

which can be written as free quark propagator times some x-independent matrix. Some complications are only connected with three flavours and imbedding of instanton SU(2) group into realistic SU(3). The (somehow lengthy) effective interaction looks as follows

$$\begin{aligned} \mathcal{L}_{eff} = \int d\rho \frac{dn(\rho)}{dz\, d\rho} \Big\{ & \prod_{i=1,2,3} \left(m_i\rho - \frac{4}{3}\pi^2\rho^3 \bar{q}_{iR} q_{iL}\right) + \\ & + \frac{3}{32}\left(\frac{4}{3}\pi^2\rho^3\right)^2 \Big[\left(j_1^a j_2^a - \frac{3}{4} j_{1\mu\nu}^a j_{2\mu\nu}^a\right)\left(m_3\rho - \frac{4}{3}\pi^2\rho^2 \bar{q}_{3R} q_{3L}\right) + \\ & + \frac{9}{40}\left(\frac{4}{3}\pi^2\rho^3\right) d^{abc} j_{1\mu\nu}^a j_{2\mu\nu}^b j_3^c + (2\, perm)\Big] + \frac{9}{320}\left(\frac{4}{3}\pi^2\rho^3\right)^3 d^{abc} \\ & j_1^a j_2^b j_3^c + \frac{9 i f^{abc}}{256}\left(\frac{4}{3}\pi^2\rho^3\right)^3 j_{1\mu\nu}^a j_{2\nu\lambda}^b j_{3\lambda\mu}^c \Big\} + (L \leftrightarrow R) \end{aligned} \qquad (2.41)$$

$$q_L = \left(\frac{1+\gamma_5}{2}\right)q, \quad q_R = \left(\frac{1-\gamma_5}{2}\right)q, \quad j_i^a = \bar{q}_{iR} t^a q_{iL}$$

$$j_{i\mu\nu}^a = \bar{q}_{iR}\, \sigma_{\mu\nu} t^a q_{iL}$$

As we will discuss in chapter 6, its applications are possible also outside the Euclidean time formulation, e.g. directly ~~for~~ for quark models of hadrons.

This effective interaction explicitly violates U(1) chiral symmetry, but not the SU(N_f) one. We return to this question in section 2.6.

2.4. Instanton interactions

So far we have considered instantons in "empty" space-time, in which fields decrease at infinity. Now we are going to discuss what happens if some external fields, gluon and quark ones, are present.

For weak external gluon field the problem was solved in Ref. [2.7,2.12], but we prefer to outline more simple derivation of later work [2.16]. It is based on standard reduction formula describing transition of some field A^a_μ to asymptotic gluon with the polarization vector ϵ

$$M = i\int dx\, e^{iqx}\, \epsilon^a_\mu\, q^2 A^a_\mu(x) \qquad (2.42)$$

Now, let the field be the instanton solution in the singular gauge*), than one immediately has

$$M = -\frac{4\pi^2}{g}\,\bar\eta^a_{\mu\nu}\, q_\nu\, \epsilon^a_\mu\, \rho^2 \qquad (2.43)$$

For n gluons it goes similarly, and exponentiation corresponds to the classical limit. As a result, the instanton density is modified by the following factor

$$\frac{dn}{d\rho\, dz} = \left.\frac{dn}{d\rho\, dz}\right|_{G=0} \exp\left(-\frac{2\pi^2}{g}\rho^2\bar\eta^a_{\mu\nu}G^a_{\mu\nu}(z)\right) \qquad (2.44)$$

So, instanton behaves in the external field as four-dimensional dipole. Note also, that only field of definite duality (e.g. caused by antiinstantons) give the nonzero contribution to (2.44).

Now, consider small instanton in the QCD vacuum, with nonzero fields of vacuum fluctuations in it characterized by [5.13]:

*) This point is important, because in this gauge field decreases at infinity in such a way that asymptotic states make sense.

$$\langle 0|(gG)^2|0\rangle \simeq 0.5\ GeV^4 \qquad (2.45)$$

Using (2.44) up to the second order we find [2.16]

$$\frac{dn}{d\rho\, dz} = \left.\frac{dn}{d\rho\, dz}\right|_{G=0}\left(1 + \frac{\pi^3\rho^2}{2g^4}\langle (gG^a_{\mu\nu})^2\rangle + \dots\right) \qquad (2.46)$$

and with available estimates for the gluon condensate (2.45) one can see that t'Hooft formulae for the instanton density are applicable for $\rho \lesssim 1\ GeV^{-1}$, or for very small instantons.

