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THEORY AND PHENOMENOLOGY OF THE QCD VACUUM

4.CORRELATORS AND SUM RULES. THE METHODS.

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НОВОСИБИРСК

ABSTRACT

This preprint contains discussion of the correlation functions in QCD vacuum. Section 4.1 deals with different forms of the sum rules and their physical meaning, it is important for the first reading. Other sections 4.2—4.4 contain more technical information on operator product expansion method, examples of the analytical evaluation of the correlators and review of numerical methods for the evaluation of propagators in arbitrary external fields.

4.CORRELATORS AND SUM RULES. THE METHODS

The standard method of investigations of the structure of any kind of matter (say, solids, liquides or atomic nuclei) is based on its perturbation by some weak external «probes» with subsequent observation of the system (linear) response to it. Results of such experiments are usually expressed in terms of the so-called «correlators», depending on the properties of the «probes» and their locations. Probably no examples are needed as far as we discuss ordinary matter. However, for the QCD vacuum the underlying physics is essentially the same, and studies of the correlation functions are the subject of this and the next chapters.

Important general feature of such approach is connected with close connections between the correlators and the properties of elementary excitations of the system. The well known Kramers-Kronig dispersion relations connects the real part of the correlator with its imaginary part, the so-called physical spectral density. The latter prescribes at which frequencies the matter may absorb energy from the «probes» and with what intensity. Since this energy is spent to produce certain elementary excitations (say, phonons in solids) we may learn a lot about them by investigations of the correlators.

These general ideas are now widely used in order to connect the information on elementary excitations of the QCD vacuum (the hadrons) available from multiple experimental data with some theoretical evaluation of the correlators. Unfortunately, the latter ones are rather limited by the fact that only Euclidean formulation of the theory can at the moment be used, so we calculate only real part of the correlators. In other terms, we can only describe experiments in which the probes are placed outside the light cone, so that only virtual intermediate states are allowed. As a result, we are able to study only the lowest excited states in each channel. However, even in this restricted region we have a lot of facts to be explained.

The general introduction into the sum rule method is contained in section 4.1. The readers which are not much interested in technical details may omit other sections of this chapter and proceed directly to applications discussed in chapter 5.

Section 4.2 is intended to be a kind of technical introduction into the operator product expansion (OPE) method originally suggested by K.G. Wilson, which is nowdays widely used for the

evaluation of the correlators. It was emphasized in the Introduction that in the review paper being so wide as the present one it is technically impossible to derive all results explicitely, so we only outline the main ideas. However, in this more technical section we somewhat deviate from this style and discuss few simplest examples in more details, demonstrating how the method works. When this paper was nearly completed, much more detailed technical review of the OPE methods have appeared [4.22] which may be recommended to interested readers.

The OPE methods discussed in section 4.2 make no assumptions about quark and gluon fields present in the QCD vacuum and affecting the correlators, but parametrize them by a set of operator average values. However, their applications are limited by sufficiently small distances. More insight into correlator behaviour and OPE applicability region can be obtained by few examples of particular field configurations for which the correlators can be found analytically for any distances. Their discussion is made in section 4.3. Let me also comment that the qualitative understanding of the correlator behaviour in different field configurations is not in well developed stage at the moment, so probably more efforts are needed at this point.

Finally, in section 4.4 we consider numerical methods for the evaluation of the propagators in (arbitrary) external fields, which are now developing very fast. With sufficiently effective algorithm at hand it becomes possible to understand what particular properties of the vacuum fields are necessary in order to reproduce available data for the correlators.

4.1. The sum rules

As it was noted above, this method is based on general relations, following from analytical properties of the correlators, being in turn the direct consequences of causality. Probably, there is no need to go in details at this point.

Considering this method from more practical side we may say that it is based on the comparison of experimental data on the correlators with their theoretical description. As we already noted above, so far theory can only provide the correlators outside the light cone, where they do not oscillate but decay exponentially. Moreover, even with the Euclidean correlators our ability is rather

restricted by some limited range of the distances. Note that experimental data are more precise at large (time-like) intervals between the positions of the currents: here the correlators can be considered as being due to the exchange by the lightest hadrons with suitable quantum numbers. To give an example, we may recall the Ukawa theory of nuclear forces. At large enough distances between the nucleons they are described by the one-pion exchange and, therefore, relatively well understood. At smaller distances one should consider more complicated exchanges, respectively the theory becomes more uncertain.

Theoretical methods used by the sum rules are based on the asymptotic freedom property of QCD, and therefore they are most accurate at small space-time intervals. In this limit the correlators can in first approximation be considered just as free propagation of quarks and gluons, and in the second one as some small corrections induced by nontrivial structure of the QCD vacuum. As it was noted in the pioneer work by Shifman, Vainstein and Zakharov [4.4], the general way to make it is provided by the operator product expansion (OPE) method originally suggested by Wilson [4.12], to be discussed in the next section.

Quite different approach is used nowdays in lattice calculations, which in principle can produce the correlators at any distances. However, in practice they are strongly limited from below by lattice spacing and from above by the lattice size and the statistical accuracy needed in order to observe very weak signal (to say nothing on other approximations made), see section 4.4.

It is probably useful to start with the explanation where the experimental data on the correlators in the QCD vacuum come from. Let me recollect in this connection more familiar case, say studies of the nucleon structure by means of inclusive deep inelastic electron scattering. It produces an amplitude of the nucleon state transition into all other states under the influence of electromagnetic current operator $j_\mu(x)$. In this amplitude squared it is possible to sum over all intermediate states and obtain the spectral density for the following two-current correlator inside the nucleon:

$$K_{\mu\nu}(q)=i\int dx\, e^{iqx}\langle N|\, T\{j_\mu(x)j_\nu(0)\}|\, N\rangle \qquad (4.1)$$

where q is the momentum transfered to the electron. Note that at large q one has small x, and that is why we are sure that

perturbative QCD methods are applicable for deep inelastic scattering. It is hardly necessary to say more on this well known example.

