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GENERAL N-th ORDER DIFFERENTIAL SPECTRAL PROBLEM: GENERAL STRUCTURE OF THE INTEGRABLE EQUATIONS, NONUNIQUENESS OF RECURSION OPERATOR AND GAUGE INVARIANCE

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НОВОСИБИРСК

GENERAL N-th ORDER DIFFERENTIAL SPECTRAL PROBLEM: GENERAL STRUCTURE OF THE INTEGRABLE EQUATIONS, NONUNIQUENESS OF RECURSION OPERATOR AND GAUGE INVARIANCE

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A b s t r a c t

The general scalar Gelfand-Dikij-Zakharov-Shabat spectral problem of arbitrary order is considered within the framework of the AKNS method. The general form of the integrable equations is found. Uncertainties which appear in the construction of the recursion operator are discussed. Transformation properties of the integrable equations under the gauge transformations are considered. The manifestly gauge-invariant formulation of the integrable equations is given. It is shown that the integrable equations under consideration are Hamiltonian ones with respect to the infinite family of Hamiltonian structures.

I. Introduction

The inverse scattering transform method is a powerful tool for the investigation of nonlinear differential equations (see e.g. [1-3]). A large number of various differential equations has been integrated by the inverse scattering transform method during the last ten years [1-3]. One of the main problems of the inverse scattering transform method is a problem of effective description of the equations to which this method is applicable. There exist different approaches to this problem. A very convenient and simple method has been proposed by Ablowitz, Kaup, Newell and Segur (AKNS) in the paper [4] for the equations integrable by the second order matrix spectral problem $\frac{\partial \psi}{\partial x} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \psi$. The so called recursion operator plays a central role in this method (AKNS method): the integrable equations can be represented in a compact form just with the use of this recursion operator. The recursion operator plays also an important role in the Hamiltonian treatment of integrable equations. The Hamiltonian structure of the equations integrable by the second-order matrix spectral problem has been investigated in [5]. Then after the papers [4,5] the AKNS method has been generalized to a number of different spectral problems [6-14].

In the present paper we consider the general Gelfand-Dikij-Zakharov-Shabat spectral problem, i.e. the general scalar N-th order spectral problem

$$\left(\partial^N + V_{N-1}(x,t)\partial^{N-1} + \ldots + V_1(x,t)\partial + V_0(x,t) \right) \psi = \lambda^N \psi \quad (1.1)$$

where $\partial = \frac{\partial}{\partial x}$, λ is a spectral parameter and $V_0(x,t)$, $V_1(x,t), \ldots, V_{N-1}(x,t)$ are scalar functions such that $V_k(x,t) \xrightarrow[|x| \to \infty]{} V_{k\infty} \neq 0$ ($k = 0, 1, \ldots, N-1$). In the frames of the inverse scattering transform method the spectral problem (1.1) has been considered by Zakharov and Shabat [15] for the first time. This problem and associated evolution equations were investigated by another technique by Gelfand and Dikij [16]. Then the problem (1.1), its properties and the nonlinear equations connected to this problem have been considered in [17-23].

In the present paper we consider the spectral problem (1.1) within the framework of AKNS method. We find the general form of the nonlinear evolution equations integrable by (1.1) and also show that these equations are Hamiltonian ones with respect to the infinite family of Hamiltonian structures.

It is shown that there exist a certain freedom in the construction of the recursion operator. The principal equation for the calculation of the recursion operator is of the form

$$\lambda^N \mathcal{G} \chi(\lambda) = \mathcal{F} \chi(\lambda) \qquad (1.2)$$

where $\mathcal{G}$ and $\mathcal{F}$ are certain matrix differential operators and $\chi(\lambda)$ is a column with N components. The main feature of equation (1.2) is that the matrix elements of $\mathcal{G}$ which belong to the first line are equal to zero and therefore equation (1.2) contains a constraint

$$\sum_{k=1}^{N} \ell_k \chi_k = 0 \qquad (1.3)$$

where $\ell_k = \sum_{\ell=1}^{N-k+1} C^{k-1}_{k+\ell-1} (-\partial)^\ell V_{k+\ell-1}$ and $C^k_n = \frac{n!}{k!(n-k)!}$.
There are two ways to deal with the constraint (1.3). The first way is to solve equation (1.3), for example, with respect to χ_α and to introduce the quantity $\chi_{(\alpha)} = (\chi_1, \ldots, \chi_{\alpha-1}, 0, \chi_{\alpha+1}, \ldots, \chi_N)$ which contains only independent variables. As a result, one obtains, from (1.2), the recursion operator which acts on the space of independent variables $\chi_{(\alpha)}$: $\lambda^N \chi_{(\alpha)}(\lambda) = L_\alpha \chi_{(\alpha)}(\lambda)$. For a given α the recursion operator L_α is unique but for different α the operators L_α are different. For such a type of taking into account the constraint (1.3) the general form of the evolution equations connected to the problem (1.1) is

$$M^+_\alpha \frac{\partial V}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^+_\alpha, t) \mathcal{L}^+_{k(\alpha)} (V - V_\infty) = 0 \qquad (1.4)$$

where $V = (V_0, V_1, \ldots, V_{N-1})^T$, M^+_α, $\mathcal{L}^+_{k(\alpha)}$ are certain matrix operators and $\Omega_k(L^+_\alpha, t)$ are arbitrary functions which are meromorphic on L^+_α.

The second way of dealing with the constraint (1.3) is not to solve it at all and define an action of the recursion operator on the whole N-dimensional space of all components $\chi_1, \ldots, \chi_N$: $\lambda^N \chi(\lambda) = L \chi(\lambda)$. One can introduce such a recursion opera-

tor but it is not defined uniquely. The uncertainty which appears in the calculation of such a recursion operator can be effectively described. The general form of the evolution equations integrable by (1.1) within this approach is

$$\frac{\partial V(x,t)}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^+, t) \mathcal{L}^+_k (V - V_\infty) - f(L^+, t) e^+ \varphi = 0 \qquad (1.5)$$

where $\Omega_k(L^+, t), f(L^+, t)$ are arbitrary functions which are meromorphic on L^+, $e^+ = (e^+_1, \ldots, e^+_N)^T$ and $\varphi(x,t)$ is an arbitrary scalar function. There exists a close relation between equations (1.4) and (1.5).

Let us emphasize that the nonuniqueness of the recursion operator is a general feature of the AKNS method: it takes place for the other spectral problems, too.

In the paper the transformation properties of the integrable equations under the gauge transformations which conserve (1.1) are considered. It is shown that equations (1.4) are gauge invariant and, what is more, these equations can be represented in the manifestly gauge invariant form

$$\frac{\partial W(x,t)}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^+_W, t) \mathcal{L}^+_{(W)k} (W - W_\infty) = 0 \qquad (1.6)$$

where $W = (W_0, W_1, \ldots, W_{N-2}, 0)^T$ and $W_0, W_1, \ldots, W_{N-2}$ are gauge invariants, i.e. the functions on $V_0, V_1, \ldots, V_{N-1}$ which are invariant under the gauge transformations. Operators L^+_W and $\mathcal{L}^+_{(W)k}$ depend only on the gauge invariants $W_0, W_1, \ldots, W_{N-2}$.

It is shown that the equations (1.5) with given functions Ω_k and different functions $f(L^+, t)$ and φ are gauge equivalent each to other. In particular, any equation of the form (1.5) is gauge equivalent to the equation

$$\frac{\partial V(x,t)}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^+_S, t) \mathcal{L}^+_{(S)k} (V - V_\infty) = 0$$

where L^+_S is a some standart recursion operator.

So the whole uncertainty which appears within the description of the integrable equations in the form (1.5) is of the pure gauge nature.

We also discuss a Hamiltonian structure of the integrable equations (1.4) and (1.5). In order to treat these equations as the Hamiltonian systems, one has to exclude the pure gauge (non-dynamical) degrees of freedom. One can achieve it either by imposing additional gauge conditions on $V_0(x,t),\dots,V_{N-1}(x,t)$ (for example, $V_\alpha = V_{\alpha\infty}$) or by projecting the equations onto the N-1 - dimensional gauge-invariant submanifold which is described by equations (1.6). After the exclusion of the gauge degrees of freedom the evolution equations integrable by (1.1) are Hamiltonian systems with respect to the infinite family of Hamiltonian structures.

The paper is organized as follows. In the second section a group of the gauge transformations which preserve (1.1) is considered. The gauge invariants are calculated. In the third section we discuss a direct scattering problem for (1.1). In section 4 the general form (1.4) of the integrable equations is found. Gauge invariance of equations (1.4) and manifestly gauge invariant formulation (1.6) are discussed in section 5. The Hamiltonian structure of the integrable equations (1.4) and (1.6) is considered in sections 6 and 7. In section 8 the integrable equations in the form (1.5) and the corresponding recursion operators are considered. The examples of equations (1.2)-(1.6) for the cases N = 2,3 are given in sections 9 and 10.

II. Gauge group

The spectral problem (1.1), as it is easy to see, is invariant under the transformations

$$\psi(x,t,\lambda) \to \psi'(x,t,\lambda) = g(x,t)\,\psi(x,t,\lambda),$$

$$V_k(x,t) \to V_k'(x,t) = g(x,t)\sum_{n=0}^{N-k} C_{k+n}^{k}\, V_{k+n}(x,t)\,\partial^n\Big(\frac{1}{g(x,t)}\Big) \tag{2.1}$$

where $g(x,t)$ is any differentiable function such that $g(x,t)\underset{|x|\to\infty}{\longrightarrow} 1$ and $C_n^k = \frac{n!}{k!(n-k)!}$. The transformations (2.1) do not change the values of $V_0(x,t), V_1(x,t), \dots, V_{N-1}(x,t)$ at the infinities ($|x| \to \infty$) and form an infinite-dimensional abelian group of gauge transformations for the problem (1.1). This group is the subgroup of the general gauge transformations group which was discussed in [24, 25].

The gauge invariance of the problem (1.1) allows us to impose additional constraints (gauge conditions) on the potentials $V_0, V_1, \dots, V_{N-1}$. For example, one can transform any linear superposition $\sum_{k=0}^{N-1}\alpha_k V_k(x,t)$ into $\sum_{k=0}^{N-1}\alpha_k V_{k\infty}$ and, in particular, any (but only one) potential $V_k(x,t)$ into $V_{k\infty}$ by an appropriate gauge transformation. We will shortly refer to the gauge condition as the gauge. The transition from one gauge to another one is performed by a certain gauge transformation.

Then, it is clear that there exist N-1 independent functions $W_0(V_0,\dots,V_{N-1}), W_1(V_0,\dots,V_{N-1}), \dots, W_{N-2}(V_0,\dots,V_{N-1})$ which are invariant under the gauge transformations (2.1), i.e. the functions such that $W_k(V_0',\dots,V_{N-1}') = W_k(V_0,\dots,V_{N-1})$ (k = 0,1,..., N-2). An explicit form of the invariants W_0, W_1, ..., W_{N-2} can be found directly from (2.1) by excluding the function $g(x,t)$. For our purpose the following set of the invariants is convenient

$$W_k = V_k - \frac{1}{N}\sum_{n=1}^{N-k} C_{k+n}^{k}\, V_{k+n}\Big(\partial - \frac{1}{N}V_{N-1}\Big)^{n-1} V_{N-1} \tag{2.2}$$

$$(k = 0, 1, \dots, N-2).$$

To prove the invariance of W_k , it is sufficient to consider the infinitesimal gauge transformations ($V_k \to V_k' = V_k + \delta V_k$)

$$\delta V_k(x,t) = -\ell_{k+1}^{+}\,\varepsilon(x,t) \tag{2.3}$$

where $\varepsilon(x,t)$ is an arbitrary function and

$$\ell_{k+1}^{+} = \sum_{\ell=1}^{N-k} C_{k+\ell}^{k}\, V_{k+\ell}\,\partial^{\ell} .$$

In the simplest cases of N = 2,3, the invariants are

<u>N = 2:</u> $\quad W_0 = V_0 - \frac{1}{2}\partial V_1 - \frac{1}{4}V_1^2,$

<u>N = 3:</u> $\quad W_0 = V_0 - \frac{1}{3}\partial^2 V_2 - \frac{1}{3}V_1V_2 + \frac{2}{27}V_2^3,$

$\quad W_1 = V_1 - \partial V_2 - \frac{1}{3}V_2^2 .$

The gauge $V_{N-1} = 0$ is, in a certain sense, a preferable one since in this gauge $W_k = V_k$.

Since for the gauge transformations $W_k(V_0',\dots,V_{N-1}') = W_k(V_0,\dots,V_{N-1})$ the invariants W_k take the same values in the different gauges. So the potentials V_k in the different gauges are connected by the relations $W_k(V_0',\dots,V_{N-1}') = W_k(V_0,\dots,V_{N-1})$ (k = 0,1,..., N-2).

Proposition (2.1) [26]. The Miura transformation is a gauge transformation.

Indeed, let N = 2 and $V_{0\infty} = V_{1\infty} = 0$. Let us consider the gauge transformation (2.1) from the gauge $V_0 = 0$ to the gauge $V_1' = 0$. Since $W_0 = V_0 - \frac{1}{2}\partial V_1 - \frac{1}{4}V_1^2$, from the equality $W_0(V_0', V_1'=0) = W_0(V_0=0, V_1)$ one has

$$V_0' = -\frac{1}{2}\partial V_1 - \frac{1}{4}V_1^2 \qquad (2.4)$$

that is the famous Miura transformation [27]. Gardner transformation is a gauge transformation, too (see section 9).

For the further purposes it is convenient to represent the spectral problem (1.1) in the well-known matrix Frobenius form

$$\frac{\partial F}{\partial x} = (A + P(x,t))F \qquad (2.5)$$

where $F = (\psi, \partial\psi, \dots, \partial^{N-1}\psi)^T$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ \lambda^N & 0 & 0 & \dots & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \\ -V_0, & -V_1, & \dots, & -V_{N-1} \end{pmatrix}.$$

The gauge transformations (2.1) have now the form

$$F \to F' = GF, \quad P \to P' = G(A+P)G^{-1} - A + \partial G \cdot G^{-1} \qquad (2.6)$$

where $G_{ik} = C_{i-1}^{k-1}\partial^{i-k}g(x,t),\ i \geq k;\quad G_{ik} = 0,\ i < k$.

Introducing N-component column $V = (V_0,\dots,V_{N-1})^T$, one can represent the gauge transformations (2.6) in the form

$$V \to V' = \tau(g)V + \mathcal{V}(g) \qquad (2.7)$$

where $\tau(g) = g(G^T)^{-1}$ and $\mathcal{V}_k(g) = C_N^k g\partial^{N-k}(1/g)$ (k = 0,1,..., N-1). Using the explicit form of $\tau(g)$ and $\mathcal{V}(g)$, it is not difficult to show that

$$\tau(g_2)\tau(g_1) = \tau(g_2g_1), \quad \mathcal{V}(g_2g_1) = \tau(g_2)\mathcal{V}(g_1) + \mathcal{V}(g_2) \qquad (2.8)$$

i.e. that the transformations (2.7) indeed form a group.

The form (27) of the gauge transformations (2.1) is useful for many purposes.

For example, the invariants W_k can be written in the form

$$W = \tau(\tilde\rho^{-1})V + \mathcal{V}(\tilde\rho^{-1}) \qquad (2.9)$$

where $W = (W_0, W_1, \dots, W_{N-2}, 0)^T$ and $\tilde\rho(x,t) = \exp(-\frac{1}{N}\int dx' V_{N-1}(x',t))$. Then the potentials V_k can be represented as the functions on invariants W_k and "gauge" variables $\rho(x,t)$

$$V(x,t) = \tau(\rho)W + \mathcal{V}(\rho). \qquad (2.10)$$

One can easily show that if under the gauge transformations

$$W \to W' = W, \quad \rho \to \rho' = g\rho \qquad (2.11)$$

then $V(x,t)$ is transformed according to (2.7).

III. Direct scattering problem

We will study the problem (1.1) in the form (2.5). We assume that $V_k(x,t)$ tend to their asymptotic values $V_{k\infty}$ so fast that all integrals which will appear in our calculations will exist. Then, we assume that all eigenvalues μ_i of the matrix $\tilde A = A + P_\infty$ where $P_\infty \overset{def}{=} \lim_{|x|\to\infty} P(x,t)$ are different.