Effect of the quark fields can be estimated with the help of effective Lagrangian (2.41), considering its average value in physical vacuum with nonzero quark fields, connected with the quark condensate $\langle\bar\Psi\Psi\rangle$ and SBCS. In Ref. [2.16] discussion of rather complicated multiquark operators en tering (2.41) was simplified by the so called factorization hypothesis, to be discussed in section 8.2. In this case only the following fermionic factor appears

$$\frac{dn}{d\rho\, dz} = \left.\frac{dn}{d\rho\, dz}\right|_{no\ quarks} \cdot \prod_i \left(1.34\,\rho\, M^{(i)}_{eff}\right) \qquad (2.47)$$

$$M_{eff}(\rho) \equiv m_i - \frac{2\pi^2}{3}\langle\bar q_i q_i\rangle\rho^2$$

where we have introduced the so called effective mass M_{eff} caused by the quark condensate. Note, that due to SBCS the instanton density in the QCD vacuum is nonzero even in the massless quark limit.

Completing our discussion of the instanton interaction with weak gluon and quark fields we may conclude, that in both cases we have found that the instanton density is increased. Using the terminology of statistical mechanics we may say that the interaction between instantons is of the attractive type.

In section 2.2 we have shown that instanton density is too large for large enough ρ , so some <u>repulsion</u> is needed in order to stabilize it. Jevicki [2.14] have suggested the existence of some repulsive core for the instantons, and Ilgenfritz and Mueller-Preussker [2.19,2.20] have developed some model based on the assumption of its existence. However, the physical nature of this repulsion was not understood.

The difficulty is mainly methodical: so far all attempts to solve the problem of <u>interacting</u> instanton gas have failed because no well defined way to introduce some collective variable with the meaning of instanton-antiinstanton separation was found.

Quite radical step was recently made by Dyakonov and Petrov [2.22] , who have left the straitforward path and have considered the problem with the help of Feynman variational principle. In such framework the question about the instanton interactions is much simplified: all one has to do is to substitute the <u>ansätz</u> used into the chosen action. Details of this approach will be discussed in the next section, and here we just present the results for the simplest ansätz, being just linear superposition of instantons and antiinstantons

$$A_\mu^{(cl)} = \sum_{i=1}^{N_+} A_\mu(x, z_i, \rho_i) + \sum_{i=1}^{N_-} A_\mu(x, z_i, \rho_i) \qquad (2.48)$$

S_{int} , defined as the difference between $\sum_i \left(\frac{8\pi^2}{g^2(\rho_i)}\right)$ and the complete action, is rather complicated function of the instanton positions and orientations. We give here only results for small and large distance R between two instantons and instanton-antiinstanton, marked II and $I\bar{I}$ respectively:

$$\left.\begin{aligned} S_{INT}^{I\bar{I}} &= 4\ln\left(\frac{\rho^2}{R^2}\right) \\ S_{INT}^{II} &= \left[3 + (\mathrm{Tr}\,O)^2\right]\ln\left(\frac{\rho^2}{R^2}\right) \end{aligned}\right\}(R \to 0) \qquad (2.49)$$

$$S_{INT}^{I\bar{I}} = \left(\bar{\eta}^a_{\mu\nu} O^{ab} \eta^b_{\mu\lambda} \frac{R_\nu R_\lambda}{R^2}\right)\left(4\frac{\rho_1^2\rho_2^2}{R^4} - \frac{15}{2}\frac{\rho_1^2\rho_2^2(\rho_1^2+\rho_2^2)}{R^6}\right) + \frac{27N}{4(N^2-1)}\frac{\rho_1^2\rho_2^2(\rho_1^2+\rho_2^2)}{R^6} + O(R^{-8}) \qquad (2.50)$$