Note that in the example considered the particular properties of the nucleon state are not important, thus it can well be substituted by any other one, say the vacuum state. Electromagnetic current excites the hadronic vacuum in the process of e^+e^- annihilation into hadrons, so these data can be converted into the two-current correlator

$$K_{\mu\nu}(x)=\langle 0|\, T\{j_\mu(x)j_\nu(0)\}|\,0\rangle \tag{4.2}$$

Introducing the standard polarization operator $\Pi(q^2)$

$$\Pi(q^2)(q_\mu q_\nu - q^2 g_{\mu\nu}) = i\int dx\, e^{iqx} K_{\mu\nu}(x) \tag{4.3}$$

we may write down the standard dispersion relation for it

$$\Pi(Q^2)=\frac{1}{\pi}\int \frac{\mathrm{Im}\,\Pi(S)dS}{S+Q^2}$$

$$Q^2=-q^2 \tag{4.4}$$

and connect its spectral density $\mathrm{Im}(\Pi)$ to physical cross section

$$\mathrm{Im}\,\Pi(S)=\frac{S}{16\pi^2\alpha^2 e_q^2}\sigma(e^+e^-\to\bar q\, q) \tag{4.5}$$

where e_q is the electric charge of the particular quark flavour (it is assumed that experimentally such cross sections can be distinguished). Unfortunately, we do not have many other «probes» as good as the electromagnetic current. However, from tau lepton decay we know some part of the spectral density of the axial current. In many other cases masses of the lowest states are well known, say for the pseudoscalar current

$$j=i\bar u\gamma_5 d \tag{4.6}$$

it is the negative pion. Moreover, we know also its coupling to pseudoscalar and axial currents from $\pi\to\mu\nu$ decay

$$\langle\pi^-(p)|\,\bar u\,\gamma_\mu\gamma_5 d|\,0\rangle = -ip_\mu f_\pi$$

$$\langle\pi^-(p)|\,i\bar u\,\gamma_5 d|\,0\rangle = \frac{m_\pi^2}{m_u+m_d} f_\pi \tag{4.7}$$

Thus, although we can not experimentally excite the vacuum by the current (4.6), we know something about the relevant correlator. Similar artificial currents can be introduced also in many other cases, say recently much attention was given to those producing the baryons. Such information can also be used for the test of theoretical calculations.

Particular form of the sum rules used are determined by the condition that they should be sensitive mostly to lowest excitations: otherwise their predictive power becomes too low. First types of sum rules [4.1—4.3] were indeed not so easy to use, and some improvements were suggested in [4.4]. The first idea is to use the so-called moments of the spectral density, connected with n-th derivative of the polarization operator at $q=0$:

$$M_n\equiv\frac{1}{\pi}\int_0^\infty \mathrm{Im}\,\Pi(s)\frac{ds}{s^{n+1}} \tag{4.8}$$

It is most useful to consider the ratio of subsequent moments, it is clear that at large enough n it becomes connected only with the mass of the lowest state m_{res}

$$M_{n+1}/M_n \underset{n\to\infty}{\to} 1/m_{res}^2 \tag{4.9}$$

which is usually very well known. This type of analysis is widely used for heavy quarkoniums.

Another useful suggestion made in [4.4] is connected with the so-called Borel transformation

$$\hat B f(s)=\lim_{\substack{s\to\infty\\ n\to\infty\\ m^2=s/n=\mathrm{const}}}\left[\frac{1}{(n-1)!}\left(-\frac{\partial}{\partial s}\right)^n f(s)\right] \tag{4.10}$$

which leads to sum rules of the type

$$\Pi(m^2)\equiv\hat B\Pi(s)=\frac{1}{\pi}\int \mathrm{Im}\,\Pi(s)\exp(-s/m^2)ds \tag{4.11}$$

where m is the so-called Borel parameter. Roughly speaking, $1/m$ plays the role of the distance between the applied currents. This form of the sum rules is traditional in applications to light quarks.

In the case when one (or more) quarks excited by the current is heavy it is possible to use more familiar nonrelativistic language [4.5, 4.6], explicitly analogous to Kramers-Kronig relation (in Euclidean time):

$$\Pi(\tau)=\hat{B}\Pi(E)=\frac{1}{\pi}\int \mathrm{Im}\,\Pi(E)\exp(-E\tau)dE \qquad (4.12)$$

Here E is the nonrelativistic energy and τ is just the space-time interval separating the moments of the applications of the currents.

In many cases, say in analytical models of the vacuum fields or in various numerical calculations it is very convenient to work directly in coordinate representation, rather than perform Fourier and other integral transformations. In such case one may apply the sum rules suggested in Ref. [4.7]

$$\Pi(x^2)=\frac{1}{\pi}\int \mathrm{Im}\,\Pi(s)\,D(\sqrt{s}\,,\sqrt{-x^2})ds$$

$$D(m,\tau)\equiv\frac{m}{4\pi^2\tau}K_1(m\tau) \qquad (4.13)$$

where $K_1(x)$ is the Bessel function of imaginary argument. Note that $D(m,\tau)$ is just the propagator of a scalar particle, so its physical meaning is selfevident.

Finally, let me mention also the sum rules suggested in Ref. [4.8]. In contrast to other ones considered above it is intended to study not the lowest states, but the spectral density in some particular region of energies. The main idea is that one may approach physical cut in the Q^2 complex plane not only along its real (negative) values, but also from other directions. Uncertainty principles tell us that being at some distance from the physical region we deal with virtual phenomena so one may hope to use perturbative arguments. Presumably, such estimates should be compared to «smoothed» cross section, in which details connected with «large time physics» are assumed to be absent. In some works (see for example [4.9]) such analysis was applied to e^+e^- data. In particular, it is shown that the average cross section above charm threshold is somewhat larger than the asymptotic value predicted by lowest order QCD. In Ref. [4.10] it was speculated that strong enough instanton-type fluctuations in vacuum may explain the physical cross section in this region but no explicite fit to data was in fact demonstrated.

Apart from other details, the main idea of this approach is rather uncertain, for we do not really know whether «large distance physics» is indeed excluded by suggested «smoothening» prescriptions. For example, in section 4.3 we demonstrate that weak (Euclidean) electric field weakly affects the «small distance

physics» and usual sum rules, while it leads to infinite mass renormalization and, therefore, vanishing spectral density in the physical region. This example (although not quite physical by itself) clearly demonstrates that attempts to calculate explicitely the (smoothed) cross sections are very uncertain.