We introduce, in a standard manner [1-3], the fundamental matrices-solutions $F^+(x,t,\lambda)$, $F^-(x,t,\lambda)$ of the problem (2.5) given by their asymptotic behaviour

$$F^+(x,t,\lambda) \xrightarrow[x\to\infty]{} \mathcal{D}(\lambda)\exp\bar A x, \quad F^-(x,t,\lambda) \xrightarrow[x\to-\infty]{} \mathcal{D}(\lambda)\exp\bar A x \qquad (3.1)$$

where $\bar A_{ik} = \mu_i\delta_{ik}$ $(i,k = 1,\dots,N)$ and $\mathcal{D}^{-1}\tilde A\mathcal{D} = \bar A$. For $V_{k\infty} = 0$ one has $\mu_i = \lambda q^{i-1}$, $\mathcal{D}_{ik} = \frac{1}{\sqrt{N}}(\lambda q^{(k-1)})^{i-1}$, $q = \exp(\frac{2\pi i}{N})$. A scattering matrix $S(\lambda,t)$ is also introduced by the standard formula

$$F^{+}(x,t,\lambda)=F^{-}(x,t,\lambda)\,S(\lambda,t)\ . \qquad (3.2)$$

Let $P(x,t)$ and $P'(x,t)$ be two different potentials and $F^{+}, F^{+\prime}, S, S'$ be corresponding solutions and scattering matrices for (2.5). One can show that

$$S'(\lambda,t)-S(\lambda,t)=-\int_{-\infty}^{+\infty} dx\,(F^{-}(x,t,\lambda))^{-1}(P'(x,t)-P(x,t))(F^{+}(x,t,\lambda))' . \qquad (3.3)$$

Formula (3.3) which relates a variation of the potential P to those of the scattering matrix plays a fundamental role in the AKNS method.

Putting $V_k=V_{k\infty}$ $(k=0,1,\ldots,N-1)$ and taking into account that in this case S = 1 we obtain the following expression for the scattering matrix through the potential (omitting the prime)

$$S(\lambda,t)=1-\int_{-\infty}^{+\infty} dx\,E^{-1}(x,\lambda)(P(x,t)-P_\infty)E(x,\lambda)\times$$
$$\times\,\mathcal{P}\Big\{\exp\int_{-\infty}^{x} dx'\,E^{-1}(x',\lambda)(P(x',t)-P_\infty)E(x',\lambda)\Big\} \qquad (3.4)$$

where $E(x,\lambda)=\mathcal{D}(\lambda)\exp\bar{A}x$ and $\mathcal{P}\{\exp\int_{-\infty}^{x} dx' Z(x')\}$ denotes the well known x -ordered exponent:

$$\mathcal{P}\Big\{\exp\int^{x} dx' Z(x')\Big\}=1+\int^{x} dx' Z(x')+\int^{x} dx' Z(x')\int^{x'} dx'' Z(x'')+\ldots$$

If one introduces an operator

$$T(x,t,\lambda)=\mathcal{P}\Big\{\exp\int_{\infty}^{x} dx'\,E^{-1}(x',\lambda)(P(x',t)-P_\infty)E(x',\lambda)\Big\}, \qquad (3.5)$$

then

$$S(\lambda,t)=T(x=-\infty,t,\lambda)=$$
$$=\mathcal{P}\Big\{\exp\Big(-\int_{-\infty}^{+\infty} dx'\,E^{-1}(x',\lambda)(P(x',t)-P_\infty)E(x',\lambda)\Big)\Big\}\ . \qquad (3.6)$$

The simple formula (3.6) for the scattering matrix is convenient for an analysis of the spectral problem (1.1). Let us note that a redefinition of the asymptotics E : $E(x,\lambda)\to E'(x,\lambda)=E(x,\lambda)K$ leads to a trivial redefinition of the scattering matrix: $S\to S'=K^{-1}SK$.

It follows from the definition (3.2) and formula (3.6) that the scattering matrix is gauge invariant quantity.

IV. General form of the integrable equations

1. Formula (3.6) establishes a mapping of the manifold of the potentials $\{P(x,t),\ P(x,t)\xrightarrow[|x|\to\infty]{} P_\infty\}$ onto the manifold of the scattering matrices $\{S(\lambda,t)\}$. A main observation of the inverse scattering transform method is that there exist a close relation between some special nonlinear evolution equations for the potentials and the linear evolution equations for the scattering matrix. We use here this idea within the framework of the AKNS method.

First of all, let us note that from (3.3) it follows

$$\frac{\partial S(\lambda,t)}{\partial t}=-\int_{-\infty}^{+\infty} dx\,(F^{-}(x,t,\lambda))^{-1}\frac{\partial P(x,t)}{\partial t}F^{+}(x,t,\lambda)\ . \qquad (4.1)$$

Then the following relation holds

$$[Y(\lambda,t),S(\lambda,t)]=-\int_{-\infty}^{+\infty} dx\,(F^{-}(x,t,\lambda))^{-1}[\tilde{Y}(\lambda,t),P(x,t)-P_\infty]F^{+}(x,t,\lambda) \qquad (4.2)$$

where $Y(\lambda,t)$ is an arbitrary diagonal matrix and $\tilde{Y}(\lambda,t)=\mathcal{D}(\lambda)Y(\lambda,t)\mathcal{D}^{-1}(\lambda)$ where $\mathcal{D}(\lambda)$ is given by (3.1). Since $\mathcal{D}\bar{A}\mathcal{D}^{-1}=\tilde{A}$ the matrix $\tilde{Y}$ is an arbitrary matrix which commutes with $\tilde{A}$. Since all eigenvalues of the matrix $\tilde{A}$ are different, any matrix $\tilde{Y}$ which commutes with $\tilde{A}$ can be represented in the form $\tilde{Y}(\lambda,t)=\sum_{k=0}^{N-1}\Omega_k(\lambda^N,t)\tilde{A}^k$ where Ω_k are certain functions and $\tilde{A}^0\equiv 1$.

Combining (4.1) with (4.2), one has

$$\frac{d\,S(\lambda,t)}{d\,t}-[Y(\lambda,t),S(\lambda,t)]=$$
$$=-\int_{-\infty}^{+\infty} dx\,(F^{-}(x,t,\lambda))^{-1}\Big(\frac{\partial P(x,t)}{\partial t}-[\tilde{Y}(\lambda,t),P(x,t)-P_\infty]\Big)F^{+}(x,t,\lambda)= \qquad (4.3)$$
$$=-S(\lambda,t)\int_{-\infty}^{+\infty} dx\,(F^{+}(x,t,\lambda))^{-1}\Big(\frac{\partial P(x,t)}{\partial t}-[\tilde{Y}(\lambda,t),P(x,t)-P_\infty]\Big)F^{+}(x,t,\lambda).$$

Let $\langle \dots \rangle \overset{def}{=} \int_{-\infty}^{+\infty} dx\, tr(\dots)$ and one introduces the quantity

$$\overset{++}{\varphi}{}^{ik}_{\ell m}(x,t,\lambda) = (F^{+}(x,t,\lambda))_{\ell k} (F^{+}(x,t,\lambda))^{-1}_{im} \qquad (4.4)$$

$$(i,k,\ell,m = 1,\dots,N).$$

Since $\det S \neq 0$, from (4.3) it follows

Proposition (4.1). If

$$\frac{dS(\lambda,t)}{dt} - [Y(\lambda,t), S(\lambda,t)] = 0 \qquad (4.5)$$

where $Y(\lambda,t)$ is an arbitrary diagonal matrix, then

$$\left\langle \left(\frac{\partial P}{\partial t} - \sum_{k=1}^{N-1} \Omega_k(\lambda^N,t)[\tilde{A}^k, P-P_\infty] \right) \overset{++}{\varphi}{}^{iN}(\lambda) \right\rangle = 0 . \qquad (4.6)$$

The inverse statement is also valid: if equation (4.6) takes place, then the scattering matrix satisfies (4.5).

Equation (4.6) contains the functions $\Omega_k(\lambda^N,t)$ and matrix $\tilde{A}$ which explicitly depend on the spectral parameter λ^N. The next step (which is a standard one for AKNS technique) consists in the converting of the relation (4.6) into such a form which does not contain the explicit dependence on λ^N. In order to do this, one must calculate the so-called recursion operator.

2. Recursion operator. So it is necessary to be able to exclude the explicit dependence on λ^N in the expressions of the form $\lambda^N \overset{++}{\varphi}{}^{iN}(\lambda,t)$. Using equation (2.5) and equation $\frac{\partial F^{-1}}{\partial x} = -F^{-1}(A+P)$, one can show that $\overset{++}{\varphi}{}^{iN}$ satisfy the equation

$$\frac{\partial \overset{++}{\varphi}{}^{iN}(x,t,\lambda)}{\partial x} = [A + P(x,t), \overset{++}{\varphi}{}^{iN}(x,t,\lambda)] . \qquad (4.7)$$

From (4.7) we get

$$\mathcal{P}^m \overset{++}{\varphi}{}^{iN} = \overset{++}{\varphi}{}^{iN}(A+P)^m \qquad (4.8)$$

where $\mathcal{P} \overset{def}{=} A + P(x,t) - \partial$ and m is any positive integer.

In virtue of the special forms of the matrices A and $P(x,t)$, all matrix elements of $\overset{++}{\varphi}{}^{iN}$ can be expressed through the N matrix elements $(\overset{++}{\varphi}{}^{iN})_{kN}$ (k = 1,2,...,N) (for $V_{N-1} = 0$ see [14]). Let us introduce the operation Δ of projection onto the last column of the matrix: $(\varphi_\Delta)_{ik} \overset{def}{=} \delta_{kN}\varphi_{iN}$ (i,k = 1,2,..., N). With the use of equations (4.7),(4.8) and the explicit forms of A and P one gets

$$\sum_{m=0}^{N} \mathcal{P}^m(\varphi_\Delta \circ V_m) = \lambda^N \varphi_\Delta \qquad (4.9)$$

where $(\varphi \circ V_m)_{ik} \overset{def}{=} \varphi_{ik} V_m$. Then it is not difficult to show that the operator $\mathcal{P}^m$ is linear on λ^N

$$\mathcal{P}^m = \lambda^N r_m + S_m \quad . \qquad (4.10)$$

Substituting (4.10) into (4.9), we obtain

$$\lambda^N \mathcal{Y} \overset{++}{\varphi}{}^{iN}_\Delta(\lambda) = \mathcal{F} \overset{++}{\varphi}{}^{iN}_\Delta(\lambda) \qquad (4.11)$$

where

$$\mathcal{Y} \cdot = \sum_{m=0}^{N} r_m(\cdot \circ V_m) - 1 , \quad \mathcal{F} = -\sum_{m=0}^{N} S_m(\cdot \circ V_m) . \qquad (4.12)$$

Using the properties of the matrix A, one can show that the matrix $\mathcal{Y}$ is a lowertriangular one:

$$\mathcal{Y} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ -N\partial & 0 & 0 & \dots & 0 & 0 \\ * & -N\partial & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ * & * & * & \dots & -N\partial & 0 \end{pmatrix} \qquad (4.13)$$

and

$$(S_k)_{1\ell} = C_k^{\ell-1}(-\partial)^{k+1-\ell}, \quad \ell = 1,\dots,k+1;$$

$$(S_k)_{1\ell} = 0, \quad \ell = k+2,\dots,N; \qquad (4.14)$$

$$(S_N)_{1\ell} = C_N^{\ell-1}(-\partial)^{N+1-\ell} - V_{\ell-1}, \quad \ell = 1,\dots,N.$$

Therefore the operator $\mathcal{Y}$ is a degenerate one. As a result, the first equation (4.11) is a relation between $\overset{++}{\varphi}_{kN}$ which does not contain λ^N, i.e. the constraint

$$\sum_{k=1}^{N} \ell_k \overset{++}{\varphi}{}^{in}_{kN} = 0 \tag{4.15}$$

where

$$\ell_k = \sum_{\ell=1}^{N-k+1} C_{k+\ell-1}^{k-1} (-\partial)^{\ell} V_{k+\ell-1} \, . \tag{4.16}$$

The degeneracy of the matrix ψ (its rank is N-1) and the existence of the constraint (4.15) are the fundamental properties of equation (4.11) which serves for the calculation of the recursion operator. Such a situation is a typical one for AKNS method. Generally used way to deal with the constraint (4.15) is to solve it expressing one of the elements φ_{kN} through the N-1 others. These last elements of φ_{Δ} are now independent ones. Usually (see e.g. [6-14]) only one standard way of solving the constraint is used. In our case this way corresponds to the choice of $\varphi_{1N}, \ldots, \varphi_{N-1\,N}$ as the independent variables [14].

In the present paper we do not confine ourselves only to this way. We will consider a wider class of possibilities by choosing as the independent any N-1 quantities from φ_{1N}, $\varphi_{2N}, \ldots, \varphi_{NN}$. One can consider also, in a similar manner, the case of the most general choice of independent variables.

So let us solve the constraint (4.15) with respect to $\varphi_{\alpha N}$ where α is any from 1,2,..., N. Taking into account that in our case the operators ℓ_k have no nontrivial kernels, we get

$$\overset{++}{\varphi}{}^{in}_{\alpha N}(x,\lambda) = -\ell_{\alpha}^{-1} \sum_{\beta \neq \alpha} \ell_{\beta} \overset{++}{\varphi}{}^{in}_{\beta N} \, , \quad (i \neq n) \, . \tag{4.17}$$

Let us denote $(E_{\alpha})_{ik} = \delta_{ik} - \delta_{i\alpha}\delta_{k\alpha}$, $\varphi_{\Delta_{\alpha}} = E_{\alpha}\varphi_{\Delta} =$ $= (\varphi_{1N}, \varphi_{2N}, \ldots, \varphi_{\alpha-1\,N}, 0, \varphi_{\alpha+1\,N}, \ldots \varphi_{NN})$ and $(M_{\alpha})_{ik} = \delta_{ik} - \delta_{i\alpha}\ell_{\alpha}^{-1}\ell_k$, i.e.

$$M_{\alpha} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ -\ell_{\alpha}^{-1}\ell_1, & -\ell_{\alpha}^{-1}\ell_2, & \ldots, & -\ell_{\alpha}^{-1}\ell_{\alpha-1}, & 0, & -\ell_{\alpha}^{-1}\ell_{\alpha+1}, \ldots, & -\ell_{\alpha}^{-1}\ell_N \\ 0 & 0 & \ldots & 0 & 0 & 1 \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & 0 & 0 & 0 \ldots & 1 \end{pmatrix} . \tag{4.18}$$

The operators M_{α} have the following properties (a summation over repeated indices is absent):

$$M_{\alpha}E_{\alpha} = M_{\alpha}, \; E_{\alpha}M_{\alpha} = E_{\alpha}, \; M_{\alpha}M_{\beta} = M_{\beta} \; (\alpha, \beta = 1, \ldots, N). \tag{4.19}$$

The solution of the constraint in the form (4.17) means that φ_{Δ} is represented as follows

$$\overset{++}{\varphi}{}^{in}_{\Delta} = M_{\alpha} \overset{++}{\varphi}{}^{in}_{\Delta\alpha} \, .$$

Substituting this expression for $\overset{++}{\varphi}{}^{in}$ into (4.11), we get

$$\lambda^N \psi M_{\alpha} \overset{++}{\varphi}_{\Delta_{\alpha}} = \mathcal{F} M_{\alpha} \overset{++}{\varphi}_{\Delta_{\alpha}} \, . \tag{4.20}$$

Then the use of the equalities (4.19) gives,

$$E_{\alpha}M_N \tilde{\psi} \psi M_{\alpha} = E_{\alpha}M_N E_N M_{\alpha} = E_{\alpha}M_N M_{\alpha} = E_{\alpha}M_{\alpha} = E_{\alpha}$$

where $\tilde{\psi}\psi = E_N$. Multiplying the left- and right-hand sides of (4.20) by $E_{\alpha}M_N\tilde{\psi}$, we therefore obtain

$$\lambda^N \overset{++}{\varphi}{}^{in}_{\Delta_{\alpha}}(\lambda) = L_{\alpha} \overset{++}{\varphi}{}^{in}_{\Delta_{\alpha}}(\lambda) \quad (i \neq n) \tag{4.21}$$

where

$$L_{\alpha} = E_{\alpha}M_N \tilde{\psi} \mathcal{F} M_{\alpha} \, . \tag{4.22}$$

The operator L_{α} is just the recursion operator which allows to convert the expressions with the explicit dependence on the spectral parameter λ^N into the expressions without such a dependence. Indeed, for any entire function $\Omega(\lambda^N)$ one has

$$\Omega(\lambda^N) \overset{++}{\varphi}{}^{in}_{\Delta_{\alpha}}(x,t,\lambda) = \Omega(L_{\alpha}) \overset{++}{\varphi}{}^{in}_{\Delta_{\alpha}} \, . \tag{4.23}$$

The index α means that the recursion operator L_{α} is calculated by the α -th way of solving the constraint (4.15) which corresponds to (4.17). This operator L_{α} has zero elements at α-th column and α-th line. Let us also note that the recursion operator L_{α} can be represented as

$$L_{\alpha} = E_{\alpha} L M_{\alpha} \tag{4.24}$$

where $L \stackrel{def}{=} M_N \mathcal{Y} \mathcal{F}$.