$$S_{INT}^{II} = \frac{\rho_1^2\rho_2^2(\rho_1^2+\rho_2^2)}{2R^6}\left[\frac{27N}{4(N^2-1)} + \mathrm{Tr}\,O\right] + O(R^{-8})$$

$$(R \to \infty)$$

Here O^{ab} is the colour matrix, depending on the relative orientation in colour space. Note that at large R the leading term in $I\bar{I}$ case is the familiar dipole term. The factor in brackets lies within the following limits

$$-3 \le \left(\bar{\eta}^a_{\mu\nu} O^{ab} \eta^b_{\mu\lambda} \frac{R_\nu R_\lambda}{R^2}\right) \le 1 \qquad (2.51)$$

so for proper orientation this term is attractive. However, the next to leading R^{-6} term is repulsive and identical both for II and $I\bar{I}$ cases, so for most attractive orientation it produces some minimum at $R \simeq 2.5\,\rho$ with depth of about $0.1 \cdot S_0$. With the interaction of such type the instanton density surely is stabilized.

However, a word of worning is needed here. First obvious remark is that apart from "classical" part of the interaction there also exist "quantum " interaction, being the log of determinant in our ansätz background field. Second, one may well take another ansatz. In particular, the multiinstanton configurations [2.11] have the property that "classical" II interaction is exactly zero. This observation suggests some better ansätz, with lower energy of the QCD vacuum.

Our last remark is that quantum effects really seem to produce the necessary repulsion between instantons, as it is found that strong enough field (also strong enough temperature or quark density - see section 7.2) suppress them, see [6.11]. Qualitative conclusion made in my work is, in fact, very simple. Large instantons have large probability because their action $S_0 = 8\pi^2/g^2(\rho)$ is small. However, in strong field $G^a_{\mu\nu}$ all radiative corrections are cut off by the field at momenta $K^2 \sim |G^a_{\mu\nu}|$

Concluding this section we may say, that small instantons with $\rho \lesssim 1\ \mathrm{GeV}^{-1}$ make the ideal gas with well defined properties, which is however too dilute to produce some noticable effects. At larger ρ instantons interact, at large distanses attractively and at small distances (presumably) repulsively.

2.5. The instanton liquid

The qualitative properties of the instanton interaction suggest that they may form some matter similar to ordinary liquids, thus the title of this section. We know from statistical mechanics that even simplest liquids are not so simple objects, to say nothing about the instanton problem with rather uncertain interaction.*) Therefore, it is natural to proceed to the problem from the phenomenological side.

The first estimate of the instanton density made along this

*) However one should remember that this problem is much less complicated than that of nonabelian quantum field theory.

line was made in Ref.[2.17] based on the phenomenological value of the gluon condensate, found by means of sum rules. Assuming that instantons dominate in this quantity one finds

$$n_+ = n_- = \int_0^{\rho} \frac{dn}{d^4z\, d\rho} d\rho \simeq \frac{\langle (gG)^2 \rangle}{64\pi^2} \qquad (2.52)$$

Roughly speaking, it means about 1 instanton or antiinstanton in a cube $(1\ \mathrm{fm})^4$.

Now, are these instantons large or small? In order to answer this question in Ref.[2.17] the dilute gas t'Hooft instanton density was used, cut off at some ρ_c at which total density is equal to (2.52). The result was

$$\rho_c = \frac{1}{170\ \mathrm{MeV}} \sim 1\ \mathrm{fm} \qquad (2.53)$$

comparable to the instanton separation. As a result, rather disappointing conclusion was drown that main fluctuations in the QCD vacuum are "soft" and instantons are completely "melted".

However, as we have discussed in the preceeding section, this expression for the instanton density is strongly modified at much smaller radii due to the interaction. Qualitatively, at $\rho \sim 1\mathrm{GeV}^{-1}$ the main effect is attractive and density exceeds the dilute gas expression, but than repulsion should make some cut off. In Ref.[2.18] I have taken into account the dipole forces and than compared the density with (2.52). As a result, (2.53) was essentially modified

$$\rho_c \simeq \frac{1}{600\ \mathrm{MeV}} \simeq 1/3\ \mathrm{fm} \qquad (2.54)$$

If this estimate is correct, completely different picture of the vacuum fields takes place.