Finally, few words about some useful parametrization being now standard in applications of the sum rules. We always have the lowest resonance and «something else», which can be parametrized as follows

$$\mathrm{Im}\,\Pi(s)=\lambda^2\delta(s-m_{res}^2)+\theta(s-W^2)\,\mathrm{Im}\,\Pi^{pert}(s) \qquad (4.14)$$

where $\mathrm{Im}(\Pi^{pert})$ stands for perturbatively evaluated spectral density. In this formula we have three free parameters: λ, m_{res} and W. At large distances all is governed by the lowest state, so two first parameters can be determined. At small distances due to the asymptotical freedom the second term produces correct asymptotic behaviour. Let us discuss this point in more details, using for example the coordinate representation (4.13). Let us write down the physical correlator at small distances as the perturbative one plus small nonperturbative correction

$$\frac{1}{\pi}\int_0^\infty \mathrm{Im}\,\Pi^{pert}(s)D(\sqrt{s}\,,\tau)ds+\Pi^{nonpert}(\tau)=$$
$$=\lambda^2D(m_{res},\tau)+\frac{1}{\pi}\int_{W^2}^\infty \mathrm{Im}\,\Pi^{pert}(s)D(\sqrt{s}\,,\tau)ds$$

It is natural to cancel integrals over large energies at both sides (related to most singular at $\tau\to 0$ contributions) and to write the remaining condition as follows

$$\frac{1}{\pi}\int_0^{W^2} \mathrm{Im}\,\Pi^{pert}(s)D(\sqrt{s}\,,\tau)ds+\Pi^{nonpert}(\tau)=\lambda^2D(m_{res},\tau) \qquad (4.15)$$

This relation reminds naive «duality condition»: the contribution of the resonance is equal to that of the gap «eaten up» by the resonance. Note however, that this relation is now better grounded. In particular, it includes the (calculatable) nonperturbative corrections. With relation like (4.15) one may easily estimate the value of W. This parameter is important, for it fixes the energy scale at which asymptotic freedom is violated by some vacuum effects.

4.2. Operator product expansion

The general idea of this method (by the way, as of many other works by K.G. Wilson) is the explicite separation of physical phenomena characterized by different scales. In the problem under consideration, evaluation of the correlation between two currents, one scale is given by distance x (assumed to be the smallest one) and the second one corresponds to some typical correlation length in the QCD vacuum. «Expansion» implies that nonlocal quantity is expressed in terms of local properties of the QCD vacuum, while the word «operator» means that no specific assumptions about the vacuum fields are made. Its general form can be written as follows:

$$j(x)j(0)=\sum C_k(x)O_k(0) \tag{4.16}$$

where (x-depending) coefficients C_k are some numbers while O_k are the (local) operators composed out of all fields of the theory, taken at the point $x=0$. With small parameter x at hand we should first consider effects leading to singular at $x\to 0$ terms in the correlators. Below we show that it can effectively be done in momentum representation as an expansion over powers of $1/Q$ where Q is large momentum transfered by the currents. Somewhat more subtle point is the evaluation of terms regular at small x, and recently there was some activity concerning these questions in the framework of exactly solvable two-dimensional models, see Refs [4.14, 4.15]. Note however, that usually the leading singularity is rather strong (say, the correlator of currents made of two massless quarks starts from $1/x^6$ term due to their free propagation), so regular terms are not very important quantitatively at sufficiently small x.

If one includes the radiative corrections (loop diagrams) he introduces all intermediate scales. Then one should prescribe more definitely which of them he includes in coefficients and which in the operators. Usually only logarithmic effects are considered, therefore only the order of magnitude of some intermediate «normalization point» μ should be fixed. By definition, all virtual momenta with $k^2<\mu^2$ we include in the operators (resulting in $\log(\mu/\Lambda)$) and all large momenta $k^2>\mu^2$ in the coefficients (resulting in $\log(x\cdot\mu)$). Since the introduced quantity μ is completely artificial, the sum (4.16) should not depend on it.

So far it was only some convention, thus it is meaningless to ask whether it was right or wrong. The situation is changed when one makes some particular calculations in which definite physical assumptions are made. In QCD we believe that «small scale physics» can be approximately controlled by the perturbation theory, while the (so far unclear) nonperturbative phenomena are connected with the «large scale physics» and can be attributed to the operators. This standard assumption has, of course, some accuracy which can be discussed.

Generally speaking, this assumption does not hold because there may exist nonperturbative fluctuations of the size $\varrho\sim 1/\mu$ and even smaller, so by definition there exist nonperturbative effects in the coefficients as well. However, at sufficiently large μ (e.g. $\geqslant 1$ GeV) they are very small (see for example the instanton density considered in chapter 2). More detailed discussion of this point can be found in Ref. [4.13].

Another side of the coin is that in pure perturbative calculations with loops one should not forget to exclude the integration over momenta $k^2<\mu^2$. This in general leads to contributions into the OPE coefficients being some powers of μ, which are sensitive to particular «scale separation» prescription. Of course, such μ-dependent terms should be canceled by similar terms in the operators, producing physical μ-independent results. The very fortunate feature of QCD (which may well be absent in other theories) is that vacuum expectations values of main operators are so large that with typical μ of the order of 1 GeV or smaller one can safely ignore powers of μ. This comment will become more clear after examples considered below, see also [4.22].

Coming to particular examples I would like to recollect the Fermi saying «What is the hydrogen atom of this problem?». It seems that in this case the simplest problem is that strongly resembling the hydrogen atom by itself, for it is connected with current producing one very heavy and one light quark (antiquark) [5.37]. The virtue of heavy quark is connected with the possibility to apply more familiar nonrelativistic language, while the virtue of the light one is that one may ignore Coulomb-type interaction between quarks, being important for quarkonium currents (see section 5.6). Thus our current looks as follows:

$$j_A=\bar{Q}\Gamma_A q \tag{4.17}$$

where Γ is some gamma matrix. At sufficiently small distances the

polarization operator is given by the loop made of free propagators

$$\Pi_{AB}^{free}(x) = \mathrm{Tr}\ [\Gamma_A S_q^{free}(0, x)\ \Gamma_B S_Q^{free}(x, 0)] \tag{4.18}$$

where x may be taken along Euclidean time axes and $x^2 = -\tau^2$, so that the relevant propagators look as follows

$$S_Q^{free}(\tau) = \left(\frac{1+\gamma_0}{2}\right) \exp(-m_Q\tau)/(2\pi m_Q\tau)^{3/2}$$

$$S_q^{free}(\tau) = -\gamma_0/(2\pi^2\tau^3) \tag{4.19}$$

At small x the first nonperturbative correction to this simple calculation is connected with the nonzero quark condensate. Roughly speaking, it implies that there are plenty of light quarks in vacuum, so the second current does not necessary pick up the quark produced by the first one. In order to account for such «exchange interaction» with the condensate one has to include in the light quark propagator the following constant term:

$$S_q = S_q^{free} - i\langle 0|\bar{q}q|0\rangle + \ldots \tag{4.20}$$

and the related correction is immediately found:

$$\Pi_{ij}(\tau) = \Pi_{ij}^{free}(\tau)\ [1 + P \cdot 2\pi^2\langle 0|\bar{q}q|0\rangle\tau^3 + \ldots] \tag{4.21}$$

So, we have found the OPE coefficient for the operator $\bar{\Psi}\Psi$. (P appearing in this relation is the state parity.) Note that splitting in parity (say between pseudoscalar and scalar channels) appears only with the account of chiral symmetry breaking (we return to this point in discussion of the quark effective mass in section 6.1). All gluon-induced interactions between quarks or with vacuum fields are in parity-conjugate channels identical. (They can produce other splittings though, say between pseudoscalar and vector channels. This is of course the «hyperfine splitting» related to spin-spin interaction being of the order of $1/M$ where M is the heavy quark mass.)