It follows from (4.21),(4.24) and (4.18) that the recursion operators L_α with different α are connected by a simple formula

$$L_\alpha = E_\alpha M_\beta L_\beta E_\beta M_\alpha - E_\alpha M_\beta L_\beta \widetilde{E_\alpha M_\beta} \qquad (4.25)$$

where $\widetilde{E_\alpha M_\beta}\, E_\alpha M_\beta \stackrel{def}{=} E_\beta$.

It is convenient to rewrite the relation (4.21) in the form

$$L_\alpha \chi_{(\alpha)}(\lambda) = \lambda^N \chi_{(\alpha)}(x,\lambda) \qquad (4.26)$$

where $\chi_{(\alpha)} \stackrel{def}{=} (\overset{++}{\varphi}{}^{in}_{1N}, \ldots, \overset{++}{\varphi}{}^{in}_{\alpha-1N}, 0, \overset{++}{\varphi}{}^{in}_{\alpha+1N}, \ldots, \overset{++}{\varphi}{}^{in}_{NN})$. In our further constructions we will need also the operators L^+_α adjoint to the operators L_α with respect to the bilinear form

$$\langle\langle \chi, \varphi \rangle\rangle \stackrel{def}{=} \sum_{i=1}^{N} \int_{-\infty}^{+\infty} dx\, \chi_i(x) \varphi_i(x) \quad . \qquad (4.27)$$

The operators L^+_α are

$$L^+_\alpha = M^+_\alpha M^+_N \mathcal{F}^+ \mathcal{Y}^+ M^+_N E_\alpha = M^+_\alpha L^+ E_\alpha \quad . \qquad (4.28)$$

One also has

$$M^+_\beta M^+_\alpha = M^+_\beta , \; M^+_\alpha E_\alpha = E_\alpha , \; E_\alpha M^+_\alpha = M^+_\alpha \;\; (\alpha,\beta = 1,\ldots,N) \qquad (4.29)$$

and

$$L^+_\alpha = M^+_\alpha E_\beta L^+_\beta M^+_\beta E_\alpha \quad . \qquad (4.30)$$

3. Further we must also exclude the explicit dependence on λ^N which is contained in $\tilde{A}^k$.

Firstly we see that $\tilde{A}^k$ is the linear function on λ^N

$$\tilde{A}^k = \lambda^N a_k + b_k \qquad (4.31)$$

where a_k and b_k are certain constant matrices. Then one can show that

$$\overset{++}{\varphi}_{\Delta_k} - \sum_{m=0}^{N-k} \mathcal{P}^{N-k-m} (\overset{++}{\varphi}_\Delta \circ V_{N-m}) A^{k-N} =$$
$$= (\lambda^N \mathcal{Y}_{(N-k)} + \mathcal{F}_{(N-k)}) \overset{++}{\varphi}_\Delta A^{k-N} \qquad (4.32)$$

where $(\varphi_{\Delta_k})_{en} \stackrel{def}{=} \delta_{nk} \varphi_{ek}$ and

$$\mathcal{Y}_{(k)} \cdot = \sum_{m=0}^{k} r_{k-m} (\cdot \circ V_{N-m}) , \; \mathcal{F}_{(k)} \cdot = \sum_{m=0}^{k} S_{k-m} (\cdot \circ V_{N-m}) \qquad (4.33)$$

where r_k and S_k are given by (4.10).

Let us denote the column with N components $V_0 - V_{0\infty}$, $V_1 - V_{1\infty}$, . . . , $V_{N-1} - V_{N-1\infty}$ as $V - V_\infty$. Using (4.31), (4.32), (4.26) (4.23) and taking into account the equalities $(A^{k-N})_{Nk} = 1$ and $a_k(P - P_\infty) = 0$ (k = 1,..., N-1), we get

$$\Omega_k(\lambda^N, t) \langle [\tilde{A}^k, P - P_\infty] \overset{++}{\varphi}{}^{in} \rangle =$$
$$= \langle\langle (V - V_\infty) \mathcal{Z}_{k(\alpha)} \Omega_k (L_\alpha, t) \chi_{(\alpha)} \rangle\rangle \qquad (4.34)$$

where

$$\mathcal{Z}_{k(\alpha)} = a_k M_\alpha L_\alpha + b_k M_\alpha - \sum_{i=1}^{N} (b_k)_{iN} (\mathcal{Y}_{(N-i)} M_\alpha L_\alpha + \mathcal{F}_{(N-i)} M_\alpha) . \qquad (4.35)$$

By virtue of (4.34) and $\langle \frac{\partial P}{\partial t} \varphi_\Delta \rangle = -\langle\langle \frac{\partial V}{\partial t} M_\alpha \chi_{(\alpha)} \rangle\rangle$, the equality (4.6) can be rewritten in the form

$$\langle\langle \frac{\partial V}{\partial t} M_\alpha \chi_{(\alpha)} - (V - V_\infty) \sum_{k=1}^{N-1} \mathcal{Z}_{k(\alpha)} \Omega_k (L_\alpha, t) \chi_{(\alpha)} \rangle\rangle = 0 . \qquad (4.36)$$

The equality (4.36) is the form of the equality (4.6) in which the explicit dependence on λ^N is eliminated. This elimination becomes possible due to the existence of the recursion operators.

At last the relation (4.36) is equivalent to the following one

$$\langle\langle \chi_{(\alpha)} \Big(M^+_\alpha \frac{\partial V}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^+_\alpha, t) \mathcal{Z}^+_{k(\alpha)} (V - V_\infty) \Big) \rangle\rangle = 0 \qquad (4.37)$$

where L^+_α is given by the formula (4.29) and

$$\mathcal{Z}^+_{k(\alpha)} = L^+_\alpha M^+_\alpha \Big(\sum_{i=1}^{N} (b_k)_{iN} \mathcal{Y}^+_{(N-i)} - a^+_k \Big) + M^+_\alpha \Big(\sum_{i=1}^{N} (b_k)_{iN} \mathcal{F}^+_{(N-i)} - b^+_k \Big) \qquad (4.38)$$

and

$$(\tilde{A}^+)^k = \lambda^N a^+_k + b^+_k ,$$
$$\mathcal{Y}^+_{(k)} = \sum_{m=0}^{k} V_{N-m} r^+_{k-m} , \quad \mathcal{F}^+_{(k)} = \sum V_{N-m} S^+_{k-m} . \qquad (4.39)$$

Since all components of $\chi_{(\alpha)}$ are independent, then the equality (4.37) takes place if

$$M_\alpha^+ \frac{\partial V}{\partial t} - \sum_{k=1}^{N-1} \Omega_k(L_\alpha^+, t)\mathcal{Z}_{k(\alpha)}^+ (V - V_\infty) = 0 \quad . \tag{4.40}$$

Thus we have obtained the nonlinear evolution equations for the potential V in the form which corresponds to the α-th way of calculation of the recursion operator.

In the process of the construction of equations (4.40) we have assumed that $\Omega_k(\lambda^N, t)$ are the entire functions on λ^N in order to be able to use the relation (4.23). It is not difficult, nevertheless, to generalize our constructions to the case when the functions $\Omega_k(\lambda^N, t)$ are meromorphic on λ^N.

Indeed, together with the relation (4.3) we have

$$f(\lambda^N, t)\frac{dS(\lambda,t)}{dt} - \Big[\sum_{k=1}^{N-1}\tilde{\Omega}_k(\lambda^N, t)\bar{A}^k, S(\lambda,t)\Big] =$$
$$= -\int_{-\infty}^{+\infty} dx \,(F^-(x,t,\lambda))^{-1}\Big(f(\lambda^N,t)\frac{\partial P(x,t)}{\partial t} - \Big[\sum_{k=1}^{N-1}\tilde{\Omega}_k(\lambda^N,t)\bar{A}^k, P(x,t) - P_\infty\Big]\Big)F^+(x,t,\lambda) \tag{4.41}$$

where $f(\lambda^N, t)$ and $\tilde{\Omega}_k(\lambda^N, t), (k = 1, \dots, N-1)$ are any functions entire on λ^N. Any meromorphic function $\Omega_k(\lambda^N, t)$ can be represented in the form $\Omega_k(\lambda^N, t) = \tilde{\Omega}_k(\lambda^N, t)/f(\lambda^N, t)$ where $\tilde{\Omega}_k(\lambda^N, t)$ and $f(\lambda^N, t)$ are the entire functions.

Let

$$\frac{dS(\lambda,t)}{dt} - \Big[\sum_{k=1}^{N-1}\Omega_k(\lambda^N, t)\bar{A}^k, S(\lambda,t)\Big] = 0$$

where $\Omega(\lambda^N, t)$ are arbitrary functions meromorphic on λ^N. As a result, the right-hand side of (4.41) is equal to zero, too. Then repeating the calculation of this section, one obtains the equations

$$f(L_\alpha^+, t)M_\alpha^+\frac{\partial V}{\partial t} - \sum_{k=1}^{N-1}\tilde{\Omega}_k(L_\alpha^+, t)\mathcal{Z}_{k(\alpha)}^+(V - V_\infty) = 0 \quad . \tag{4.42}$$

Equations (4.42) are equivalent to equations (4.40) with the meromorphic functions Ω_k.

So we have the following

<u>Theorem 4.1</u>

If the scattering matrix evolves according to the linear equation

$$\frac{dS(\lambda,t)}{dt} - \Big[\sum_{k=1}^{N-1}\Omega_k(\lambda^N, t)\bar{A}^k, S(\lambda,t)\Big] = 0$$

where $\Omega_k(\lambda^N, t)$ are arbitrary functions meromorphic on λ^N, then the potential $V(x,t)$ evolves according to the nonlinear equation

$$M_\alpha^+\frac{\partial V}{\partial t} - \sum_{k=1}^{N-1}\Omega_k(L_\alpha^+, t)\mathcal{Z}_{k(\alpha)}^+(V - V_\infty) = 0$$

and vice versa.

Matrices M_α^+ and L_α^+ are the degenerated ones (their rank is N-1) and the elements of M_α^+ which belong to the α-th line are equal to zero. Therefore, the system of equations (4.40) contains only N-1 nontrivial equations and the α-th equation is of the form $0 \equiv 0$.

The fact that within the framework of AKNS method we obtain N-1 equations for N potentials $V_0, \dots, V_{N-1}$ is a typical situation for this method. It is related to the fact that solving the constraint (4.15) we proceed to a subspace of the independent variables $\chi_{(\alpha)}$ which is N-1-dimensional one.

The different ways of solving the constraint (4.15) are interrelated: $\chi_{(\alpha)} = E_\alpha M_\beta \chi_{(\beta)}$. Therefore the systems of equations (4.40) which correspond to different α are not independent. The relation between equations (4.40) with different α can be found either from (4.37) using the equality $\chi_{(\alpha)} = E_\alpha M_\beta \chi_{(\beta)}$ or directly from the equations (4.40) with the use of the relations

$$M_\beta^+(L_\alpha^+)^n = (L_\beta^+)^n M_\beta^+ E_\alpha, \quad E_\alpha L_\alpha^+ = L_\alpha^+, \quad M_\beta^+ E_\alpha \mathcal{Z}_{k(\alpha)}^+ = M_\beta^+ \mathcal{Z}_{k(\alpha)}^+ = \mathcal{Z}_{k(\beta)}^+ . \tag{4.43}$$

We get

$$M_\beta^+\Big(M_\alpha^+\frac{\partial V}{\partial t}-\sum_{k=1}^{N-1}\Omega_k(L_\alpha^+,t)\mathcal{Z}_{k(\alpha)}^+(V-V_\infty)=$$
$$=M_\beta^+\frac{\partial V}{\partial t}-\sum_{k=1}^{N-1}\Omega_k(L_\beta^+,t)\mathcal{Z}_{k(\beta)}^+(V-V_\infty) \tag{4.44}$$

for any $\alpha,\beta=1,\dots,N$.

The formula (4.44) is very simple, but it establishes an important fact: definite combinations of equations (4.40) with a given recursion operator have the same form (4.40) but with the other recursion operators.

Thus the nonlinear evolution equations, connected with the spectral problem (1.1), can be represented in different (but equivalent) forms with different recursion operators. More general ways of solving the constraint lead to the same situation. For the other spectral problems there exists an analogous phenomenon, too.

Equations (4.40) for $\alpha=N$, $V_{k\infty}=0$ and $V_{N-1}=0$ coincide with those constructed in [14].

V. Gauge invariance and manifestly gauge-invariant formulation

1. Let us firstly consider the transformation properties of equations (4.40) under the gauge transformations (2.1). Let us obtain the transformation laws for the quantities which have appeared in the previous section.

From the definition of $\varphi^{in}(x,t,\lambda)$ and (2.6) it follows that

$$\varphi^{in}(x,t,\lambda)\xrightarrow{g}\varphi^{in\prime}(x,t,\lambda)=G(x,t)\,\varphi^{in}(x,t,\lambda)\,G^{-1}(x,t)\,. \tag{5.1}$$

For the column $\chi(x,t,\lambda)\overset{def}{=}(\varphi_{1N},\dots,\varphi_{NN})$ the law (5.1) gives

$$\chi\xrightarrow{g}\chi'=\pi(g)\,\chi \tag{5.2}$$

where $\pi(g)=g^{-1}G(g)$. Further, for the quantity $\chi_{(\alpha)}$ the transformation law is

$$\chi_{(\alpha)}\xrightarrow{g}\chi'_{(\alpha)}=\pi_\alpha(g)\chi_{(\alpha)} \tag{5.3}$$

where

$$\pi_\alpha(g,V)=E_\alpha\pi(g)M_\alpha\ . \tag{5.4}$$

In particular, $\pi_N=E_N\pi(g)$ is the function only on $g(x,t)$.

Using the explicit form of the operators ℓ_k and the transformation properties of V_k and $\pi(g)$, we get

<u>Lemma 5.1.</u> Let $\ell_k'\overset{def}{=}\ell_k(V')$. Then for gauge transformations

$$\sum_{n=1}^N\ell_n'\pi_{nm}=\ell_m\ . \tag{5.5}$$

Using this Lemma, one can prove

<u>Proposition 5.1.</u> The constraint (4.15) is the gauge invariant one:

$$\sum_{k=1}^N\ell_k'\varphi_{kN}'=\sum_{k=1}^N\ell_k\varphi_{kN}\ . \tag{5.6}$$

It follows from (5.6) that

$$\varphi_{\alpha N}'+\ell_\alpha'^{-1}\sum_{\beta\neq\alpha}\ell_\beta'\varphi_{\beta N}'=\ell_\alpha'^{-1}\ell_\alpha\Big(\varphi_{\alpha N}+\ell_\alpha^{-1}\sum_{\beta\neq\alpha}\ell_\beta\varphi_{\beta N}\Big)\,. \tag{5.7}$$

In particular, $\ell_N'^{-1}\ell_N=1$.

Therefore, not only the constraint (4.15) but also the procedure of its solving which has been considered in the previous section are both gauge invariant ones for any α.

Then the relations

$$\chi'=M_\alpha'\chi_{(\alpha)}'=M_{(\alpha)}'\pi_\alpha\chi_{(\alpha)}=\pi\chi=\pi M_\alpha\chi_{(\alpha)}$$

give

$$M_\alpha\xrightarrow{g}M_\alpha'=\pi M_\alpha\tilde{\pi}_\alpha \tag{5.8}$$

where $\pi_\alpha\tilde{\pi}_\alpha\overset{def}{=}E_\alpha$.

<u>Proposition 5.2.</u> The transformation law of the recursion operator L_α^+ under the gauge transformations is the following

$$L_\alpha^+(V)\longrightarrow L_\alpha^{+\prime}(V)\overset{def}{=}L_\alpha^+(V')=\tilde{\pi}_\alpha^+L_\alpha^+(V)\pi_\alpha^+ \tag{5.9}$$

Proof. One can derive the law (5.9) either by the straightforward calculation or by using the gauge invariance of the relation (4.26). Indeed, using (5.3) and $L'_\alpha \chi'_{(\alpha)} = \lambda^N \chi'_{(\alpha)}$, one obtains $L_\alpha(V') = \pi_\alpha L_\alpha \tilde{\pi}_\alpha$ and therefore (5.9). In particular, $L'_N = E_N \pi(g) L_N \pi^{-1}(g)$.

Lemma 5.2. The following relations

$$\left\langle \frac{\partial P'}{\partial t} \overset{++}{\varphi}{}' \right\rangle = \left\langle \frac{\partial P}{\partial t} \overset{++}{\varphi} \right\rangle \tag{5.10}$$

and

$$\left\langle [\tilde{A}^k, P' - P_\infty] \overset{++}{\varphi}{}' \right\rangle = \left\langle [\tilde{A}^k, P - P_\infty] \overset{++}{\varphi} \right\rangle \tag{5.11}$$

hold for the gauge transformations.