Its main qualitative feature is <u>the vacuum diluteness</u>, the ratio of ρ_c to avarage spacing between the fluctuations is relatively small:

$$\rho_c / \bar{R} \simeq \frac{1}{3} \ll 1 \qquad (2.55)$$

At first sight it is not so small parameter, but it mainly enters in the form of the standard packing fraction f [2.12]

$$f \equiv 2\left(\frac{\pi^2}{2}\rho_c^4\right) n_+ \sim \frac{1}{20} \qquad (2.56)$$

(first factor 2 stands for account of instantons and antiinstantons, while $\frac{\pi^2}{2}\rho_c^4$ is the volume of the four-dimensional sphere).

Moreover, the value of the typical action of the instantons is in this case rather large

$$S_0 = \frac{8\pi^2}{g^2(\rho_c)} \simeq 15 \gg 1 \qquad (2.57)$$

so that one may ignore quantum fluctuations around instantons and justify the semiclassical framework used.

Finally, the correction to the instanton action due to the interaction with other instantons is of the order*)

$$S_{INT} \simeq -(3 \div 5) \qquad (2.58)$$

so that it is reasonably smaller than S_0 (2.57) and instantons are not in fact "melted" by the interaction.

On the other hand, S_{INT} is large enough so that it is important for the estimates of the absolute probabilities

$$\exp(-S_{INT}) \sim 20 \div 100 \qquad (2.59)$$

It means that the dilute gas approximation is inadequate.

*) Although the matter is rather dilute (2.56), in four dimensions each instanton has about 8 closest neighbours, which explains why this quantity is relatively large.

The vacuum picture outlined above was tested using some correlators in Ref. [2.18] , which we discuss in details in chapter 5. Monte-Carlo data on beta-function, first compared with the instanton calculations in Ref. [2.15], where than treated by Ilgenfritz and Mueller-Preusske r [2.19] , with fitted core size leading to nearly the same parameters for the instanton size and spacing. Discussion of SBCS in this model is given in the next section, while its relation to "constituent quark" model of hadrons is discussed in chapter 6. With the increase of the number of successful applications, the confidence to such picture is growing.

It is instructive to compare the instanton density which follow from these considerations to parameters obtained by Callan, Dashen and Gross in Ref[2.13]. In the absence of repulsive interaction or empirical limitations like the $\langle (gG)^2 \rangle$ value, they have predicted too much instantons. We remind that in Ref. [2.13] considerations start with the simplest case of strong applied field with small instanton density, and than the field decreases. With larger density of instantons it is claimed that some instability is observed, so that weak enough field is expelled from vacuum. At Fig.2 we compare the instanton density at various stages of this process with empirical data considered above. It is seen, that for this instability to take place too many instantons are needed, much larger than it is allowed by known value of the gluon condensate. (See also remarks in [2.20]).

Now we pass from the general discussion of the empirical information to recent variational approach suggested by Dyakonov and Petrov [2.22]. Previously Feynman variational principle [2.21] was used in somehow similar problems, say that of liquid helium.

It is based on simple and very general relation

$$\langle e^{x}\rangle \geq e^{\langle x\rangle} \qquad (2.60)$$

where x is random variable with arbitrary distribution. It is just a reflection of the fact that exponent is the convex curve, it is most instructive to check (2.60) for two arbitrary points.

Now, suppose we substitute the exact action S by approximate one S_1 , than from (2.60) it follows that

$$\int \mathcal{D}\varphi\, e^{-S} \geq \left(\int \mathcal{D}\varphi\, e^{-S_1}\right) \exp\left[-\langle S-S_1\rangle_1\right] \qquad (2.61)$$

where φ is some field and the index "1" near the brackets means averaging with S_1. Remember that statistical sum can be understood as

$$\int \mathcal{D}\varphi\, e^{-S} \sim \exp\left[-\mathcal{E}\cdot TV\right] \qquad (2.62)$$

where $\mathcal{E}$ is the vacuum energy density and $T\cdot V$ is the four volume considered. Therefore, one may rewrite (2.61) in more transparant way

$$\mathcal{E} \leq \mathcal{E}_1 + \frac{\langle S-S_1\rangle_1}{TV} \qquad (2.63)$$

which means that we estimate the vacuum energy from above.