Perturbative evaluation of the OPE coefficients for gluonic operators can in principle be made with ordinary Feynman diagrams containing external gluonic lines. However, it is extremely inadequate way of making such calculations. First of all, separate diagrams are not gauge invariant, only their complete sum is. Second, covariant operators contain terms of different order in coupling constant, so in case of several operators one should make next order calculations in order to distinquish them. Finally,

familiar diagrammatic calculations are made in momentum representation and one should be careful in separation of small and large scale physics. For example, soft gluon emission from external lines is, of course, the large distance effect. However, such diagrams also contain «contact terms» which should be included in the coefficients. Because of all these complications it turns out that it is more convenient to use other methods, based on the ideas previously suggested by Schwinger in QED framework (see e.g. the classical paper [4.16]).

The so-called operator formalism considers particle propagation in some external «background» field G which is arbitrary (in particular, it is not assumed to be weak). Its very useful features are gauge invariance of the calculations at all stages of the calculations and the fact that calculations are made in coordinate representation. Its more detailed discussion is made in Ref. [4.22].

Let us introduce the formal basis of coordinate states $|x\rangle$, being the eigenvectors of the coordinate operator X

$$X_\mu|x\rangle = x_\mu|x\rangle \tag{4.22}$$

as well as momentum operator P_μ acting in this basis as a covariant derivative

$$\langle x|P_\mu|y\rangle = -i\frac{\partial}{\partial x_\mu}\delta(x-y) + gT^aA_\mu^a(x)\delta(x-y) \tag{4.23}$$

where the colour generator T^a here is assumed to be in the representation corresponding to particle under consideration. This operator satisfies the following commutation rules:

$$[P_\mu X_\nu] = ig_{\mu\nu}, \qquad [P_\mu P_\nu] = igT^aG_{\mu\nu}^a \tag{4.24}$$

In such formalism the propagator can be written as

$$S(x, y) = \langle x|\frac{1}{\hat{P}-m}|y\rangle, \qquad \hat{P} \equiv \gamma_\mu P_\mu \tag{4.25}$$

Its Fourier transform can be written as follows:

$$S(q) = \int dx\, e^{iqx}\langle|\frac{1}{\hat{P}-m}|0\rangle = \int dx\,\langle x|\frac{1}{\hat{P}+\hat{q}-m}|0\rangle \tag{4.26}$$

where the following identities were used:

$$e^{iqX}P_\mu = (P_\mu + q_\mu)e^{iqX}$$

$$e^{iqX}=|0\rangle=|0\rangle$$

The formula (4.26) is very convenient for OPE, one may say that there is large (numerical) part of the momentum q and small (operator) one P, so it is tempting to expand in P/q (for simplisity we put $m=0$)

$$\frac{1}{\hat{P}+\hat{q}}=\frac{1}{\hat{q}}-\frac{1}{\hat{q}}\hat{P}\frac{1}{\hat{q}}+... \tag{4.27}$$

Note that as soon as P is the differential operator we have indeed expanded the nonlocal object, the propagator, over the local ones. All what is left to do are some algebraic manipulations in order to rewrite the result in more familiar notations. Let us demonstrate how this method works with the example considered above, the correlator for a current made of light and heavy quark. Introducing also the operator Ψ acting as

$$\Psi|x\rangle=\psi(x)|x\rangle \tag{4.28}$$

where $\psi(x)$ is some wave function satisfying the equation of motion in the background field

$$i\hat{D}_x\psi(x)=0 \tag{4.29}$$

while its particular form is irrelevant. Substituting into the correlator light quark field operators and heavy quark propagator in simbolic form one obtains

$$K_{ij}=\langle x|\overline{\Psi}\Gamma_i\frac{1}{\hat{P}+\hat{q}-m_Q}\Gamma_j\Psi|0\rangle \tag{4.30}$$

as our starting point. First, let us for simplisity consider heavy quark mass to be very large and use the nonrelativistic approximation

$$S_Q\simeq\frac{(1+\gamma_0)}{2\cdot(P_0-E)} \tag{4.31}$$

where E is the nonrelativistic energy, $E=q_0-m_Q$.

Expanding (4.31) in P_0/E we observe that zero order term was already used above while the next nontrivial correction appears only in second order. It is clear that nonzero vacuum average values are present only for Lorents scalars, thus

$$\langle\overline{\Psi}P_0^2\Psi\rangle=\frac{1}{4}\langle\overline{\Psi}P_\mu^2\Psi\rangle=-\frac{1}{16}\langle\overline{\Psi}(ig\sigma_{\alpha\beta}G^a_{\alpha\beta}t^a)\Psi\rangle \tag{4.32}$$

where in the last step we have used Dirac equation in its squared form. Further manipulations of this kind produce the following result [5.37] (for $Qi\gamma_5q$ current):

$$\Pi=-\frac{\langle\overline{\Psi}\Psi\rangle}{2E}+\frac{\langle\overline{\Psi}(igG_{\mu\nu}G^a_{\mu\nu}t^a)\Psi\rangle}{32E^3}-\frac{\langle\overline{\Psi}(\gamma_\mu t^a gD^{ab}_\nu G^b_{\mu\nu})\Psi\rangle}{2^6 3E^4}+... \tag{4.33}$$

It is true, one should spend some time in order to master these formalism, but he will spend more making diagramatic derivation of the same results.

Another useful method of evaluation of OPE coefficients is based on the propagator expansion in the so-called fixed point gauge:

$$x_\mu A_\mu(x)=0 \tag{4.34}$$

which was independently suggested by Fock, Schwinger and many others [4.18, 4.19]. In this gauge there is very convenient formula expressing potential in terms of the field and its covariant derivatives:

$$A_\mu(x)=x_\nu\int_0^1 d\alpha\,\alpha G_{\nu\mu}(x\alpha)=$$
$$=\sum_{k=0}^{\infty}\frac{1}{k!(k+2)}x_\nu x_{\alpha_1}...x_{\alpha k}(D_{\alpha_1}...D_{\alpha k}G_{\mu\nu})|_{x=0} \tag{4.35}$$

(see its detailed discussion in Shifman work [4.20] or in [4.22], a little bit complicated point here is the substitution of ordinary derivatives in (4.35) by the covariant ones). In particular, from (4.35) it follows that $A_\mu(0)=0$.