This lemma is proved by direct calculation with the use of (2.6), (5.1) and combining certain terms into the total derivatives over x .

From the relation (5.10) we have $\left\langle\!\left\langle \frac{\partial V'}{\partial t} \chi' \right\rangle\!\right\rangle = \left\langle\!\left\langle \frac{\partial V}{\partial t} \chi \right\rangle\!\right\rangle$. Therefore, $\left\langle\!\left\langle \pi^+_\alpha M^{+\prime}_\alpha \frac{\partial V'}{\partial t} \chi_{(\alpha)} \right\rangle\!\right\rangle = \left\langle\!\left\langle M^+_\alpha \frac{\partial V}{\partial t} \chi_{(\alpha)} \right\rangle\!\right\rangle$. As a result, taking into account (2.7), one gets

$$\pi^+_\alpha M^{+\prime}_\alpha \frac{\partial V'}{\partial t} = \pi^+_\alpha M^{+\prime}_\alpha \left(\frac{\partial \tau}{\partial t} V + \tau \frac{\partial V}{\partial t} + \frac{\partial \mathcal{V}}{\partial t} \right) =$$

$$= M^+_\alpha \pi^+ \left(\tau \frac{\partial V}{\partial t} + \frac{\partial \tau}{\partial t} V + \frac{\partial \mathcal{V}}{\partial t} \right) = M^+_\alpha \frac{\partial V}{\partial t} .$$

Hence

$$M^+_\alpha \left(\tau^{-1} \frac{\partial \tau}{\partial t} V + \tau^{-1} \frac{\partial \mathcal{V}}{\partial t} \right) = 0 . \tag{5.12}$$

Analogously, from (5.11) we have

$$\left\langle\!\left\langle \chi_{(\alpha)} \mathcal{Z}^+_{k(\alpha)} (V - V_\infty) \right\rangle\!\right\rangle = \left\langle\!\left\langle \chi'_{(\alpha)} \mathcal{Z}^{+\prime}_{k(\alpha)} (V' - V_\infty) \right\rangle\!\right\rangle$$

and therefore

$$\pi^+_\alpha \mathcal{Z}^{+\prime}_{k(\alpha)} (V' - V_\infty) = \mathcal{Z}^+_{k(\alpha)} (V - V_\infty) . \tag{5.13}$$

Using (5.12), (5.13) and (5.8), (5.9), we obtain

$$M^{+\prime}_\alpha \frac{\partial V'}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^{+\prime}_\alpha, t) \mathcal{Z}^{+\prime}_{k(\alpha)} (V' - V_\infty) =$$

$$= M^{+\prime}_\alpha \tau \left(\tau^{-1} \frac{\partial \tau}{\partial t} V + \tau^{-1} \frac{\partial \mathcal{V}}{\partial t} + \frac{\partial V}{\partial t} \right) -$$

$$- \tilde{\pi}^+_\alpha \sum_{k=1}^{N-1} \Omega_k (L^+_\alpha, t) \pi^+_\alpha \mathcal{Z}^{+\prime}_{k(\alpha)} (V' - V_\infty) =$$

$$= \tilde{\pi}^+_\alpha \left(M^+_\alpha \frac{\partial V}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^+_\alpha, t) \mathcal{Z}^+_{k(\alpha)} (V - V_\infty) \right) .$$

So we have proved the following

Theorem 5.1. Equations (4.40) are invariant under the gauge transformations (2.1) and

$$\begin{aligned} M^{+\prime}_\alpha \frac{\partial V'}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^{+\prime}_\alpha, t) \mathcal{Z}^{+\prime}_{k(\alpha)} (V' - V_\infty) &= \\ = \tilde{\pi}^+_\alpha \left(M^+_\alpha \frac{\partial V}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^+_\alpha, t) \mathcal{Z}^+_{k(\alpha)} (V - V_\infty) \right) &. \end{aligned} \tag{5.14}$$

In the particular case $\alpha = N$ we have $\tilde{\pi}^+_N = \tau(g) E_N$ and therefore

$$\begin{aligned} M^{+\prime}_N \frac{\partial V'}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^{+\prime}_N, t) \mathcal{Z}^{+\prime}_{k(N)} (V' - V_\infty) &= \\ = \tau(g) \left(M^+_N \frac{\partial V}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^+_N, t) \mathcal{Z}^+_{k(N)} (V - V_\infty) \right) &. \end{aligned} \tag{5.15}$$

2. Manifestly gauge invariant formulation. Equations (4.40) are the equations for the potentials $V_0, V_1, \ldots, V_{N-1}$ which are transformed under the gauge transformations according to the law (2.1). The left-hand side of these equations is transformed, too (according to (5.14)). On the other hand, the scattering matrix $S(\lambda, t)$ and equation (4.5) is not transformed at all under the gauge transformations. So it is quite natural to formulate the integrable equations in the manifestly gauge invariant form. It is possible due to the existence of the gauge invariants $W_0, W_1, \ldots, W_{N-2}$.

Theorem 5.2. The equations (4.40) for any α and $V_{N-1\infty} = 0$ are equivalent to the equations

$$\frac{\partial W(x,t)}{\partial t} - \sum_{k=1}^{N-1} \Omega_k(L_W^+, t)\mathcal{Z}^+_{(W)k}(W-W_\infty) = 0 \qquad (5.16)$$

where $W=(W_0, W_1, \dots, W_{N-2}, 0)^T$ and operators L_W^+ and $\mathcal{Z}^+_{(W)k}$ are given by formulas (4.29),(4.38) (with $\alpha = N$) in which one must make the substitution $V_i \to W_i$ (i = 0,1,...,N-2), $V_{N-1} \to 0$.

The Proof is based on formulas (2.7) and (2.8). It follows from these formulas that the transition from the potentials $(V_0, V_1, \dots, V_{N-1})$ to the invariants $(W_0, W_1, \dots, W_{N-2}, 0)$ is just a special gauge transformations with the gauge function $g=\beta^{-1}=\exp(\frac{1}{N}\int_{-\infty}^{x} dx' V_{N-1}(x'))$. For such a gauge transformation, $V_i' = W_i(V_0, V_1, \dots, V_{N-1})$, $(i=0,1,\dots,N-2)$ and $V'_{N-1}=0$. Then from formula (5.15) for this gauge transformation we have

$$M_W^+ \frac{\partial W}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L_W^+, t)\mathcal{Z}^+_{(W)k}(W-W_\infty) =$$

$$= \tau(\beta^{-1})\Big(M_N^+ \frac{\partial V}{\partial t} - \sum_{k=1}^{N-1} \Omega_k(L_N^+, t)\mathcal{Z}^+_{k(N)}(V-V_\infty)\Big). \qquad (5.17)$$

At last since $M_W^+ \frac{\partial W}{\partial t} = \frac{\partial W}{\partial t}$, $\det \tau(\beta^{-1}) \neq 0$, and equations (4.40) with different α are equivalent, then equation (5.17) results in the statement of the theorem.

Equations (5.16) contain only the gauge invariants W_k and represent the manifestly gauge invariant form of the evolution equations integrable by the problem (1.1). In different gauges, equations (5.16) look like the completely different equations but they are all gauge equivalent to each other [26]. The manifestly gauge invariant description (5.16) is also important for the Hamiltonian treatment of the evolution equations integrable by (1.1).

VI. Prehamiltonian form of the integrable equations

For the study of the Hamiltonian structure of equations (4.40) one must firstly express the nonlinear part of these equations through the variational derivatives. The relation

(3.3) is very useful for this purpose. It gives the relation between the infinitesimal variation of the potential and those for the scattering matrix:

$$\delta S_{in}(\lambda, t) = - \langle \delta P^T \vec{\varphi}^{+in}(\lambda) \rangle = \langle\langle \delta V \vec{\chi}^{+in} \rangle\rangle \qquad (6.1)$$

where $\vec{\varphi}^{+in}_{ke} \overset{def}{=} (F^+(x,t,\lambda))_{kn} (F^-(x,t,\lambda))^{-1}_{ie}$ and $\vec{\chi}^{+in} \overset{def}{=} (\vec{\varphi}^{+in}_{1N}, \vec{\varphi}^{+in}_{2N}, \dots, \vec{\varphi}^{+in}_{NN})^T$. The equality (6.1) means

$$\vec{\chi}^{+in}_e(x,t,\lambda) = \frac{\delta S_{in}(\lambda,t)}{\delta V_{e-1}(x,t)} \qquad (6.2)$$

where $\delta/\delta V_e$ denotes a variational derivative. However, since $\sum_{k=1}^{N} \ell_k \vec{\chi}_k = 0$, not all variational derivatives $\frac{\delta S}{\delta V_e}$ are independent.

Proposition 6.1. The variational derivatives of the scattering matrix satisfy the constraint

$$\sum_{k=1}^{N} \ell_k \frac{\delta S(\lambda,t)}{\delta V_{k-1}} = 0 \qquad (6.3)$$

where ℓ_k are given by (4.16).

The existence of the constraint (6.3) is essential for the Hamiltonian treatment of equations (4.40). This constraint is also closely connected to the gauge invariance.

Proposition 5.2. The constraint (6.3) is a consequence of the gauge invariance of the scattering matrix.

Proof. For the infinitesimal gauge transformations we have

$$\delta V_k(x,t) = -\ell^+_{k+1} \varepsilon(x,t), \; (k=0,1,\dots,N-1). \qquad (6.4)$$

Then, by the definition,

$$\delta S(\lambda,t) = \int_{-\infty}^{+\infty} dx \sum_{k=1}^{N} \frac{\delta S(\lambda,t)}{\delta V_{k-1}(x,t)} \delta V_{k-1}(x,t). \qquad (6.5)$$

This formula is also valid for the variations δV_k and δS induced by the gauge transformations. Substituting (6.4) into (6.5), we get

$$\delta S(\lambda,t) = -\int_{-\infty}^{+\infty} dx\, \varepsilon(x,t) \sum_{k=1}^{N} \ell_k \frac{\delta S(\lambda,t)}{\delta V_{k-1}(x,t)} = 0. \qquad (6.6)$$

Since $\mathcal{E}(x,t)$ is an arbitrary function, the equality (6.6) leads to (6.3).

The constraint (6.3) can be solved by differents ways. We will confine ourselves to those which have been considered in section 4.

The quantity $\vec{\chi}^{+in}$ plays an important role. Analogously to section 4, one can get

$$\lambda^N \Psi \vec{\chi}^{+in}(\lambda) = \mathcal{F} \vec{\chi}^{+in}(\lambda) \tag{6.7}$$

where Ψ and $\mathcal{F}$ are given by formulas (4.12). Equation (6.7) contains a constraint

$$\sum_{k=1}^{N} \ell_k \chi_k^{+in} = 0 \quad . \tag{6.8}$$

Let us choose the quantities $\chi_1^{+in}, \ldots, \chi_{\alpha-1}^{+in}, \chi_{\alpha+1}^{+in}, \ldots, \chi_N^{+in}$ i.e. the column $\vec{\chi}_{(\alpha)}^{+in} \overset{def}{=} (\chi_1^{+in}, \ldots, \chi_{\alpha-1}^{+in}, 0, \chi_{\alpha+1}^{+in}, \ldots, \chi_N^{+in})^T$ as the independent variables. From (6.8) we have

$$\chi_\alpha^{+in}(x,t,\lambda) = \delta_{\alpha N} \chi_{N(0)}^{+in} - \ell_\alpha^{-1} \sum_{\beta \neq \alpha} \ell_\beta \chi_\beta^{+} \tag{6.9}$$

where $\chi_{N(0)}^{+in}$ is a kernel of the operator ℓ_N. Therefore

$$\vec{\chi}^{+in} = \bar{M}_\alpha \vec{\chi}_{(\alpha)}^{+in} + \delta_{\alpha N} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \chi_{N(0)}^{+in} \end{pmatrix} \tag{6.10}$$

where

$$(\bar{M}_\alpha)_{ik} = \delta_{ik} - \delta_{i\alpha} \ell_\alpha^{-1} \ell_k \ . \tag{6.11}$$

Substitution of (6.10) into (6.7) gives

$$\lambda^N \Psi \left(\bar{M}_\alpha \vec{\chi}_{(\alpha)}^{+in} + \delta_{\alpha N} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \chi_{N(0)}^{+in} \end{pmatrix} \right) = \mathcal{F} \bar{M}_{(\alpha)} \vec{\chi}_{(\alpha)}^{+in} \quad . \tag{6.12}$$

Multiplying the relation (6.12) by $E_\alpha \bar{M}_N \tilde{\Psi}$ where $\tilde{\Psi}\Psi = E_N$, we obtain

$$\lambda^N \vec{\chi}_{(\alpha)}^{+in} = \tilde{L}_\alpha \vec{\chi}_{(\alpha)}^{+in} + \tilde{\Psi}_\alpha \vec{\chi}^{+in}(x=-\infty) \tag{6.13}$$

where $\tilde{\Psi}_\alpha \overset{def}{=} -E_\alpha \bar{M}_N N(\tilde{\Psi}\partial) R^T$, $R_{ik} \overset{def}{=} \delta_{i+1\,k}, (i,k=1,\ldots,N)$

and

$$\tilde{L}_\alpha = E_\alpha \bar{M}_N \tilde{\Psi} \mathcal{F} \bar{M}_\alpha \ . \tag{6.14}$$

Note that in the operators $\bar{M}_\alpha, \tilde{\Psi}, \ell_\alpha^{-1}$ the operator ∂^{-1} means $(\partial^{-1}f)(x) = \int_{-\infty}^{x} dy f(y)$.

Let us now introduce the quantities

$$\Pi^{(k)}(x,t,\lambda) \overset{def}{=} \sum_{n=1}^{N} (\bar{A}^k)_{nn} \frac{\vec{\chi}^{+n}(x,t,\lambda)}{S_{nn}(\lambda)} \ , \quad \Pi^{(k)}_{(\alpha)}(x,t,\lambda) = E_\alpha \Pi^{(k)}(x,t,\lambda) . \tag{6.15}$$

It follows from (6.13) that

$$\lambda^N \Pi^{(k)}_{(\alpha)} = \tilde{L}_\alpha \Pi^{(k)}_{(\alpha)} + \lambda^N \tilde{\Psi}_\alpha \Pi^{(k)}_0 \tag{6.16}$$

where $\Pi^{(k)}_0 = \sum_{n=1}^{N} (\bar{A}^k)_{nn} \frac{\vec{\chi}^{+n}(x=-\infty)}{S_{nn}(\lambda)}$.

Further, by virtue of (6.2), we have

$$\Pi^{(k)}_{(\alpha)}(x,t,\lambda) = E_\alpha \frac{\delta\, tr(\bar{A}^k \ell_n S_{\mathcal{D}}(\lambda))}{\delta V(x,t)} \tag{6.17}$$

where $(S_{\mathcal{D}})_{ik} \overset{def}{=} \delta_{ik} S_{ii}$. Note that right-hand side of (6.17) does not contain $\delta/\delta V_{\alpha-1}$. The substitution of (6.17) into (6.10) gives us the equations with the variational derivatives we are needed. The shortcoming of these equations is that they contain the operator $\tilde{L}_\alpha$ instead of the operator L_α^+ which defines the nonlinear part of equations (4.40).

However, one can prove, by straighforward calculations, the following important

<u>Theorem 6.1.</u> The relation

$$\mathcal{J}_\alpha \tilde{L}_\alpha = L_\alpha^+ \mathcal{J}_\alpha \tag{6.18}$$

holds where $\mathcal{J}_\alpha$ are N x N matrix integro-differential operators of the form

$$\mathcal{J}_\alpha = M_\alpha^+ \mathcal{J} \bar{M}_\alpha \tag{6.19}$$

where

$$\mathcal{J}_{ik}=\sum_{\ell=0}^{N+1-i-k}\{C_{\ell+i-1}^{i-1}V_{\ell+i+k-1}\partial^{\ell}-C_{k+\ell-1}^{k-1}(-\partial)^{\ell}(\cdot V_{\ell+i+k-1})\}+$$

$$+N\ell_N^{-1}\ell_k\delta_{i1}-N\ell_i^{+}\ell_N^{+-1}\delta_{k1}\,;$$

$$\mathcal{J}_{ik}=0\,,\quad i+k>N+1\,. \tag{6.20}$$

The relation (6.18) allows us to express the terms $(L_\alpha^+)^n\mathcal{L}_{k(\alpha)}^+(V-V_\infty)$ through $\Pi_{(\alpha)}^{(k)}$. Indeed, let us substitute the asymptotic expansion

$$\Pi_{(\alpha)}^{(k)}(x,t,\lambda)=\sum_{n=0}^{\infty}\lambda^{-nN}\Pi_{(\alpha)n}^{(k)}(x,t)$$

into (6.16). Solving the obtained system of recurrent relations and taking into account the equality $\mathcal{J}_\alpha\tilde{L}_\alpha\tilde{\tilde{\mathcal{G}}}_\alpha\Pi_0^{(k)}=$ $=\mathcal{L}_{k(\alpha)}^+(V-V_\infty)$, we obtain

$$(L_\alpha^+)^n\mathcal{L}_{k(\alpha)}^+(V-V_\infty)=(L_\alpha^+)^q\mathcal{J}_\alpha\Pi_{(\alpha)(n-q+1)}^{(k)} \tag{6.21}$$

where q is any integer.