This idea was used in [2.22] as follows. Let us take the ansätz (2.48) for classical field $A_\mu^{(cl)}$ and write

$$A_\mu^a = A_\mu^{a(cl)} + a_\mu^a \qquad (2.64)$$

The action is now expanded in quantum field a_μ^a as follows

$$S = S(A^{(cl)}) + \int dx\, j_\mu a_\mu + \int dx\,dx'\, a_\mu(x') W_{\mu\nu}(x',x) a_\nu(x) + \ldots \qquad (2.65)$$

$$j_\mu = (\mathcal{D}_\mu G_{\mu\nu})_{cl}, \quad W_{\mu\nu} = (-\mathcal{D}^2\delta_{\mu\nu} + 2i\,G_{\mu\nu})_{cl}$$

Existence of the linear term shows that $A_\mu^{(cl)}$ is not a solution of the classical equations. The radical idea is to omit this linear term and consider new action S_1 without it, for which gaussian integration can be made. In particular, such choice of S_1 leads to

$$\langle S-S_1\rangle_1 = \int dx\, j_\mu \langle a_\mu\rangle = 0 \qquad (2.66)$$

so the additional term in (2.63) does not arize. Approximately determinants for multiinstanton configurations is substituted by the product of determinants: diluteness a posteriori justifies it. Now the problem is formulated as that of statistical mechanics, with only "classical" interaction discussed in the preceeding section depending on the instanton configurations.

Of course, such problem is very nontrivial by itself. In particular, in [2.18] I came across the uncomfortable fact that the minimum of energy is definitely given by some cubic crystall of NaCl type, with alterating instantons and antiinstantons. Obviously, we do not need spontaneous violation of Lorentz and colour symmetry in QCD*). However, the instanton problem has some analog of nonzero temperature in it, $g^2(\rho_c)$, so the minimum of free energy is in fact relevant. As it is shown in [2.22], it safely corresponds to the liquid phase for $N \lesssim 20$.

Moreover, some features of the problem turns out to be similar to those previously obtained from the phenomenology, say

$$\rho_c/\bar{R} \simeq 0.3 \qquad 8\pi^2/g^2(\rho_c) \simeq 15 \qquad (2.67)$$

*) This phenomenon may be wellcomed in other context, say in theories of gravitation in which zero mass of the graviton is connected to Goldstone theorem. In this case one needs tensor condensate $\langle\theta_{\mu\nu}\rangle \neq 0$ in space-time.

Evaluation of the absolute value of the vacuum energy by this method is not so easy to make, and they have used another variational principle. Note, that also variation of the instanton shape $f(x^2)$

$$A_\mu = \frac{2}{g} \eta_{a\mu\nu} \frac{x_\nu}{x^2} f\left(\frac{x^2}{\rho^2}\right) \tag{2.68}$$

was shown to increase the vacuum energy by the factor 2. The final estimate of Ref. [2.22] is as follows

$$\langle (gG)^2_{\mu\nu} \rangle \simeq (3.26 \Lambda_{PV})^4 \tag{2.69}$$

which coinsides with the phenomenological one at $\Lambda_{PV} = 250$ MeV. Recent lattice calculations to be discussed in chapter 3 give for gluodynamics $\Lambda_{PV} \simeq 150$-200 MeV, also they have produced $\langle (gG)^2 \rangle$ one order larger than the empirical one in QCD with quarks. Therefore, the result (2.69) can be said to be too small.

Two possibilities are therefore open. The first one is that with more accurate calculation, better ansatz and inclusion of quarks we will be able to obtain more reasonable value for the vacuum energy. Another case is that some other fluctuations are dominant in QCD vacuum. Anyway, the variational principle is very valuable methodical tool, and its better understanding and more wide applications is needed.