In order to demonstrate utility of these relations let me start with the simplest example, the propagator of scalar massless particle in external field [4.21]:

$$D(q)=\int dx\,\langle x|\frac{1}{(P+q)^2}|0\rangle \tag{4.36}$$

Expanding as above in P/Q

$$\frac{1}{(P+q)^2}=\frac{1}{q^2}-\frac{2Pq}{q^4}+\frac{4(Pq)^2}{q^6}-\frac{P^2}{q^4}+$$

$$+\frac{P^2(2Pq)+(2Pq)P^2}{q^6}-\frac{8(Pq)^3}{q^8}+O(P^4) \tag{4.37}$$

we obtain some polinomials in P. Note that each covariant derivative consists of two parts, the ordinary derivative and the term with A_μ. The former gives zero acting to the left in (4.36)

$$\int dx\,\langle x|\,\partial_\mu ...|\,0\rangle=\int dx\,dy\,\partial_\mu\delta(x-y)\,\langle y|\,...|\,0\rangle \tag{4.38}$$

while A_μ gives zero acting to the right

$$A_\mu(x)|0\rangle=A_\mu(0)|0\rangle \tag{4.39}$$

All one has to do is to move all derivatives to the left and all potentials to the right, with only the commutators (derivatives of A) surviving. For their evaluation (4.35) is of great use, giving covariant expressions for these derivatives. For example, one finds:

$$\int dx\,\langle x|\,P^2P_\alpha|\,0\rangle=\frac{g}{6}(D_\beta G_{\beta\alpha})$$

$$\int dx\,\langle x|\,P_\alpha P^2|\,0\rangle=-\frac{g}{3}(D_\beta G_{\beta\alpha}) \tag{4.40}$$

which leads to the following result

$$D(q)=\frac{1}{q^2}-\frac{g}{3q^6}(D_\alpha G_{\alpha\beta}q_\beta)-\frac{q^2}{2q^8}\Big[q_\alpha q_\gamma G_{\alpha\beta}G_{\beta\gamma}+$$

$$+\frac{1}{4}G^2q^2\Big]-\frac{ig}{q^8}(qD)(DGq)+O(P^5) \tag{4.41}$$

Now, let us also give for applications analogous formulae for spinor and gluon propagators derived by this method in Ref. [4.21]:

$$S(q)=\frac{1}{\hat q}-\frac{g}{2q^4}(q\tilde G\gamma)\gamma_5+\frac{g}{3q^6}[q^2(DG\gamma)-\hat q(DGq)-(qD)(qG\gamma)-$$

$$-3i(qD)(q\tilde G\gamma)\gamma_5]+\frac{g}{q^8}[iq^2(qD)(DG\gamma)-i\hat q(qD)(DGq)-$$

$$-i(qD)^2(qG\gamma)+2(qD)^2(q\tilde G\gamma)\gamma_5-\frac{1}{2}D^2(q\tilde G\gamma)\gamma_5-\frac{g}{2}\hat q(qGGq)+$$

$$+\frac{g}{4}q^2q_\alpha\{G_{\alpha\beta}G_{\beta\delta}\}_+\gamma_\delta-\frac{g}{4}q^2q_\alpha[G_{\alpha\beta}G_{\beta\delta}]_-\gamma_\delta]+O(P^5) \tag{4.42}$$

where $G_{\mu\nu}=G^a_{\mu\nu}t^a$ is colour matrix. In gluonic propgator it is more convenient to use other notation $G^{ab}_{\mu\nu}=f^{abc}G^c_{\mu\nu}$. The result looks as follows:

$$D_{\mu\nu}(q)\equiv\int dx\,\langle x|\,\frac{1}{(q+P)^2g_{\mu\nu}-2gG_{\mu\nu}}|\,0\rangle=$$

$$=\frac{g_{\mu\nu}}{q^2}+\frac{2g}{q^4}G_{\mu\nu}+\frac{4ig}{q^6}(qD)G_{\mu\nu}-\frac{2ig}{3q^6}g_{\mu\nu}(DGq)+$$

$$+\frac{2g}{q^8}(qD)(DGq)g_{\mu\nu}+\frac{2g}{q^8}[q^2D^2G_{\mu\nu}-4(qD)^2G_{\mu\nu}]+$$

$$+\frac{g^2}{2q^8}g_{\mu\nu}[q^2G^2_{\alpha\beta}-4(q_\alpha G_{\alpha\beta})^2]+\frac{4g^2}{q^6}G_{\mu\alpha}G_{\alpha\nu}+O(P^5) \tag{4.43}$$

Let us ask the following (somewhat strange) question: what happens if one try to average these formulae over fields, considering them as vacuum ones so that only scalar combinations survive? It can be easily seen that no corrections in the approximation considered so far appear in scalar and spinor cases, while in the gluon case it appears due to «magnetic moment» effect. Analogous cancellation takes place in the selfdual field $G^a_{\mu\nu}=i\tilde G^a_{\mu\nu}$ for which the stress tensor

$$G^a_{\mu\sigma}G^a_{\nu\sigma}-\frac{1}{4}g_{\mu\nu}(G^a_{\alpha\beta})^2 \tag{4.44}$$

is identically zero. We return to these observations in the next section, while here it can be used in order to simplify the calculations of «gluon condensate» effects, very important for applications. As the simplest example we take vector current made of light quarks $j_\mu=\overline{\Psi}\gamma_\mu\Psi$ and consider the two-current correlator in coordinate representation. Effects of the order of G^2 can appear either from $O(G)$ effects in both propagators, or from $O(G^2)$ correction in one of them. The observation made above shows that the second possibility produces zero effect, so it is sufficient to use $O(G)$ correction. Rewriting (4.42) in coordinate representation:

$$S(x)=-\frac{\hat x}{2\pi^2x^4}-\frac{g}{16\pi^2x^2}(x\tilde G\gamma)\gamma_5+O(\ln x) \tag{4.45}$$

One easily obtains

$$\Pi_{\mu\nu}(x)=\langle 0|\,T\{j_\mu(x)j_\nu(0)\}|\,0\rangle=-\frac{3i}{\pi^4x^8}(2x_\mu x_\nu-g_{\mu\nu}x^2)-$$

$$-\frac{2i}{3(16\pi^2)^2}\langle 0|(gG)^2|0\rangle(2x_\mu x_\nu+g_{\mu\nu}x^2) \qquad (4.46)$$

(Note that $\partial_\mu\Pi_{\mu\nu}=0$ as it should be.) This result corresponds to that originally obtained in Ref. [4.4] by much more complicated diagrammatic calculation, while that given above [4.21] takes only few minutes! This method was later used for rather complicated calculations [5.49] of $O(G^3)$ and $O(G^4)$ corrections for heavy quarks.