With the use of (6.21) and (6.17) one can show that valid is the following

Theorem 6.2. Equations (4.40) with $\Omega_k(L_\alpha^+,t)=$ $=\sum_{n=0}^{\infty}\omega_{kn}(t)(L_\alpha^+)^n$ where $\omega_{kn}(t)$ are arbitrary functions can be represented in the form

$$M_\alpha^+\frac{\partial V}{\partial t}=(L_\alpha^+)^q\mathcal{J}_\alpha\frac{\delta\mathcal{H}_{-q}}{\delta V} \tag{6.22}$$

where q is any integer and

$$\mathcal{H}_{-q}=\sum_{k=1}^{N-1}\sum_{n=0}^{\infty}\frac{\omega_{kn}(t)}{(n+q-1)!}\left.\frac{\partial^{n+q-1}tr(\bar{A}^k\ln S_{\mathcal{D}}(\lambda))}{\partial(\lambda^{-N})^{n+q-1}}\right|_{\lambda=\infty}. \tag{6.23}$$

Note that the right-hand side of (6.22) does not contain $\delta/\delta V_{\alpha-1}$ and the set of the functionals $\mathcal{H}_{-q}$ is the same for the different α.

Since the operator M_α^+ is a nonconvertable one we will refer to the form (6.22) of the equations (4.40) as the prehamiltonian form.

Equations (6.22) possess the same transformation properties under the gauge group as equations (4.40). The functionals $\mathcal{H}_{-q}$ are the gauge invariant ones. One can also show that under the gauge transformations

$$\mathcal{J}_\alpha\to\mathcal{J}_\alpha'=\tilde{\pi}_\alpha^+\mathcal{J}_\alpha\tilde{\tilde{\pi}}_\alpha\quad. \tag{6.24}$$

In particular, $\mathcal{J}'=\tau\mathcal{J}\pi^{-1}$.

Note that, by virtue of (4.5), $\frac{dS_{\mathcal{D}}}{dt}=0$ and therefore the coefficients $C_n^{(k)}$ of the asymptotic expansion $tr(\bar{A}^k\ln S_{\mathcal{D}}(\lambda))=\sum_{n=0}^{\infty}\lambda^{-nN}C_n^{(k)}$ are integrals of motion for equations (4.40). So

$$\mathcal{H}_{-q}=\sum_{k=1}^{N-1}\sum_{n=0}^{\infty}\omega_{kn}(t)C_{n+q-1}^{(k)}\quad. \tag{6.25}$$

Emphasize also that the gauge invariance of the $C_n^{(k)}$ and the condition (6.3) for $C_n^{(k)}$ mean that all integrals of motions are the functionals only on the gauge invariants $W_0,W_1,\ldots,W_{N-2}$.

VII. Hamiltonian structure of the integrable equations

The configuration space for the evolution equations (4.40) has a functional dimension N. However, the gauge group of symmetry with the functional dimension 1 act on this space. The space with N variables $V_0(x,t),V_1(x,t),\ldots,V_{N-1}(x,t)$ therefore contains only N-1 dynamical (nongauge) degrees of freedom.

Further, the functionals $\mathcal{H}_{-q}$ which are candidates to the Hamiltonians are gauge-invariant ones and, therefore, their "functional" gradients satisfy the constraint

$$\sum_{k=1}^{N}\ell_k\frac{\delta\mathcal{H}_{-q}}{\delta V_{k-1}}=0\quad. \tag{7.1}$$

All this leads to the conclusion that for Hamiltonian treatment of equations (4.40) one must exclude the pure gauge (nondynamical) degrees of freedom.

The first, more traditional, way is to fix a gauge. For a fixed gauge, the gauge freedom is absent and the equations contain only N-1 independent dynamical variables.

For equations (4.40) with given α, the gauge $V_{\alpha-1} = V_{\alpha-1\infty}$ is convenient. In such a gauge $M_\alpha^+ \frac{\partial V}{\partial t} = \frac{\partial V}{\partial t}$ and equations (6.22) become

$$\frac{\partial V_{(\alpha)}}{\partial t} = (L_\alpha^+)^q J_\alpha \frac{\delta \mathcal{H}_{-q}}{\delta V_{(\alpha)}} \tag{7.2}$$

where $V_{(\alpha)} \stackrel{def}{=} (V_0, V_1, \ldots, V_{\alpha-2}, 0, V_\alpha, \ldots, V_{N-1})$ and in the operators L_α^+ and J_α one must put $V_{\alpha-1} = V_{\alpha-1\infty}$.

Theorem 7.1. Equations (4.40) in the gauge $V_{\alpha-1} = V_{\alpha-1\infty}$ are Hamiltonian ones, i.e. they are representable in the form

$$\frac{\partial V_{(\alpha)}(x,t)}{\partial t} = \{V_{(\alpha)}(x,t), \mathcal{H}_{-q}\}_{(\alpha)q} \tag{7.3}$$

with respect to the infinite family of Hamiltonians $\mathcal{H}_{-q}$ (6.23) and infinite family of Poisson brackets $\{\ ,\ \}_{(\alpha)q}$ where

$$\{\mathcal{F}, \mathcal{H}\}_{(\alpha)q} = \int_{-\infty}^{+\infty} dx \sum_{i,k=1}^{N} \frac{\delta \mathcal{F}}{\delta V_{i-1}(x,t)} \left((L_\alpha^+)^q J_\alpha\right)_{ik} \frac{\delta \mathcal{H}}{\delta V_{k-1}} = \tag{7.4}$$

$$= \left\langle \frac{\delta \mathcal{F}}{\delta V_{(\alpha)}} (L_\alpha^+)^q J_\alpha \frac{\delta \mathcal{H}}{\delta V_{(\alpha)}} \right\rangle .$$

The fact that the brackets (7.4) are indeed the Poisson brackets is verified by direct calculations.

For different α (i.e. for different gauges) we have different families of Poisson brackets. In the particular case $\alpha = N$ the brackets (7.4) coincide with those calculated earlier in [14]. The bracket $\{\ ,\ \}_{(N)0}$ is the well-known Gelfand--Dikij bracket [16].

For the first time the existence of the infinite family of Hamiltonian structures for the equations integrable by the inverse scattering transform method has been pointed out in [28,29]. Then the hierarchies of the Hamiltonian structures have been discussed in [30,31,20,21,11,9,13]. In particular, $\{\ ,\ \}_{(N)1}$ is the second Hamiltonian structures which has been considered in [18,31].

Thus, the first way of exclusion of the gauge degrees of freedom consists in the choice of only one representative from

each class of the gauge equivalent potentials. This choice is performed by crossing the classes of gauge equivalent potentials by the surface $V_\alpha = V_{\alpha\infty}$.

There exists another way of the exclusion of the gauge degrees of freedom. It consists in conversion of equations (4.40) into the manifestly gauge invariant form. Indeed, if one parametrize the potential V by the invariant W and "gauge" variable ρ : $V = \tau(\rho)W + \mathcal{V}(\rho)$ (see (2.10)), then by the use of the theorem 5.2 one gets that equations (4.40) are equivalent to equations (5.10). The latter do not contain any gauge degrees of freedom and they are Hamiltonian ones.

Theorem 7.2. Manifestly gauge invariant equations (5.16) can be represented in the Hamiltonian form

$$\frac{\partial W(x,t)}{\partial t} = \{W(x,t), \mathcal{H}_{-q}(W)\}_q \tag{7.5}$$

with respect to the infinite family of the Hamiltonians $\mathcal{H}_{-q}$, which are the functionals only on the gauge invariants W_0, $W_1, \ldots, W_{N-2}$, and the infinite family of the manifestly gauge invariant Poisson brackets $\{\ ,\ \}_q$ where

$$\{\mathcal{F}, \mathcal{H}\}_q = \int_{-\infty}^{+\infty} dx \sum_{i,k=1}^{N} \frac{\delta \mathcal{F}}{\delta W_{i-1}} \left((L_W^+)^q J_W\right)_{ik} \frac{\delta \mathcal{H}}{\delta W_{k-1}} \tag{7.6}$$

where J_W is given by formula (6.19) for $\alpha = N$ in which one must make a change

$$V_i \to W_i\ ,\ (i = 0, 1, \ldots, N-2)\ ;\ V_{N-1} \to 0\ .$$

The Proof of this theorem is analogous to that for the theorem 7.1 for $\alpha = N$. The only difference is the substitution $V_i \to W_i$.

Theorem 7.2 gives the manifestly gauge invariant description of the Hamiltonian structures which correspond to the evolution equations integrable by (1.1). In different gauges the brackets (7.6) give different Hamiltonian structures in terms of the potentials V_k. But all of them are gauge equivalent each to other (see [26]).

In virtue of the manifest gauge invariance of the Poisson brackets (7.6), any gauge transformations $V \to V'$ is a canonical transformation. Therefore, the gauge transformation from one gauge $V_\alpha = V_{\alpha\infty}$ to the other such gauge is the canonical transformation, too. In particular, by virtue of the Proposition 2.1, the Miura transformation is a canonical one.

In conclusion let us emphasize that the Hamiltonian structures (7.4) and (7.6) are universal ones, i.e. they are the Poisson brackets for equations (4.40) and (5.16) with any functions Ω_k.

VIII. Recursion operator and general form of the integrable equations in the N x N matrix form without solving the constraint

A starting point for calculation of the recursion operator is equation (4.11) which we rewrite here in the form

$$\lambda^N \mathcal{Y} \chi = \mathcal{F} \chi \tag{8.1}$$

where $\chi = (\chi_1, \chi_2, \ldots, \chi_N) \stackrel{def}{=} (\overset{++}{\varphi}{}^{in}_{1N}, \overset{++}{\varphi}{}^{in}_{2N}, \ldots, \overset{++}{\varphi}{}^{in}_{NN}), (i \neq n)$.
Since the rank of the matrix $\mathcal{Y}$ is equal to N-1 (see (4.13)), then equation (8.1) contains the constraint (4.15) $\sum_{k=1}^{N} \ell_k \chi_k = 0$. This constraint plays a fundamental role for the calculation of the recursion operator. In section 4, solving this constraint, we introduced the N-1 dimensional spaces of the independent variables $\chi_{(\alpha)}$. As a result, the nontrivial parts of the recursion operators L_α (L^+_α) which are N x N matrices have the matrix dimension N-1 and the integrable equations (4.40) are the systems of N-1 equations for N functions $V_0, V_1, \ldots, V_{N-1}$. Such a situation is a typical one for the AKNS method.

Here we show that the recursion operator can be defined in the N-dimensional space of all variables $(\chi_1, \chi_2, \ldots, \chi_N)$. Using such a recursion operator we present the equations integrable by (1.1) as the system of N equations for N functions $V_0, V_1, \ldots, V_{N-1}$.

First of all, note that, in virtue of (4.13), equation (8.1) is equivalent to the equation

$$\lambda^N \mathcal{Y} \chi = E_1 \mathcal{F} \chi \tag{8.2}$$

supplemented by the constraint

$$\sum_{k=1}^{N} \ell_k \chi_k = 0 . \tag{8.3}$$

From (8.2) we have

$$\lambda^N \chi = M_N \tilde{\mathcal{Y}} \mathcal{F} \chi \stackrel{def}{=} L_S \chi \tag{8.4}$$

where $\tilde{\mathcal{Y}} \mathcal{Y} = E_N$. The operator $L_S = M_N \tilde{\mathcal{Y}} \mathcal{F}$ is just the recursion operator which acts in the whole N-dimensional space $(\chi_1, \chi_2, \ldots, \chi_N)$. Equation (8.4) is compatible with (8.3).

However, L_S is not the most general recursion operator which can be defined on the whole N-dimensional space.

Proposition 8.1. The general form of the recursion operator which acts on the whole N-dimensional space $(\chi_1, \ldots, \chi_N)$

$$L \chi(\lambda) = \lambda^N \chi(\lambda) \tag{8.5}$$

is

$$L = L_S + Q \otimes \ell \tag{8.6}$$

where $\ell \stackrel{def}{=} (\ell_1, \ell_2, \ldots, \ell_N)$, $Q \stackrel{def}{=} (Q_1, Q_2, \ldots, Q_N)^T$ where $Q_1, Q_2, \ldots, Q_N$ are arbitrary operators and $\otimes$ denotes a tensor product.

Proof. The difference $L - L_S = \Delta$ should satisfy the condition $\Delta \chi = 0$. Since χ has N-1 independent components, the rank of the matrix Δ is equal to 1. As a result, taking into account (8.3) we have $\Delta_{ik} = Q_i \ell_k$ where Q_i are arbitrary operators.

So there exists a large freedom for the construction of the recursion operator in the whole N-dimensional space. The following calculation gives the examples of the operators Q_i:

$$\begin{aligned} \lambda^N \chi(x,\lambda) &= \lambda^N E_\beta \chi + \lambda^N (1 - E_\beta) \chi = \\ &= E_\beta M_N \tilde{\mathcal{Y}} \mathcal{F} \chi + (1 - E_\beta) M_N \tilde{\mathcal{Y}} \mathcal{F} \chi = \\ &= E_\beta M_N \tilde{\mathcal{Y}} \mathcal{F} \sum_{\alpha=0}^{N} c_\alpha M_\alpha \chi + (1 - E_\beta) M_N \tilde{\mathcal{Y}} \mathcal{F} \sum_{\alpha=0}^{N} d_\alpha M_\alpha \chi = L \chi \end{aligned} \tag{8.7}$$

where $M_0 = 1$ and c_α, d_α are arbitrary constants which satisfy the condition $\sum_{\alpha=0}^{N} c_\alpha = \sum_{\alpha=0}^{N} d_\alpha = 1$. The operators Q_ℓ which correspond to (8.7) are

$$Q_i = \sum_{\alpha=0}^{N} c_\alpha (E_\beta M_N \tilde{\mathcal{G}}\mathcal{F})_{i\alpha} \ell_\alpha^{-1} + \sum_{\alpha=0}^{N} d_\alpha ((1-E_\beta)M_N \tilde{\mathcal{G}}\mathcal{F})_{i\alpha} \ell_\alpha^{-1} \quad . \tag{8.8}$$

The general recursion operator L can also represented in the form

$$L = L_s + \sum_k Q_{(k)} \sum_{\alpha=0}^{N} (c_{(k)\alpha} - d_{(k)\alpha}) M_\alpha \tag{8.9}$$

where $Q_{(k)}$ are arbitrary matrix operators and $c_{(k)\alpha}$, $d_{(k)\alpha}$ are arbitrary constants which obey $\sum_{\alpha=0}^{N} c_{(k)\alpha} = \sum_{\alpha=0}^{N} d_{(k)\alpha} = 1$. One can show that the forms (8.6) and (8.9) of the recursion operator L are equivalent.

Taking into account that $\sum_{k=1}^{N} \ell_k (M_\alpha)_{k\ell} = 0$, we have

Corollary 8.1. Operator L^n has the structure analogous to (8.6), i.e.

$$L^n = L_s^n + Q_{(n)} \otimes \ell \tag{8.10}$$

where $Q_{(n)}$ are certain operators.

Corollary 8.2. For the adjoint operators we have

$$\begin{gathered} L^+ = L_s^+ + \ell^+ \otimes Q^+ , \\ (L^+)^n = (L_s^+)^n + \ell^+ \otimes Q_{(n)}^+ , \ (n=2,3,\ldots) \quad . \end{gathered} \tag{8.11}$$

The analogous freedom appears in the calculation of the operators $\mathcal{L}_k^+$, too. These are of the form

$$\mathcal{L}_k^+ = \mathcal{L}_{(s)k}^+ + \ell^+ \otimes \tilde{Q} \tag{8.13}$$

where $\tilde{Q} \stackrel{def}{=} (\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_N)$ and $\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_N$ are arbitrary operators and

$$\mathcal{L}_{(s)k}^+ = L_s^+ \Big(\sum_{i=1}^{N} \mathcal{G}_{(N-i)}^+ (b_k)_{iN} - a_k^+ \Big) + \\ + \sum_{i=1}^{N} \mathcal{F}_{(N-i)}^+ (b_k)_{iN} - b_k^+$$

where $\mathcal{G}_{(k)}^+$ and $\mathcal{F}_{(k)}^+$ are given by formulas (4.33).