In connection with this, one more remark is in order here. It is possible to evaluate corrections to variational energy density by the evaluation of $\langle (S-S_1)^2 \rangle$ which, if small, tell us that we are really near the true vacuum energy.

2.6. Instantons and SBCS

We have already mentioned in section 2.3 that t'Hooft effective interaction induced by instantons in the presence of light quarks violates the U(1) chiral symmetry, but not the SU(N_f) one, so in order to explain its spontaneous breakdown in QCD vacuum (SBCS) we need some other mechanism.

Callan, Dashen and Gross [2.24] have suggested the second order interaction induced by instanton-antiinstanton pair (see diagram at Fig.3) as a candidate for the attraction between quark and antiquark in the scalar channel, leading to the instability of the symmetric vacuum. Its contribution to Bethe-Salpeter kernel was found to be as follows

$$\Sigma_p(\rho,\rho') = \left[\frac{dn}{d^4z\, d\rho} \frac{dn}{d^4z'\, d\rho'}\right]^{1/2} 2^{10} \pi^4 \rho^3 \rho'^3 \cdot \tag{2.70}$$
$$\cdot \int \frac{d^4q}{(2\pi)^4} \exp\left[-(\rho+\rho')(|q| + |p-q|)\right] \frac{q(p-q)}{q^2 (p-q)^2}$$

where p is the total momentum of the pair. The instanton density enters in power 1/2 because each instanton belongs to two loops, - see Fig.3 . Using some approximate formula for the integral over q (valid at $p\rho \lesssim 1$) one has

$$\Sigma_p(\rho,\rho') \simeq 32\pi^2 \left[\frac{dn}{d^4z\, d\rho} \frac{dn}{d^4z'\, d\rho'}\right]^{1/2} \frac{\rho^3 \rho'^3}{(\rho+\rho')^2} \exp\left[-p(\rho+\rho')\right] \tag{2.71}$$

The condition for the instability reads as the condition for the eigenvalues of the kernel under consideration

$$\mathcal{E}(p) > 1 \tag{2.72}$$
$$\int d\rho' \, \Sigma_p(\rho,\rho') \Psi_\mathcal{E}(\rho') = \mathcal{E}(p) \Psi_\mathcal{E}(\rho)$$

and the question we adress now is whether the instanton parameters considered above are sufficient to do the job.

Both in the work by Callan, Dashen and Gross [2.24] and in my paper [2.27] the answer to this question is positive: the condition (2.72) is satisfied. However, there is important quantitative difference.

In Ref. [2.24] the instability was found to appear at rather large $\rho \lesssim 1$ GeV, while in my work much smaller instanton density has resulted in more modest effect, with the instability present only for $\rho \lesssim 200$ MeV. The latter case implies that the condensate (developing due to the instability) is rather homogeneous, with important support from evaluation of four-fermion operator averages by the sum rules, see section 8.2.

The next point deals with the evaluation of the quark condensate value. The necessary equations were first considered by Caldi [2.23], with also important contribution by Carlitz and Creamer [2.25]. In our condensed notations, there are two equations. First, given instanton density generates the following contribution to the condensate

$$\langle \bar{\psi}\psi \rangle = - 2 \int \frac{dn}{dz\, d\rho} \frac{d\rho}{M_{eff}(\rho)} \tag{2.73}$$

On the other hand, the instanton density for massless quarks is nonzero only due to effective mass M_{eff} (2.47)

$$M_{eff}(\rho) = - \frac{2\pi^2}{3} \langle \bar{\psi}\psi \rangle \rho^2 \tag{2.74}$$

In general, we have some integral equations, but with instantons of the same size $\rho = \rho_c$ it can easily be solved [2.27]

$$\langle \bar{\psi}\psi \rangle = - \left(\frac{3n_+}{\pi \rho_c}\right)^{1/2}, \quad M_{eff} = 2\pi \rho_c \left(\frac{n_+}{3}\right)^{1/2} \tag{2.75}$$

With the instanton density (2.52) and $\rho_c \simeq 1/3$ fm it gives

$$\langle \bar{\psi}\psi \rangle \simeq - 10^{-2}\, GeV^3, \quad M_{eff} \simeq 200\, MeV \tag{2.76}$$

which is slightly smaller than phenomenological values, but obviously of reasonable order of magnitude.