Another sufficiently simple application of the fixed-point gauge was made by Shifman [4.20] and it corresponds to OPE expansion of the average value of the Wilson loop:

$$W(c)=\left\langle\frac{1}{N_c}\,\mathrm{Tr}\,\mathrm{Pexp}\left[\frac{ig}{2}\oint_c A_\mu^a t^a\,dx_\mu\right]\right\rangle \qquad (4.47)$$

which is assumed to be small as compared to the correlation length in QCD vacuum. These formulae are now used in order to extract local properties of the QCD vacuum from lattice calculations.

Substituting into (4.47) potential in the form (4.35) we expand in terms of the fields. The first nontrivial correction [4.20] looks as follows

$$W(c)=1-\frac{\langle(gG)^2\rangle}{192N_c^2}\iint_c[(xy)(dx\,dy)-(x\,dx)(y\,dy)] \qquad (4.48)$$

For plane contour the latter integral is just $4A^2$ where A is its area. It was also pointed out in [4.20] that further corrections also depend on the area rather than on the contour shape, say the next correction is as follows

$$\Delta W=\frac{\langle g^3f^{abc}G_{\mu\nu}^aG_{\nu\sigma}^bG_{\sigma\mu}^c\rangle}{768N_c}\iint[x^2(xy)+y^2(xy)-2x^2y^2(dx\,dy)] \qquad (4.49)$$

where the integral is equal to const$\cdot A^3$ where const$=4/\pi$ for round contour and 4/3 for the square one. Unfortunately, simplest factorization hypothesis concerning vacuum fields leads to oscillating rather than exponentially decreasing behaviour at large A (Wilson criterium for confinement), see details in [4.20].

Our last pedagogical example is simple derivation of the famous «triangular anomaly» relation for divergence of the axial

current:

$$\partial_\mu j_\mu^A=\frac{\partial}{\partial x_\mu}\left\{\overline{\Psi}(x+\varepsilon)\gamma_\mu\gamma_5\left[\exp\int_{x-\varepsilon}^{x+\varepsilon}igt^aA_0^a(y)\,dy_0\right]\Psi(x-\varepsilon)\right\} \qquad (4.50)$$

where Schwinger splitting prescription was used. Applying the fixed-point gauge and the quark propagator in $O(G)$ approximation (4.45) one easily finds the well known relation

$$\partial_\mu j_\mu^A=\frac{g^2}{16\pi^2}G_{\alpha\beta}^a\tilde{G}_{\alpha\beta}^a$$

4.3. Analytical estimates of the correlators

In this section we discuss two examples, in which the correlators can be found analytically. The first one is rather simple, it is connected with the nonrelativistic motion in homogeneous electric field. The second one is more complicated, it deals with instantons, but presumably it can be used in realistic applications. I hope that these consideration provide some insight into qualitative behaviour of the correlators.

The first problem [4.23] considers heavy quark-antiquark pair in abelian homogeneous electric field $E(\tau)$ depending on Euclidean time τ. The corresponding action looks as follows:

$$S=\int_0^{\tau_0}\left[\frac{m\dot{x}_q^2}{2}+\frac{m\dot{x}_{\bar{q}}^2}{2}+\frac{g}{2}E(\tau)(x_q-x_{\bar{q}})\right]d\tau \qquad (4.51)$$

It is well known that path integral method can produce the analytic solution for the propagator, because the relevant integrals are Gaussian. All one has to make is to find the extreme path from classical (Euclidean) equation of motion

$$x_{cl}(\tau,E)=\frac{g}{m}\left[\int_0^\tau d\tau'\int_0^{\tau'}d\tau''\,E(\tau'')-\frac{\tau}{\tau_0}\int_0^{\tau_0}d\tau'\int_0^{\tau'}d\tau''\,E(\tau'')\right] \qquad (4.52)$$

and to calculate the action $S[x_{cl}]$, which directly provides the modification factor for free propagators:

$$\Pi(x)=\Pi^{free}(x)\,\exp\{-2S[x_{cl}(\tau E)]+2S[x_{cl}(\tau,0)]\} \qquad (4.53)$$

The simplest case is that of the constant field

$$\Pi(x)=\Pi^{free}(x)\cdot\exp\left(\frac{g^2E^2\tau_0^3}{48m}\right) \qquad (4.54)$$

With these expressions at hand, we may make some physical observations. Recall that the field considered is assumed to model that of nonperturbative fluctuations in the QCD vacuum. The known sign of the gluon condensate corresponds (in ordinary Minkowski space-time) to real magnetic but imaginary electric field:

$$\langle E_m^2\rangle = -\langle H_m^2\rangle = -\langle G_{\mu\nu}^2\rangle/2 < 0 \tag{4.55}$$

With such sign one observes that in constant field the correlator is rapidly decreasing at large distances. Comparing this result with exp (—energy·τ) one concludes that such field excites the quark to arbitrarily large energy. (For real constant field such conclusion simply corresponds to the fact that in linear potential the energy is not limited from below.) One may say that it is «superconfinement» or, more precisely, an infinite mass renormalization. Obviously, this result is due to unphysical assumption that field is the same at arbitrarily large distances.

At sufficiently small distances compared to the correlation length in vacuum such model may be reasonable. One may use it in order to test convergence of the OPE series, which in this simple case are just the Tailor expansion of $E(\tau)$ in powers of τ, with application of (4.52, 4.53). We do not give here general formulae for they are somewhat lengthy, and only make few remarks. It turns out that it is very difficult to reconstruct the correlator from the OPE series at distances at which $E(\tau)$ is significantly changed: many terms are comparable, of different sign and, worst of all, very sensitive to the field used. One may think that the correlator is also sensitive to such details, but it is not true. Three subsequent integrations in (4.52, 4.53) very effectively wash out all inhomogeneities, so the correlator depends mainly on the average intensity of the field. The moral is that OPE works well if one correction is reasonably small and all the rest negligible, otherwise one should look for other methods.

Finally let me comment that in slightly inhomogeneous field the approach used above can be converted into approximate (semiclassical) calculations, in which Gaussian fluctuations around the extreme path are considered.