Further, using the recursion operators L^+ and $\mathcal{L}_k^+$, we analogously to the section 3 obtain

$$\Big\langle\!\Big\langle \chi \Big(\frac{\partial V}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^+, t) \mathcal{L}_k^+ (V - V_\infty) \Big) \Big\rangle\!\Big\rangle = 0 . \tag{8.14}$$

However, the relation (8.14) essentially differs from the analogous relation (4.37). Indeed, the relation (4.37) contains only independent variables $\chi_{(\alpha)}$ and, as a result, the equality (4.37) leads to (4.40). Now the variables $\chi_1, \chi_2, \ldots, \chi_N$ are not independent and obey the constraint (8.3). As a result, the equality (8.14) is not equivalent to the equality to zero of the expression in the round brackets.

Theorem 8.1. The general form of the evolution equations connected to the problem (1.1) is the following

$$\frac{\partial V(x,t)}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^+, t) \mathcal{L}_k^+ (V - V_\infty) - f(L^+, t) \ell^+ \varphi = 0 \tag{8.15}$$

where $\Omega_k(L^+, t)$, $f(L^+, t)$ are arbitrary functions meromorphic on L^+, and L^+, $\mathcal{L}_k^+$ are any operators of the form (8.11), (8.13) and $\varphi(x,t)$ is an arbitrary scalar function. For equations (8.15) the scattering matrix evolves according to the equation (4.5).

Proof. It follows from (8.14) that

$$\frac{\partial V}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^+, t) \mathcal{L}_k^+ (V - V_\infty) = Z^+ \tag{8.16}$$

where Z^+ is any column for which its adjoint Z obeys the condition $Z\chi = 0$. It is not difficult to see that the general form of Z is $Z_k = f(\lambda^n, t) \varphi(x,t) \ell_k$, $(k=1,\ldots,N)$ where $\varphi(x,t)$ and $f(\lambda^n, t)$ are arbitrary scalar functions. Using $\lambda^n \chi = L\chi$, we have $Z_k \chi_k(\lambda) = \varphi(x,t) \ell_k (f(L,t)\chi)_k$. Therefore, $Z_k^+ = (f(L^+,t)\ell^+)_k \varphi$.

Theorem 8.2. Any equation of the form (8.15) with the arbitrary functions Ω_k, f, φ and operators L^+, $\mathcal{L}_k^+$ of the form (8.11), (8.13) is equivalent to the equation

$$\frac{\partial V(x,t)}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L_s^+, t) \mathcal{L}_{(s)k}^+ (V - V_\infty) - \ell^+ \varphi = 0 \tag{8.17}$$

where $\varphi(x,t)$ is a scalar function.

Proof. Let us consider any equation of the form (8.15) and substitute the expressions (8.12),(8.13) into (8.15). Since $(L_s^+)^n \ell^+ = 0$, we obtain equation (8.17). The function φ in (8.17) is calculated through Ω_k , f and φ from equation (8.15).

By virtue of the theorem 8.2, equations (8.17) give the general form of the equations integrable by (1.1). Now the recursion operator L_s^+ and $\mathcal{L}_{(s)k}^+$ are defined uniquelly and all uncertainties are contained in the term $\ell^+\varphi$ only.

Equations (8.15) and (8.17) are the equations integrable by (1.1) in the form which is natural for the AKNS method. For the entire functions Ω_k these equations can be represented, of course, in the standard Lax form $\frac{\partial L}{\partial t}=[L,A]$ where $L=\partial^N+V_{N-1}\partial^{N-1}+\ldots+V_0$ and A is the operator of the form $A=\sum_{k=0}^{M}u_k\partial^k$. The uncertainty which is contained in equations (8.15),(8.17) corresponds to the freedom in the choice of u_0.

Let us now consider some properties of equations (8.15) and (8.17). Firstly we discuss their transformation properties under the gauge transformations.

<u>Lemma 8.1.</u> For the gauge transformations $V\to V'$ (2.7)

$$\ell^{+\prime}\stackrel{def}{=}\ell^+(V')=\tau(g)\ell^+(V) \tag{8.18}$$

and

$$\frac{\partial V'}{\partial t}=\tau(g)\frac{\partial V}{\partial t}+\ell^{+\prime}\phi \tag{8.19}$$

where $\phi=g\frac{\partial}{\partial t}\left(\frac{1}{g}\right)$.

Formulas (8.18) and (8.19) follow directly from (5.5) and (2.7)

<u>Lemma 8.2.</u> For the gauge transformations $V\to V'$ we have

$$(L^+(V'))^n=\tau(g)(L^+(V))^n\tau^{-1}(g)+\ell^{+\prime}\otimes\widetilde{Q}_{(n)}, \tag{8.20}$$

$$\mathcal{L}_k^+(V')(V'-V_\infty)=\tau(g)\mathcal{L}_k^+(V)(V-V_\infty)+\ell^{+\prime}\otimes\widetilde{\phi}_k \tag{8.21}$$

where $\widetilde{Q}_{(n)}=(\widetilde{Q}_{(n)1},\ldots,\widetilde{Q}_{(n)N})$ are certain scalar operators and $\widetilde{\phi}_k$ are scalar functions the form of which are defined by the structure of L^+ and $\mathcal{L}_k^+$.

For some operators L^+ it may occur that $\widetilde{Q}_{(n)}=0$. The example of such an operator is $L^+=M_N^+\mathcal{F}^+\widetilde{\varphi}^+M_N^+$.

<u>Proposition 8.2.</u> Equations (8.15) and (8.17) are noninvariant under the gauge transformations. Gauge transformations convert the equations of the form (8.15) into the equations of the form (8.15).

<u>Proof.</u> Using (8.18)-(8.21), one can show that for the gauge transformations

$$\begin{aligned}&\frac{\partial V'}{\partial t}-\sum_{k=1}^{N-1}\Omega_k(L^{+\prime},t)\mathcal{L}_k^{+\prime}(V'-V_\infty)-f'(L^{+\prime},t)\ell^{+\prime}\varphi'=\\&=\tau(g)\left(\frac{\partial V}{\partial t}-\sum_{k=1}^{N-1}\Omega_k(L^+,t)\mathcal{L}_k^+(V-V_\infty)-f(L^+,t)\ell^+\varphi\right)\end{aligned} \tag{8.22}$$

where $L^{+\prime}\stackrel{def}{=}L^+(V')$, $\mathcal{L}_k^{+\prime}\stackrel{def}{=}\mathcal{L}_k^+(V')$. Functions f', φ' are expressed through f, φ and Ω_k and in the general case $f'\neq f$, $\varphi'\neq\varphi$.

For equations (8.17) with $\Omega_k(L_s^+,t)=\sum_{n=0}^{\infty}\omega_{kn}(t)(L_s^+)^n$ we have under the gauge transformations

$$\begin{aligned}&\frac{\partial V'}{\partial t}-\sum_{k=1}^{N-1}\Omega_k(L_s^{+\prime},t)\mathcal{L}_{(s)k}^{+\prime}(V'-V_\infty)-\ell^{+\prime}\varphi'=\\&=\tau(g)\left(\frac{\partial V}{\partial t}-\sum_{k=1}^{N-1}\Omega_k(L_s^+,t)\mathcal{L}_{(s)k}^+(V-V_\infty)-\ell^+\varphi\right)\end{aligned} \tag{8.23}$$

where

$$\varphi'=\varphi+g\frac{\partial}{\partial t}\left(\frac{1}{g}\right)-\sum_{k=1}^{N-1}\sum_{n=0}^{\infty}\omega_{kn}\widetilde{Q}_{(n)}\tau(g)\left(\mathcal{L}_{(s)k}^+(V-V_\infty)+\ell^+\widetilde{\phi}_k\right). \tag{8.24}$$

<u>Theorem 8.3.</u> Equations (8.17) and, therefore, equations (8.15) with given functions Ω_k and different functions φ are gauge equivalent each to other, i.e. the gauge transformations act in a transitive manner on the whole class of equations (8.17). In particular, any equation of the form (8.17) is gauge equivalent to the equation

$$\frac{\partial V(x,t)}{\partial t}-\sum_{k=1}^{N-1}\Omega_k(L_s^+,t)\mathcal{L}_{(s)k}^+(V-V_\infty)=0\,. \tag{8.25}$$

<u>Proof.</u> It follows from (8.24) that for a given φ it is always possible to find such a gauge function $g(x,t)$ to obtain any function φ' given in advance. Therefore, if V obeys equation (8.17) with some function φ, then V' obeys equation (8.17) with the function φ' which is given by (8.24). By appropriate gauge transformation one can transform a given

φ into an arbitrary function φ'. In particular, one can always convert any φ to $\varphi' = 0$.

Thus, the whole freedom which appears in the description of the integrable equations in the form (8.15) or (8.17) is of the pure gauge nature. Equation (8.25) is in a certain sence the standard representative of the whole class of the gauge equivalent equations (8.15) or (8.17). Note that the different recursion operators of the form (8.6) are not, in general, gauge equivalent to each other.

Let us now consider the relation between equations (8.15) and (4.40).

From the definitions (4.29),(8.11) and properties of the operators M_α^+ it follows

$$L_\alpha^+ = M_\alpha^+ L^+ E_\alpha = M_\alpha^+ L_s^+ E_\alpha \ . \qquad (8.26)$$

Using (4.28) and (8.13), one can also show that for any operator L^+ of the form (8.11)

$$M_\alpha^+ L^+ = L_\alpha^+ M_\alpha^+, \ M_\alpha^+ \mathcal{L}_k^+ = \mathcal{L}_{k(\alpha)}^+, \ M_\alpha^+ f(L^+, t) e^+ \varphi = 0 \ . \qquad (8.27)$$

<u>Proposition 8.3.</u> The system of N equations (8.15), for any recursion operators L^+ and $\mathcal{L}_k^+$ contains a subsystem of N-1 equations which coincides with the system (4.40).

<u>Proof.</u> Multiplying the left-hand side of (8.15) by M_α^+ and using (8.27), we obtain (4.40)

$$M_\alpha^+\Big(\frac{\partial V}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L^+, t) \mathcal{L}_k^+ (V - V_\infty) - f(L^+, t) e^+ \varphi\Big) =$$
$$= M_\alpha^+ \frac{\partial V}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L_\alpha^+, t) \mathcal{L}_{k(\alpha)}^+ (V - V_\infty) = 0 \ . \qquad (8.28)$$

Emphasize that in the projection of the system of N equations (8.15) onto the system of N-1 equations (4.40) the whole uncertainty in the definition of the recursion operator and the other freedom disappear.

The system of equations (4.40) contains N-1 nontrivial equations. This system can be supplemented to the system of N equations, for example, by the equation for $V_{\alpha-1}$.

<u>Proposition 8.4.</u> The system of the equations (8.15) is equivalent to the system of N-1 equations:

$$M_\alpha^+ \frac{\partial V}{\partial t} - \sum_{k=1}^{N-1} \Omega_k (L_\alpha^+, t) \mathcal{L}_{k(\alpha)}^+ (V - V_\infty) = 0 \qquad (8.29)$$

plus the equation

$$\frac{\partial V_{\alpha-1}}{\partial t} - \sum_{k=1}^{N-1} \big(\Omega_k (L^+, t) \mathcal{L}_k^+ (V - V_\infty)\big)_\alpha - \big(f(L^+, t) e^+\big)_\alpha \varphi = 0 \qquad (8.30)$$

where α is a any integer from 1,2,.., N.

It is easy to see that one can always choose the function φ such that the equation (8.30) is satisfied identically in the gauge $V_{\alpha-1} = V_{\alpha-1\infty}$.

For $\alpha = N$, the system (8.29), in view of the theorem 5.2, is equivalent to the system (5.16) for the gauge invariants and equation (8.30) for V_{N-1}:

$$\frac{\partial V_{N-1}}{\partial t} - \sum_{k=1}^{N-1} \big(\Omega_k (L^+, t) \mathcal{L}_k^+ (V - V_\infty)\big)_N - \big(f(L^+, t) e^+\big)_N \varphi = 0 \qquad (8.31)$$

is equivalent to the equation for the "gauge" variable which was introduced in (2.10): $\partial\rho/\rho = -\frac{1}{N} V_{N-1}$;

So we have the following.

<u>Proposition 8.5.</u> The system of N equations (8.15) is equivalent to the system of N-1 manifestly gauge invariant equations (5.16) supplemented by the equation for the "gauge" variable $\rho(x, t)$.

Therefore, the whole N (functionally) - dimensional configuration space ($V_0, V_1, \ldots, V_{N-1}$) for equations (8.15) (or (8.17)) contains the N-1 - dimensional invariant subspace on which the gauge group acts by the identical transformations, i.e. does not act at all.

This circumstance is important for the Hamiltonian treatment of equations (8.15). Indeed, for the interpretation of equations (8.15) as the Hamiltonian systems, one must exclude pure gauge (nondynamical) degrees of freedom. One can do it by the two ways. The first way is to fix a gauge, for example, $V_{\alpha-1} = V_{\alpha-1\infty}$. In this gauge, equations (8.29) are Hamiltonian ones due to the theorem (7.1). Another way is to project the N-dimensional system (8.15) onto the N-1 - dimensional system (5.16). In view of the theorem (7.2), the latter is a Hamiltonian system.

Thus we have

<u>Theorem 8.3.</u> Let we have any equation of the form (8.15). In our N-dimensional space of the variables $V_0, V_1, \dots, V_{N-1}$ let us separate out the N-1 dimensional space of the independent dynamical variables either by fixing a gauge or by projecting onto the invariant subspace described by equations (5.16). In both cases the corresponding nonlinear N-1 - dimensional systems are Hamiltonian systems with respect to the infinite family of Hamiltonian structures.

For the Hamiltonian treatment of equations (8.15) we exclude the pure gauge degrees of freedom. Let us now consider an opposite case when the system (8.15) contains only pure gauge variables, i.e. when all gauge invariants W_k are equal to their asymptotic values $W_{k\infty}$.

<u>Corollary 8.3.</u> In the case $W_k = W_{k\infty}$ the system (8.15) is equivalent to the only equation (8.31) for V_{N-1} in which $V_0, V_1, \dots, V_{N-2}$ are expressed through V_{N-1}, according to (2.2).

Indeed, in view of the proposition 8.5, the system (8.15) is equivalent to the system (5.16) plus equation (8.31). For $W_k = W_{k\infty}$, equations (5.16) are satisfied identically and equation (8.31) is an equation for the "gauge" variable ρ. The examples of equations (8.31) for N = 2,3 will be given in the next sections. Note also that for $W_k = 0$ we have $\partial^N + V_{N-1}\partial^{N-1} + \dots + V_0 = (\partial + \varphi)^N$ where $\varphi = \frac{1}{N} V_{N-1}$.

IX. The examples: N = 2.

In this and the next sections we consider the examples of equations (4.40), (8.15), (8.17) for the simplest cases N = 2,3.