As it follows from our discussion above, we are rather far from accurate evaluation of the instanton parameters from first principles. It is also far from being clear whether they really are dominant in SBCS phenomenon, but the results given now shows that it may well be the case.

Apart from some numbers, not very reliable at the moment, these estimates demonstrate possible existence of one more unexpected small parameter in the QCD vacuum, namely

$$\rho_c\, M_{eff}(\rho_c) \sim \frac{1}{3} \ll 1 \tag{2.77}$$

Important, that its smallness seems to show up in real world, say it makes the instanton-induced effects in the pseudoscalar channel to be much smaller than in vector or axial one (see chapter 5).

Interesting, that its smallness is in agreement with vacuum diluteness (2.55) in the instanton liquid model: M_{eff} is caused not by single instanton, but by the instanton- antiinstanton pair, so its smallness reflects large spacing in vacuum between them. On the other hand, diluteness may well follow from the power of (2.77) in the fermionic factor of the instanton density. At the moment, we do not understand well enough all these numbers, but "empirical" small parameters like (2.77) are wellcomed.

2. SEMICLASSICAL METHODS

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FIGURE CAPTION

1. Splitting between two first states of the nonlinear two-well oscillator E_1-E_0 in unites $\hbar=c=2m=1$ as a function of S_0 , the action for the one-instanton solution. The dashed, dash-dotted and solid lines correspond to the instanton dilute gas approximation, the standard WKB and exact depemdence, respectively. Points marked "x" and "•" are taken from Monte-Carlo calculations [3.44, 3.45] .

2. Instanton density versus their radius ρ (GeV^{-1}). t'Hooft dilute gas density for gluodynamics is shown by the dashed curve, the solid curve marked "SVZ" corresponds to QCD with quarks, it includes the effect of quark and gluon condensates [2.16] . Three dotted lines correspond to Ref. [2.13] for (a) "dilute phase" in strong field, (b) instability point and (c) "meron ionization". The shaded histogramm corresponds to phenomenological estimates [2.18] with $\Delta\rho/\rho = 0.2$, similar parameters correspond to Ref. [2.19].

3. Second-order interaction in terms of t'Hooft effective interaction caused by the instanton-antiinstanton pair. It results in attraction in scalar $\bar{q}q$ channel, presumably leading to SBCS.

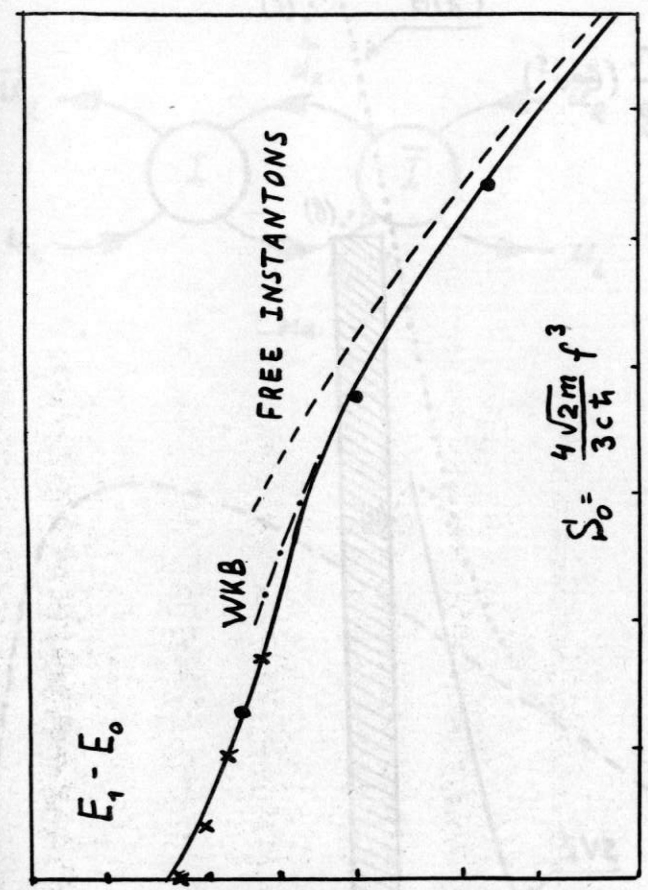

Fig.1.