The semiclassical ideas can also be applied to gauge fields, as discussed in chapter 2. We remind that it implies that they relatively weakly fluctuate around some particular configurations (the instantons). In this case calculations of the correlators are

essentially simplified.

The simplest case is the evaluation of the correlators for operators constructed out of gluonic field. In such case the first approximation is given just by the substitution of the «classical» field into the operators in question. In section 5.8 we will return to this point.

More important applications are connected with light quark correlators. In this case the role of classical field is played by t'Hooft zero mode. In order to see this we remind that in the leading order in quark mass m the quark propagator in the instanton field [2.24] is simply given by

$$S_0(x,y) = -\frac{\Psi_0(x)\Psi_0^+(y)}{m} \tag{4.56}$$

(at first sight singularity at $m\to 0$ looks strange, but one should not forget that such expression contribute to physical effects only in combination with the instanton density, proportional to some power of m). Note the specific chiral structure of this expression. Due to it in the approximation (4.56) the polarisation operator

$$\Pi_{ij}(x) = \mathrm{Tr}\ [\Gamma_i S_0(0,x)\Gamma_j S_0(x,0)] \tag{4.57}$$

is zero for vector and axial currents.

Nonzero effect for pseudoscalar current will be considered in detailes in section 5.7, and now we proceed to further approximations in quark mass for the (practically important) vector channel. This calculations was made in Refs [4.25], and it includes two next terms in the propagator

$$S(x,y) = S_0(x,y) + S_1(x,y) + m\int dz\, S_1(x,z)S_1(z,y) + O(m^2) \tag{4.58}$$

which were found in Ref. [4.24]. Here the propagator part connected with nonzero modes is equal to

$$S_1 = \hat{D}_x\Delta(x,y)\left(\frac{1+\gamma_5}{2}\right) + \Delta(x,y)\hat{D}_y\left(\frac{1-\gamma_5}{2}\right)$$

$$\Delta(x,y) = [(x-y)^{-2} + \varrho^2(x^2+y^2+i\bar{\eta}^a_{\mu\nu}x_\mu y_\nu\tau^a)/2x^2y^2(x-y)^2 -$$

$$-\varrho^2/2x^2y^2]/[4\pi^2(1+\varrho^2/x^2)^{1/2}(1+\varrho^2/y^2)^{1/2}] \tag{4.59}$$

where $\Delta(x, y)$ is the propagator of scalar particle in the instanton field. Calculations with these formulae are rather cumbersome, and the only nontrivial test is current conservation relation $\partial_\mu \Pi_{\mu\nu}=0$. One more comment is that colour trace should correspond to SU(2) group, that is why the result of [4.25] contains wrong extra factor 3/2. Finally, the polarization operator is equal to

$$\Pi^{inst}(Q^2)=-\sum_{\pm}\int dn^{\pm}(\varrho)\left[\frac{2}{3Q^4}-\frac{2\varrho^2}{Q^2}\int_0^1 dx\, K_2\left(\frac{2Q\varrho}{\sqrt{1-x^2}}\right)\right] \tag{4.60}$$

Note first the presence of power $(1/Q^4)$ correction (the free particle loop produces $\Pi\sim \log(Q)$). In the preceeding section we have calculated correction due to gluon condensate G^2 operator, which is of the same dimension, so it is tempting to connect these quite different calculations. This can indeed be done, but with small complication due to another operator of dimension 4, namely $m\cdot\bar{\Psi}\Psi$. Its OPE coefficient is equal to [5.13]

$$\Delta\Pi(Q^2)=\frac{2m\bar{\Psi}\Psi}{Q^4} \tag{4.61}$$

(derivation of this result in Schwinger formalism is just simple exercise). One should also substitute average values of these operators in the instanton field

$$\langle 0|(gG)^2|0\rangle \Rightarrow 32\pi^2\sum_{\pm}\int dn^{\pm}$$

$$\langle 0|m\bar{\Psi}\Psi|0\rangle \Rightarrow -\sum_{\pm}\int dn^{\pm} \tag{4.62}$$

in (4.61) and the final conclusion is that power correction is indeed exactly equal to that given by the OPE calculation. I emphasize this point because historically it was a source of some controversies.

One more puzzle connected with (4.60) is more striking: the second term depends exponentially on Q, and no other power effects are seen! So, where are all operators of higher dimensions? There were even doubts about relevance of OPE analysis in general.

Rather unexpected answer was given in Ref. [4.19]. Everything is O.K. with OPE (its coefficients are nonzero), but in selfdual field all matrix elements of all operators are canceled among themselves! This statement follows from the following theorem

[4.19], valid in the fixed-point gauge: in any selfdual (antiselfdual) field the propagators for massless spin 0, 1/2 particles can be written as

$$D(x)=\frac{1}{4\pi^2x^2}+D^{(reg)}(x)$$

$$S(s)=-\frac{\hat{x}}{2\pi^2x^4}-\frac{g}{16\pi^2x^2}(x\tilde{G}\gamma)\gamma_5+S^{(reg)}(x) \tag{4.63}$$

where the former terms correspond to free propagators and the latter ones are regular at the origin. For gluon propagator there is no so strong theorem.

If the instanton-type fluctuations are indeed important in QCD vacuum than the Dubovikov-Smilga theorem may have real, not only academic significance. It demonstrates how light quarks may propagate in strong selfdual field without much effect (due to cancellation), while it is not so for gluons. It is possible that this consideration explains much stronger nonperturbative effects in gluonic channels (say, relatively heavy glueball masses), etc.

The final (rather obvious) comment deals with instanton corrections at large distances. The propagators seem to be strongly modified, but they are not gauge invariant quantities and in singular gauge this artifact disappears. Evidently, individual instanton is not effecive at large distances. The «instanton liquied» made of uncorrelated instantons produces finite mass renormalization and no confinement. Too much correlated ones may lead to «superconfinement», as the constant field considered above. What are the correlations among instantons in real vacuum we do not know.

4.4. Numerical evaluation of the propagators

The methods considered above are rather restricted, so it is desirable to develope more universal numerical methods for the calculation of particle propagators in arbitrary external gauge and quark fields. Of course, it is especially true for lattice calculations, in which gauge field configurations are themselves found numerically. We start with this case and then proceed to continuous case.

In lattice formulation coordinates are discretized, so Dirac equation for the quark propagator

$$\hat{M}\,S(x,x')=\delta(x-x')$$
$$M=i\hat{D}_x+im \tag{4.64}$$

can be considered as a problem of inversion of Dirac matrix: $S=1/M$. Many numerical methods for matrix inversion are known, but the problem is that the matrix M is very large: for n^4 lattice it has n^8 elements, to say nothing on spin and colour degrees of freedom.