For N = 2, $\alpha = 2$, and $V_{0\infty} \neq 0$, $V_{1\infty} \neq 0$ from the general formulas (4.16) and (4.18), we have

$$\ell_1 = \partial^2 - \partial(V_1 \cdot) \quad , \quad \ell_2 = -2\partial \, ,$$

$$M_2^+ = \begin{pmatrix} 1 & \tilde{\ell}_1^+ \\ 0 & 0 \end{pmatrix} \quad , \quad \tilde{\ell}_1^+ = -\ell_1^+ \ell_2^{+-1} = -\frac{1}{2}(\partial + V_1) \, . \tag{9.1}$$

The recursion operator is

$$L_2^+ = \begin{pmatrix} L_{11}^+ , & 0 \\ 0 , & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\partial V_0 + V_0\partial)\partial^{-1} + \frac{1}{4}(\partial + V_1)(\partial^2 - \partial V_1)\partial^{-1} , & 0 \\ 0 & , 0 \end{pmatrix}$$

and

$$\mathcal{L}_{1(2)}^+ = \begin{pmatrix} \partial + V_1 - V_{1\infty} , & -V_0 + V_{0\infty} - \frac{1}{2}\partial^2 - \frac{1}{2}V_1\partial \\ 0 , & 0 \end{pmatrix} . \tag{9.2}$$

For N = 2, equations (4.40) contain one nontrivial equation

$$\frac{\partial V_0}{\partial t} - \frac{1}{2}(\partial + V_1)\frac{\partial V_1}{\partial t} - \Omega(L_{11}^+, t)\left(\partial(V_0 - V_{0\infty}) - \frac{1}{2}\partial^2 V_1 - \frac{1}{2}V_1\partial V_1\right) = 0 \tag{9.3}$$

where $\Omega(L_{11}^+, t)$ is an arbitrary function meromorphic on L_{11}^+ and L_{11}^+ is given by (9.2). For $\Omega = \omega_{10} - 4L_{11}^+$, equation (9.3) is

$$\frac{\partial V_0}{\partial t} - \frac{1}{2}(\partial + V_1)\frac{\partial V_1}{\partial t} - \left(\omega_{10} + 2V_{0\infty} - \frac{V_{1\infty}^2}{2}\right)\left(\partial V_0 - \frac{1}{2}\partial^2 V_1 - \frac{1}{2}V_1\partial V_1\right) + (\partial^3 V_0 + 6V_0\partial V_0 - V_1^2\partial V_0) - \frac{1}{2}(\partial + V_1)\left(\partial^3 V_1 - \frac{3}{2}V_1^2\partial V_1\right) - 2V_0(\partial + V_1)\partial V_1 - 3(\partial V_0)(\partial V_1) - V_0\partial^2 V_1 - V_0 V_1 \partial V_1 - \frac{1}{2}V_1^2\partial V_0 = 0. \tag{9.4}$$

Equations (9.3) are gauge invariant and coincide with the manifestly gauge invariant equations

$$\frac{\partial W_0}{\partial t} - \Omega(L_W^+, t)\partial(W_0 - W_{0\infty}) = 0 \tag{9.5}$$

where $W_0 = V_0 - \frac{1}{2}\partial V_1 - \frac{1}{4}V_1^2$ is the gauge invariant and $L_W^+ = \frac{1}{4}\partial^2 + \frac{1}{2}(\partial W_0 + W_0\partial)\partial^{-1}$. The simplest equation (9.5) corresponds to $\Omega = \omega_{10} - 4L_W^+$ and it is the Korteweg - de Vries (KdV) equation in the manifestly gauge invariant form

$$\frac{\partial W_0}{\partial t} + \partial^3 W_0 + 6W_0\partial W_0 - (\omega_{10} + 2W_{0\infty})\partial W_0 = 0 \quad .$$

In the gauge $V_1 = 0$ equations (9.3) and (9.5) coincide with the KdV-family of equations (see e.g. [5]): $\frac{\partial V_0}{\partial t} - \Omega(L_{KdV}^+, t) \times \partial(V_0 - V_{0\infty}) = 0$ where $L_{KdV}^+ = L_W^+|_{V_1=0} = \frac{1}{4}\partial^2 + V_0 + \frac{1}{2}(\partial V_0)\partial^{-1}$ and the family of Poisson brackets (7.6) ($\mathcal{J}_W = 2\partial$) gives the well known family of Poisson brackets for the KdV-family equations.

In the gauge $V_0 = V_{0\infty}$, $(W_0 = V_{0\infty} - \frac{1}{2}\partial V_1 - \frac{1}{4} V_1^2)$ equations (9.5), as it is easy to see, are equivalent to the mKdV-family equations [5]: $\frac{\partial V_1}{\partial t} - \Omega(L^+_{mKdV}, t)\partial V_1 = 0$ where

$$L^+_{mKdV} = (\partial + V_1)^{-1} L^+_W \big|_{V_0 = V_{0\infty}} (\partial + V_1) = \frac{1}{4}\partial^2 - \frac{1}{4}\partial(V_1\partial^{-1}V_1\cdot) + V_{0\infty}.$$

For the gauge transformation from the gauge $V_0 = V_{0\infty}$ to the gauge $V_1' = 0$ we have $W_0(V_0', V_1') = W_0(V_0, V_1)$, i.e. $V_0' = V_{0\infty} - \frac{1}{2}\partial V_1 - \frac{1}{4} V_1^2$, that is the generalization of the Miura transformation [27].

Let us consider also the general linear gauge $\alpha_0 V_0 + \alpha_1 V_1 = 0$ where α_0 and α_1 are constants. For simplicity we here assume that $V_{0\infty} = V_{1\infty} = 0$. One can introduce the function $u(x,t)$ such that $V_0 = \beta_0 u$, $V_1 = \beta_1 u$ where β_0 and β_1 are some constants ($\alpha_0\beta_0 + \alpha_1\beta_1 = 0$). In this case, $W_0 = \beta_0 u - \frac{1}{2}\beta_1\partial u - \frac{1}{4}\beta_1^2 u^2$ and equations (9.5) are equivalent to the following

$$\frac{\partial u}{\partial t} - \Omega(L^+_u, t)\partial u = 0 \tag{9.6}$$

where

$$L^+_u = \left(\frac{\mathcal{D} W_0}{\mathcal{D} u}\right)^{-1} \cdot L^+_W \Big|_{\substack{V_0 = \beta_0 u \\ V_1 = \beta_1 u}} \frac{\mathcal{D} W_0}{\mathcal{D} u}$$

where $\frac{\mathcal{D} W_0}{\mathcal{D} u}$ is Frechet Jacobian of W_0 with respect to u : $\frac{\mathcal{D} W_0}{\mathcal{D} u} = \beta_0 - \frac{1}{2}\beta_1\partial - \frac{1}{2}\beta_1^2 u$.

Equations (9.6) are the equations of the combined KdV-mKdV type. For $\beta_1 = 0$ they reduce to the KdV-family and for $\beta_0 = 0$ - to the mKdV-family. The Hamiltonian structure of equations (9.6) is given by (7.6) at $V_0 = \beta_0 u$, $V_1 = \beta_1 u$.

The transformation from one general gauge ($V_0 = \beta_0 u$, $V_1 = \beta_1 u$) to another one ($V_0' = \beta_0' u'$, $V_1' = \beta_1' u'$) is a gauge transformation and

$$\beta_0' u' - \frac{1}{2}\beta_1'\partial u' - \frac{1}{4}\beta_1'^2 u'^2 = \beta_0 u - \frac{1}{2}\beta_1\partial u - \frac{1}{4}\beta_1^2 u^2 .$$

In particular, for the gauge transformation into the gauge with $\beta_1' = 0$ we have

$$u' = \frac{\beta_0}{\beta_0'} u - \frac{1}{2}\frac{\beta_1}{\beta_0'}\partial u - \frac{1}{4}\frac{\beta_1^2}{\beta_0'} u^2$$

that is just a Gardner transformation [27].

Since Miura and Gardner transformations are gauge transformations they obviously are the canonical transformations.

Now let us consider equations (8.15). The operators ℓ_1 , ℓ_2 , M_2^+ and $\tilde{\ell}_1^+$ are given by formula (9.1) and

$$\tilde{\mathcal{G}}^+ = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}\partial^{-1} & 0 \end{pmatrix} , \quad \mathcal{F}^+ = \begin{pmatrix} \partial^2 - V_1\partial , & \partial V_0 + V_0\partial \\ -2\partial , & -\partial^2 + \partial V_1 \end{pmatrix} . \tag{9.7}$$

The standard recursion operator L^+_s and $\mathcal{L}^+_{(s)1}$ are of the form

$$L^+_s = \frac{1}{2}\begin{pmatrix} (\partial V_0 + V_0\partial)\partial^{-1}, & -\frac{1}{2}(\partial V_0 + V_0\partial)\partial^{-1}(\partial + V_1) \\ (-\partial^2 + \partial V_1)\partial^{-1}, & -\frac{1}{2}(-\partial^2 + \partial V_1)\partial^{-1}(\partial + V_1) \end{pmatrix},$$

$$\mathcal{L}^+_{(s)1} = \begin{pmatrix} \partial + V_1 - V_{1\infty} , & -V_0 + V_{0\infty} \\ 0 , & \partial \end{pmatrix} . \tag{9.8}$$

The examples of the operator L^+ with the operators Q_k of the form (8.8):

$$L^+_{(1)} = \mathcal{F}^+\tilde{\mathcal{G}}^+ M_2^+ + M_2^+\mathcal{F}^+\tilde{\mathcal{G}}^+ - \mathcal{F}^+\tilde{\mathcal{G}}^+ =$$

$$= \frac{1}{2}\begin{pmatrix} (\partial V_0 + V_0\partial)\partial^{-1} - \frac{1}{2}(\partial + V_1)(-\partial^2 + \partial V_1)\partial^{-1}, & -\frac{1}{2}(\partial V_0 + V_0\partial)\partial^{-1}(\partial + V_1) \\ 0 & , -\frac{1}{2}(-\partial^2 + \partial V_1)\partial^{-1}(\partial + V_1) \end{pmatrix} \tag{9.9}$$

and

$$L^+_{(2)} = M_2^+\mathcal{F}^+\tilde{\mathcal{G}}^+ M_2^+ + \mathcal{F}^+\tilde{\mathcal{G}}^+ - M_2^+\mathcal{F}^+\tilde{\mathcal{G}}^+ =$$

$$= \frac{1}{2}\begin{pmatrix} (\partial V_0 + V_0\partial)\partial^{-1}, & -\frac{1}{2}(\partial V_0 + V_0\partial)\partial^{-1}(\partial + V_1) + \frac{1}{4}(\partial + V_1)(-\partial^2 + \partial V_1)(\partial + V_1) \\ (-\partial^2 + \partial V_1)\partial^{-1}, & 0 \end{pmatrix} . \tag{9.10}$$

The operator $\mathcal{J} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$.

Equations (8.17) are

$$\frac{\partial}{\partial t}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} - \Omega(L_s^+, t)\partial\begin{pmatrix} V_0 - V_{0\infty} \\ V_1 - V_{1\infty} \end{pmatrix} - \begin{pmatrix} \ell_1^+ \varphi \\ \ell_2^+ \varphi \end{pmatrix} = 0 \qquad (9.11)$$

where φ is an arbitrary scalar function. For $\Omega = \omega_{10} - 4L_s^+$ the system is

$$\frac{\partial V_0}{\partial t} - (\omega_{10} + 2V_{0\infty} - \tfrac{1}{2}V_{1\infty}^2)\partial V_0 + 6V_0\partial V_0 - (\partial V_0)(\partial V_1) -$$

$$- 2V_0\partial^2 V_1 - 2V_0 V_1 \partial V_1 - \tfrac{1}{2}V_1^2\partial V_0 - \partial^2\varphi - V_1\partial\varphi = 0, \qquad (9.12)$$

$$\frac{\partial V_1}{\partial t} - (\omega_{10} + 2V_{0\infty} - \tfrac{1}{2}V_{1\infty}^2)\partial V_1 + \partial^3 V_1 - \tfrac{3}{2}V_1^2\partial V_1 - 2\partial^2 V_0 + 2\partial(V_1 V_0) - 2\partial\varphi = 0.$$

For $\varphi = 0$ the system (9.12) exemplifies the standard system (8.25). For some concrete φ, the system (9.12) reduces to a certain well-known equation. For $V_1 = V_{1\infty}$ and $\varphi = -\partial V_0 + V_{1\infty}V_0$ the second equation (9.12) is satisfied identically and the first equation is the KdV equation. If one chooses the gauge $V_0 = V_{0\infty}$ and $\varphi = -2V_{0\infty}V_1$ then the first equation is 0 = 0 and the second is the mKdV equation.

For the system of equations (9.12) it is easy directly to check that if one multiplies the second equation by $\tilde{\ell}_1^+$ and sums the obtained equation with the first equation (9.12), one obtain equation (9.3) with $\Omega = \omega_{10} - 4L_{11}^+$.

Note that in the Lax form $\frac{\partial L}{\partial t} = [L, A]$ for equations (9.12) the operator

$$A = -4\partial^3 - 6V_1\partial^2 - (3(\partial V_1) + \tfrac{3}{2}V_1^2 + 6V_0 - \omega_{10} - 2V_{0\infty} + \tfrac{1}{2}V_{1\infty}^2)\partial -$$

$$- \partial V_0 - 2V_1 V_0 - \varphi \ .$$

Let us also give the examples of equations (8.15) with $f = 0$ and the operator L^+ of the form (9.9) and (9.10). For the operator $L_{(1)}^+$ and $\Omega(L_{(1)}^+, t) = \omega_{10} - 4L_{(1)}^+$ the system (8.15) is

$$\frac{\partial V_0}{\partial t} - (\omega_{10} + 2V_{0\infty} - \tfrac{1}{2}V_{1\infty}^2)\partial V_0 + V_{0\infty}\partial^2 V_1 + V_{0\infty}V_1\partial V_1 + \partial^3 V_0 +$$

$$+ 6V_0\partial V_0 - 3(\partial V_0)(\partial V_1 + \tfrac{1}{2}V_1^2) - 3V_0(\partial^2 V_1 + V_1\partial V_1) = 0, \qquad (9.13)$$

$$\frac{\partial V_1}{\partial t} - (\omega_{10} - \tfrac{1}{2}V_{1\infty}^2)\partial V_1 + \partial^3 V_1 - \tfrac{3}{2}V_1^2\partial V_1 = 0.$$

The system (9.13) is interest because if one substitutes the solutions of the second equation (mKdV equation) into the first equation, we obtain

$$\frac{\partial V_0}{\partial t} + \partial^3 V_0 + 6V_0\partial V_0 + V_0 F_1(x,t) + (\partial V_0)F_2(x,t) + F_3(x,t) = 0$$

where $F_1(x,t), F_2(x,t), F_3(x,t)$ are some concrete functions.

Equation (8.15) with the operator $L_{(2)}^+$ and $\Omega = \omega_{10} - 4L_{(2)}^+$ is the system

$$\frac{\partial V_0}{\partial t} - (\omega_{10} + 2V_{0\infty} - \tfrac{1}{2}V_{1\infty}^2)\partial V_0 + \tfrac{1}{2}V_{1\infty}^2\partial^2 V_1 + \tfrac{1}{2}V_{1\infty}^2 V_1\partial V_1 +$$

$$+ \partial^4 V_1 - 6V_0\partial V_0 - (\partial V_0)(\partial V_1) - 2V_0\partial^2 V_1 - \tfrac{1}{2}V_1^2\partial V_0 - 2V_0 V_1\partial V_1 +$$

$$+ V_1\partial^3 V_1 - \tfrac{3}{2}V_1^2\partial^2 V_1 - 3V_1(\partial V_1)^2 - \tfrac{3}{2}V_1^3\partial V_1 = 0, \qquad (9.14)$$

$$\frac{\partial V_1}{\partial t} - (\omega_{10} + 2V_{0\infty})\partial V_1 - 2\partial^2 V_0 + 2\partial(V_1 V_0) = 0 \ .$$

Emphasize that, according to the theorem (8.3), equations (9.12), (9.13) and (9.14) are gauge equivalent to each other.

At last, equations (8.31), for V_1, are the linear equations $\frac{\partial V_1}{\partial t} + \Omega_1\partial V_1 = 0$ in case $W_0 = W_{0\infty}$.