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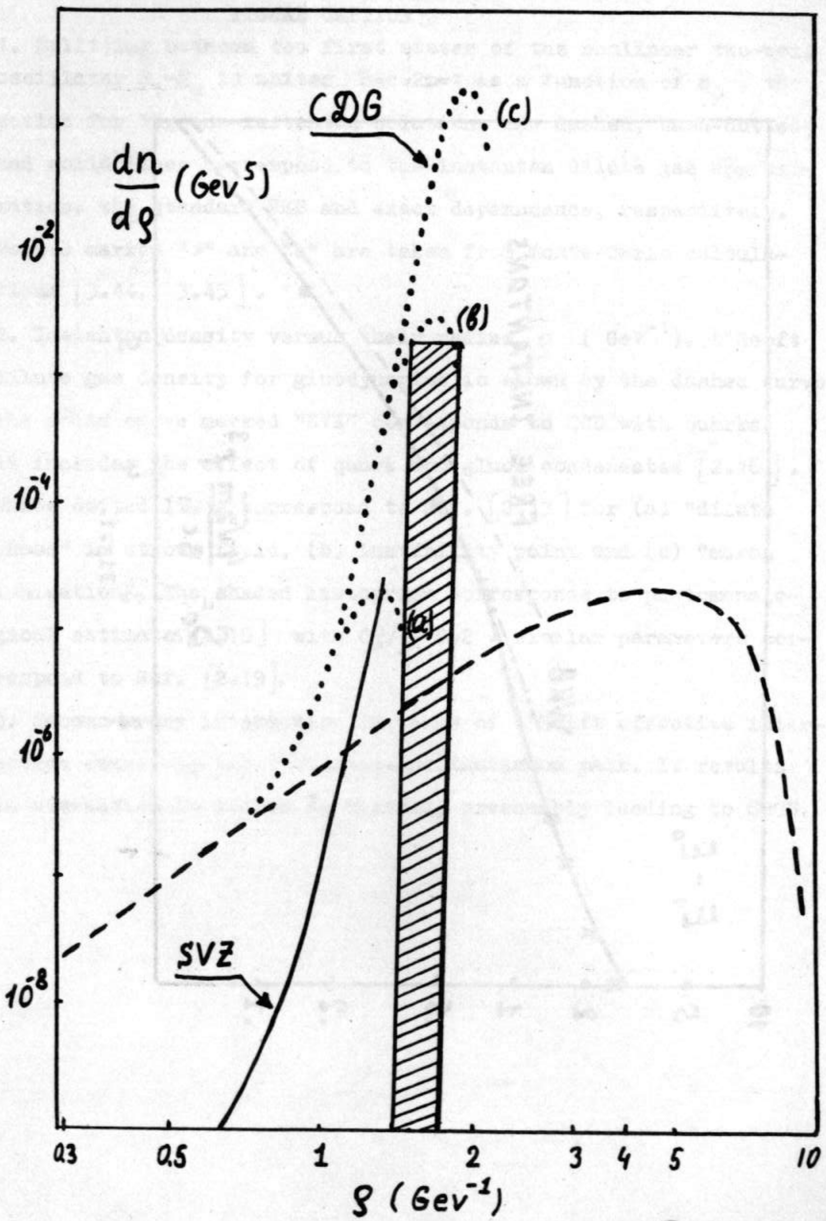

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Fig.2!

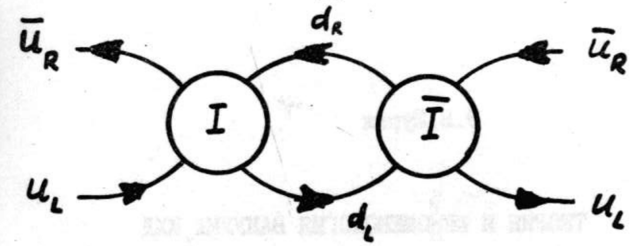

Fig.3