The most powerful standard methods are the variants of Jacobi relaxation methods, based on the auxiliary equation

$$\frac{dS_{ij}}{dt}=-[M_{ik}S_{kj}-\delta_{ij}] \tag{4.65}$$

With «time» going on, we approach the inverse matrix. Gauss-Ziegel method substitute new $S(i, j)$ into this process before the whole matrix is summed up, and this makes convergence somewhat faster. There are also more engineous variants, say the «conjugate gradient» method advocated in [7.44]. The general defect of them all is that convergence is not garanteed.

Matrix inversion is too complicated method, even at supercomputers it takes about an hour per one matrix! On the other hand, the inverse matrix contains a lot of information: it describes propagation amplitude from each point to any other one. Often we do not need it, and are interested only in one particular initial and final point (or only one matrix element). Obviously, large information can partly be used by averaging, but still it is desirable to have more effective algorithms.

The general idea is that instead of making some regular straightforward calculations one should introduce some random element in the calculation. One particular example based on classical Neumann-Ulam expansion was suggested by J. Kuti [4.28]. Let us write $M=1-K$ and expand it in powers of K:

$$S_{ij}=\left(\frac{1}{1-K}\right)_{ij}=\delta_{ij}+K_{ij}+K_{il}K_{lj}+\ldots \tag{4.66}$$

This formula can be understood as some random work over lattice with the propagation amplitude K, while $S(i, j)$ is a sum over all paths leading from i to j. Similar idea is used in hopping parameter expansion [4.30, 4.31] for Wilson quarks, the amplitude for one step is here

$$k(1-\gamma_\mu)U_{x\mu} \tag{4.67}$$

So contour of length L contains the factor k^L.

The general defect of all such methods is that paths are chosen completely randomly, so for large enough L (many integration variables) their efficiency becomes rather poor, as for «naive» Monte-Carlo integration. Unfortunately, more effective methods like Methropolis algorithm can not be used because the measure is not positive. Still I think some partial importance sampling may turn very useful here.

Now we return to continuous space-time and start with the simplest case of the nonrelativistic particle without internal degrees of freedom moving in some potential well. In this case stochastic algorithms applied to Feynman path integrals give very good results. The simplest possibility is to generate ensemble of Gaussian free paths with subsequent averaging of the interaction factor

$$G(x)=G^{free}(x)\langle\exp\left(-\int_0^{\tau_0}V(x)d\tau\right)\rangle \tag{4.68}$$

where

$$G^{free}=\exp(-m\tau_0)/(2\pi m\tau_0)^{3/2} \tag{4.69}$$

However, if the averaged factor is too large or too small, such averaging may need too large statistics. In such case one may use the «adiabatic switching» method suggested in Ref. [4.32]. Let us consider an action of the type:

$$S_\eta=\int d\tau\left[\frac{m\dot{x}^2}{2}+\eta V(x)\right] \tag{4.70}$$

where parameter η varies between 0 and 1. The average potential can be obtained by derivative over η of the statistical sum:

$$\langle V\rangle_\eta=-\frac{\partial}{\partial\eta}[\ln\int Dxe^{-S_\eta}] \tag{4.71}$$

Integrating over η backward one finds the relation

$$G(0,0,\tau)=G^{free}(0,0,\tau)\exp\left(-\int_0^1 d\eta\langle V\rangle_\eta\right) \tag{4.72}$$

where the subscript η mean averaging with the action (4.70). It is convenient to increase η gradually, so that no relaxation be needed. Since in this case the exponent is averaged, not the whole correction factor, good results can be obtained even for very large effects.

Now we proceed to particles with internal degrees of freedom, spin or colour, which make the propagator to become a matrix. Say, for nonrelativistic particle in coloured field (7.68) is generalized as

$$S(x)=S^{free}(x)\langle \operatorname{Pexp}\left(\frac{ig}{2}\oint A^a_\mu t^a\, dx_\mu\right)\rangle \tag{4.73}$$

Unfortunately, the second more effective method can not be used because Metropolis-type algorithms are inapplicable for non-positive weights.

For relativistic propagators the immediate problem is that there is no simple substitute for Feynman path-integral representation. This question is a kind of «theoretical folklore», and it was many times considered in literature (see papers mentioned in [4.33] and many others). The most useful formalism is probably that using proper time, as suggested by Fock and Schwinger long ago. Attempts to make effective numerical calculations in such framework was recently made by Zhirov [4.34], based on the following path-integral representation

$$S=(i\hat{\partial}+im)\int_0^\infty ds\int Dx_\mu(\tau)\operatorname{Pexp}\{-\int_0^s d\tau\, L[x(\tau)]\}$$

$$L[x(\tau)]=m^2+\frac{\dot{x}^2}{4}-\dot{x}\mu A^a_\mu t^a-\frac{1}{2}\sigma_{\mu\nu}G^a_{\mu\nu}t^a \tag{4.74}$$

(for the spinor case).

Finally we comment on very interesting idea suggested by Parisi [4.35] which may allow to reduce the statistics needed in stochastic calculation of sufficiently small correlators. This problem is very severe for hadronic spectroscopy on the lattice, because accurate evaluation of masses can only be made from the correlator measurements at large enough distances.

Consider a system with the action

$$S_1=S_0-\delta\cdot O(t_0) \tag{3.75}$$

where S_0 and S_1 are two actions which differ by some small term

and the operator O is averaged over all space:

$$O(t_0)=\int dx\, O(x,\, t_0) \tag{4.76}$$

and δ is just some small parameter. Expanding the average of $O(t_1)$ with action S_1 one obtains

$$\langle O(t_1)\rangle_{S_1}=\langle O(t_1)\rangle_{S_0}+\delta[\langle O(t_1)O(t_0)\rangle_{S_0}-$$

$$-\langle O(t_1)\rangle_{S_0}\langle O(t_0)\rangle_{S_0}]+O(\delta^2) \tag{4.77}$$

So at sufficiently small δ the correlator may be estimated as the difference of $\langle O(t_1)\rangle 1$ and $\langle O(t_1)\rangle 0$ divided by δ. Now, if one make the calculation of both quantities in the same stochastic process, the statistical noise can be essentially canceled. Evidently, in order it to take place one should hold two processes to be as close as possible, so small (Langevin) shifts should be made. More on this method and its applications see in Refs [3.50, 3.68].

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4. Correlators and sum rules. The methods

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