X. The examples: N = 3.

The general formulas (4.16), (4.18) give

$$\ell_1 = -\partial^3 + \partial^2 V_2 - \partial V_1 + V_0, \ \ell_2 = 3\partial^2 - 2\partial V_2, \ \ell_3 = 3\partial,$$

$$M_3^+ = \begin{pmatrix} 1 & 0 & \tilde{\ell}_1^+ \\ 0 & 1 & \tilde{\ell}_2^+ \\ 0 & 0 & 0 \end{pmatrix} \qquad (10.1)$$

where $\tilde{\ell}_1^+ = -\frac{1}{3}(\partial^2 + V_2\partial + V_1)$, $\tilde{\ell}_2^+ = -\partial - \frac{2}{3}V_2$. We consider only the case $\alpha = 3$. The operators $\mathcal{L}_{1(3)}^+$ and $\mathcal{L}_{2(3)}^+$ are

$$
\mathcal{L}^{+}_{1(3)} = \begin{pmatrix} \partial + V_2 - V_{2\infty}\,, & 0\,, & -V_0 + V_{0\infty} - \frac{1}{3}\partial^3 - \frac{1}{3}V_2\partial^2 - \frac{1}{3}V_1\partial \\ 0\,, & \partial + V_2 - V_{2\infty}\,, & -V_1 + V_{1\infty} - \partial^2 - \frac{2}{3}V_2\partial \\ 0\,, & 0\,, & 0 \end{pmatrix},
$$

$$
\mathcal{L}^{+}_{2(3)} = \begin{pmatrix} \partial^2 + (V_2 - V_{2\infty})\partial + V_1 - V_{1\infty}, & -\frac{2}{3}\partial^3 - \frac{2}{3}V_2\partial^2 - \frac{2}{3}V_1\partial - V_0 + V_{0\infty}, & (\mathcal{L}^{+}_{2(3)})_{13} \\ 2\partial + V_2 - V_{2\infty}, & -\partial^2 + (V_2 - V_{2\infty})\partial - \frac{4}{3}V_2\partial - V_{2\infty}(V_2 - V_{2\infty}), & (\mathcal{L}^{+}_{2(3)})_{23} \\ 0\,, & 0 & ,\ 0 \end{pmatrix} \quad (10.2)
$$

where

$$
(\mathcal{L}^{+}_{2(3)})_{13} = \frac{1}{3}\partial^4 + \left(\frac{4}{9}V_2 + \frac{1}{3}V_{2\infty}\right)\partial^3 + \frac{1}{3}\left(V_1 + (\partial V_2) + \frac{1}{3}V_2^2 + V_2 V_{2\infty}\right)\partial^2 +
$$

$$
+ \frac{1}{3}\left((\partial^2 V_2) + \frac{2}{3}V_2(\partial V_2) + \frac{1}{3}V_1 V_2 + \frac{1}{3}V_1 V_{1\infty}\right)\partial - \frac{1}{3}(\partial V_0) +
$$

$$
+ \frac{1}{9}(\partial^3 V_2) + \frac{1}{9}V_2(\partial^2 V_2) + \frac{1}{9}V_1(\partial V_2) + V_{2\infty}(V_0 - V_{0\infty}),
$$

$$
(\mathcal{L}^{+}_{2(3)})_{23} = \frac{1}{3}\partial^3 + \left(\frac{1}{3}V_2 + V_{2\infty}\right)\partial^2 + \frac{1}{3}\left(-2V_1 + 2(\partial V_2) +\right.
$$

$$
\left. + \frac{2}{3}V_2^2 + 2V_{2\infty}V_2\right)\partial - \frac{1}{3}(\partial V_1) + \frac{1}{3}(\partial^2 V_2) + \frac{2}{9}V_2(\partial V_2) -
$$

$$
- V_0 + V_{0\infty} + V_2(V_1 - V_{1\infty}) \quad .
$$

The matrix elements of the recursion operator L^{+}_{3} are

$$
(L^{+}_{3})_{11} = \frac{1}{3}\left[2V_0\partial + \partial V_0 + \frac{1}{3}(\partial^2 + V_2\partial + V_1)(3\partial^2 - \partial V_2)\right]\partial^{-1},
$$

$$
(L^{+}_{3})_{12} = -\frac{1}{9}\left[2V_0\partial + \partial V_0 + \frac{1}{3}(\partial^2 + V_2\partial + V_1)(3\partial^2 - \partial V_2)\right]\partial^{-1} \times
$$

$$
\times (3\partial^2 - 2\partial V_2 + V_2\partial)\partial^{-1} + \frac{1}{3}\left[\partial^2 V_0 + \partial V_0\partial + V_0\partial^2 - \partial V_0 V_2 - V_0\partial V_2 +\right.
$$

$$
\left. + V_2\partial V_0 + \frac{1}{3}(\partial^2 + V_2\partial + V_1)(\partial^3 - 2\partial V_1 - \partial^2 V_2 - \partial V_2\partial + \partial V_2^2)\right]\partial^{-1},
$$

$$
(L^{+}_{3})_{21} = \frac{1}{3}\left[-\partial^3 + V_1\partial + \partial V_1 - V_2\partial^2 + \left(\partial + \frac{2}{3}V_2\right)(3\partial^2 - \partial V_2)\right]\partial^{-1},
$$

$$
(L^{+}_{3})_{22} = -\frac{1}{9}\left[-\partial^3 + V_1\partial + \partial V_1 - V_2\partial^2 + \left(\partial + \frac{2}{3}V_2\right)(3\partial^2 - \partial V_2)\right]\partial^{-1} \times
$$

$$
\times (3\partial^2 - 2\partial V_2 + V_2\partial)\partial^{-1} + \frac{1}{3}\left[2\partial V_0 + V_0\partial + \partial^2 V_1 + \partial V_1\partial +\right.
$$

$$
+ V_1\partial^2 - \partial V_1 V_2 - V_1\partial V_2 + V_2\partial V_1 + \left(\partial + \frac{2}{3}V_2\right)(\partial^3 - 2\partial V_1 -
$$

$$
\left. - \partial^2 V_2 - \partial V_2\partial + \partial V_2^2)\right]\partial^{-1},
$$

$$
(L^{+}_{3})_{13} = (L^{+}_{3})_{23} = (L^{+}_{3})_{31} = (L^{+}_{3})_{32} = (L^{+}_{3})_{33} = 0 \ .
$$

The simplest equation (4.40) for $\alpha = 3$ and $\Omega_1 = \omega_{10} = \mathrm{const}$, $\Omega_2 = \omega_{20} = \mathit{const}$ is of the form

$$
\frac{\partial V_0}{\partial t} - \frac{1}{3}(\partial^2 + V_2\partial + V_1)\frac{\partial V_2}{\partial t} = \omega_{10}\left\{\partial V_0 - \frac{1}{3}\partial^3 V_2 - V_2\partial^2 V_2 - \frac{1}{3}V_1\partial V_2\right\} +
$$

$$
+ \omega_{20}\left\{\partial^2 V_0 - \frac{2}{3}\partial^3 V_1 - \frac{2}{3}V_1\partial V_1 + \frac{1}{3}\partial^4 V_2 + (V_2 - V_{2\infty})\partial V_0 - \frac{2}{3}V_2\partial^2 V_1 +\right.
$$

$$
+ \left(\frac{4}{9}V_2 + \frac{1}{3}V_{2\infty}\right)\partial^3 V_2 + \frac{1}{3}\left(V_1 + (\partial V_2) + \frac{1}{3}V_2^2 + V_2 V_{2\infty}\right)\partial^2 V_2 +
$$

$$
+ \frac{1}{3}\left((\partial^2 V_2) + \frac{2}{3}V_2(\partial V_2) + \frac{1}{3}V_1 V_2 + \frac{1}{3}V_1 V_{1\infty}\right)\partial V_2 +
$$

$$
\left. + \left(-\frac{1}{3}(\partial V_0) + \frac{1}{9}(\partial^3 V_2) + \frac{1}{9}V_2(\partial^2 V_2) + \frac{1}{9}V_1(\partial V_2) + V_{2\infty}(V_0 - V_{0\infty})\right)(V_2 - V_{2\infty})\right\},
$$

$$
\frac{\partial V_1}{\partial t} - \left(\partial + \frac{2}{3}V_2\right)\frac{\partial V_2}{\partial t} = \omega_{10}\left\{\partial V_1 - \partial^2 V_2 - \frac{2}{3}V_2\partial V_2\right\} + \quad (10.4)
$$

$$
+ \omega_{20}\left\{2\partial V_0 - \partial^2 V_1 + \frac{1}{3}\partial^3 V_2 + \left(\frac{1}{3}V_2 + V_{2\infty}\right)\partial^2 V_2 + (V_2 - V_{2\infty})\partial V_1 -\right.
$$

$$
- \frac{4}{3}V_2\partial V_1 + \frac{1}{3}\left(-2V_1 + 2(\partial V_2) + \frac{2}{3}V_2^2 + 2V_2 V_{2\infty}\right)\partial V_2 -
$$

$$
- \left(-\frac{1}{3}(\partial V_1) + \frac{1}{3}(\partial^2 V_2) + V_2(V_1 - V_{1\infty}) + \frac{2}{9}V_2(\partial V_2)\right)(V_2 - V_{2\infty}) -
$$

$$
\left. - V_{2\infty}(V_2 - V_{2\infty})(V_1 - V_{1\infty})\right\} .
$$

The operators $\mathcal{L}^{+}_{(s)k}$ are

$$
\mathcal{L}^{+}_{(s)1} = \begin{pmatrix} \partial + V_2 - V_{2\infty}, & 0 & , -V_0 + V_{0\infty} \\ 0 & , \partial + V_2 - V_{2\infty}, & -V_1 + V_{1\infty} \\ 0 , & 0 , & \partial \end{pmatrix},
$$

$$
\mathcal{L}^{+}_{(s)2} = \begin{pmatrix} \partial^2 + (V_2 - V_{2\infty})\partial + V_1 - V_{1\infty} - V_{2\infty}(V_2 - V_{2\infty}), & -V_0 + V_{0\infty}, & -\frac{1}{3}(\partial V_0) + V_{2\infty}(V_0 - V_{0\infty}) \\ 2\partial + V_2 - V_{2\infty}, & \partial^2 + (V_2 - V_{2\infty})\partial - V_{2\infty}(V_2 - V_{2\infty}), & -\frac{2}{3}\partial^3 - \frac{2}{3}V_2\partial^2 - \frac{1}{3}(\partial V_1) + \\ & & + V_{2\infty}(V_1 - V_{1\infty}) - \frac{2}{3}V_1\partial - V_0 + V_{0\infty} \\ 0 , & 2\partial , & -\partial^2 - \frac{1}{3}V_2\partial - V_{2\infty}\partial - \frac{1}{3}(\partial V_2) \end{pmatrix}. \tag{10.5}
$$

The matrix elements of the operator L^{+}_{s} are

$$
\begin{aligned}
&(L^{+}_{s})_{11} = (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{11}, \quad (L^{+}_{s})_{12} = (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{12}, \\
&(L^{+}_{s})_{13} = (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{11}\tilde{e}^{+}_{1} + (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{12}\tilde{e}^{+}_{2}, \\
&(L^{+}_{s})_{21} = (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{21}, \quad (L^{+}_{s})_{22} = (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{22}, \\
&(L^{+}_{s})_{23} = (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{21}\tilde{e}^{+}_{1} + (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{22}\tilde{e}^{+}_{2}, \\
&(L^{+}_{s})_{31} = (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{31}, \quad (L^{+}_{s})_{32} = (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{32}, \\
&(L^{+}_{s})_{33} = (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{31}\tilde{e}^{+}_{1} + (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{32}\tilde{e}^{+}_{2}
\end{aligned} \tag{10.6}
$$

where

$$
\begin{aligned}
&(\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{11} = V_0 + \tfrac{1}{3}(\partial V_0)\partial^{-1}, \\
&(\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{21} = -\tfrac{1}{3}\partial^2 - \tfrac{1}{3}V_2\partial + \tfrac{2}{3}V_1 + \tfrac{1}{3}(\partial V_1)\partial^{-1}, \\
&(\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{31} = -\partial + \tfrac{1}{3}V_2 + \tfrac{1}{3}(\partial V_2)\partial^{-1}, \\
&(\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{12} = \tfrac{2}{3}(\partial V_0) + \tfrac{1}{3}(\partial^2 V_0)\partial^{-1} + \tfrac{1}{9}(\partial V_0)\partial^{-1}V_2\cdot + \tfrac{2}{9}(\partial V_0)\partial^{-1}(\partial V_2)\partial^{-1}\cdot,
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{22} = {} & \tfrac{1}{3}\partial^3 + \tfrac{2}{9}V_2\partial^2 + \tfrac{1}{3}V_1\partial - \tfrac{4}{9}(\partial V_2)\partial - \tfrac{1}{9}V_2^2\partial + \\
& + V_0 + \tfrac{2}{3}(\partial V_1) - \tfrac{5}{9}(\partial^2 V_2) - \tfrac{1}{9}V_1V_2 - \tfrac{1}{3}V_2(\partial V_2) + \\
& + \left(\tfrac{2}{3}(\partial V_0) + \tfrac{1}{3}(\partial^2 V_1) - \tfrac{2}{9}(\partial^3 V_2) - \tfrac{2}{9}V_1(\partial V_2) - \tfrac{2}{9}V_2(\partial^2 V_2)\right)\partial^{-1} + \\
& + \tfrac{1}{9}(\partial V_1)\partial^{-1}V_2\cdot + \tfrac{2}{9}(\partial V_1)\partial^{-1}(\partial V_2)\partial^{-1}\cdot , \\
(\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{32} = {} & \tfrac{2}{3}\partial^2 + \tfrac{2}{3}V_1 - \tfrac{1}{3}(\partial V_2) - \tfrac{2}{9}V_2^2 + \left(\tfrac{2}{3}(\partial V_1) - \right. \\
& \left. - \tfrac{1}{3}(\partial^2 V_2) - \tfrac{4}{9}V_2(\partial V_2)\right)\partial^{-1} + \tfrac{1}{9}(\partial V_2)\partial^{-1}V_2\cdot + \tfrac{2}{9}(\partial V_2)\partial^{-1}(\partial V_2)\partial^{-1}\cdot , \\
(\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{13} = {} & (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{23} = (\mathcal{F}^{+}\tilde{\mathcal{G}}^{+})_{33} = 0.
\end{aligned}
$$

Equations (8.17) with $\varphi = 0$, $\Omega_1 = \omega_{10} = const$, $\Omega_2 = \omega_{20} = const$ are of the form

$$
\begin{aligned}
\frac{\partial V_0}{\partial t} &= (\omega_{10} - \tfrac{2}{3}V_{2\infty}\omega_{20})\partial V_0 + \omega_{20}(\partial^2 V_0 - \tfrac{2}{3}V_2\partial V_0), \\
\frac{\partial V_1}{\partial t} &= (\omega_{10} - \tfrac{2}{3}V_{2\infty}\omega_{20})\partial V_1 + \omega_{20}(-\tfrac{2}{3}\partial^3 V_2 + \partial^2 V_1 + \\
&\quad + 2\partial V_0 + \tfrac{2}{3}V_2\partial V_1 - \tfrac{2}{3}V_1\partial V_2 - \tfrac{2}{3}V_2\partial^2 V_2), \\
\frac{\partial V_2}{\partial t} &= (\omega_{10} - \tfrac{2}{3}V_{2\infty}\omega_{20})\partial V_2 + \omega_{20}(-\partial^3 V_2 + 2\partial V_1 - \tfrac{2}{3}V_2\partial V_2).
\end{aligned} \tag{10.7}
$$

Among equations (8.15) with

$$
L^{+} = \mathcal{F}^{+}\tilde{\mathcal{G}}^{+}M^{+}_{N} + M^{+}_{N}\mathcal{F}^{+}\tilde{\mathcal{G}}^{+} - \mathcal{F}^{+}\tilde{\mathcal{G}}^{+},
$$

$$
\begin{aligned}
\mathcal{L}^{+}_{k} = {} & \Big\{L^{+}M^{+}_{N}\Big(\sum_{i=1}^{N}\mathcal{G}^{+}_{(N-i)}(b_k)_{iN} - a^{+}_{k}\Big) + M^{+}_{N}\Big(\sum_{i=1}^{N}\mathcal{F}^{+}_{(N-i)}(b_k)_{iN} - b^{+}_{k}\Big)\Big\}E_N + \\
& + \Big\{L^{+}\Big(\sum_{i=1}^{N}\mathcal{G}^{+}_{(N-i)}(b_k)_{iN} - a^{+}_{k}\Big) + \Big(\sum_{i=1}^{N}\mathcal{F}^{+}_{(N-i)}(b_k)_{iN} - b^{+}_{k}\Big)\Big\}(1 - E_N)
\end{aligned}
$$

the equations with $\varphi = 0$, $\Omega_1 = \omega_{10} = const$, $\Omega_2 = \omega_{20} = const$ are of interest. These equations are

$$\frac{\partial V_0}{\partial t} = (\omega_{10} - V_{2\infty}\omega_{20})\partial V_0 + \omega_{20}\Big(\partial^2 V_0 - \frac{2}{3}\partial^3 V_1 + \frac{2}{3}\partial^4 V_2 +$$

$$+ \frac{2}{3}V_2\partial V_0 - \frac{2}{3}V_1\partial V_1 - \frac{2}{3}V_2\partial^2 V_1 + \frac{2}{3}V_1\partial^2 V_2 + \frac{2}{9}V_2\partial^3 V_2 -$$

$$- \frac{4}{3}(\partial V_2)\partial^2 V_2 - \frac{4}{9}V_2^2\partial^2 V_2 - \frac{4}{9}V_2(\partial V_2)^2 - \frac{4}{9}V_1V_2(\partial V_2)\Big),$$

$$\frac{\partial V_1}{\partial t} = (\omega_{10} - V_{2\infty}\omega_{20})\partial V_1 + \omega_{20}\Big(2\partial V_0 + \frac{4}{3}\partial^3 V_2 - \qquad (10.8)$$

$$- \frac{2}{3}\partial(V_1V_2) - \frac{4}{3}(\partial V_2)^2 - \frac{8}{9}V_2^2\partial V_2 - \frac{2}{3}V_2\partial^2 V_2\Big),$$

$$\frac{\partial V_2}{\partial t} = (\omega_{10} - V_{2\infty}\omega_{20})\partial V_2 + \omega_{20}\big(\partial^2 V_2 - \partial(V_2^2)\big) .$$

For arbitrary N and an analogous choice of L^+, $\mathcal{L}_k^+$ and $\Omega_1 = \omega_{10} = const, \Omega_2 = \omega_{20} = const, \Omega_3 = \ldots = \Omega_{N-1} = 0$, the Nth equation in the system (8.15) is the Burgers equation, too:

$$\frac{\partial V_{N-1}}{\partial t} = (\omega_{10} - \omega_{20}V_{N-1\infty})\partial V_{N-1} + \omega_{20}\big(\partial^2 V_{N-1} - \partial(V_{N-1}^2)\big). \qquad (10.9)$$

Let us consider also equations (4.40) for α = 3 in the gauge V_2 = 0. In this gauge the expressions $\mathcal{L}^+_{k(3)}(V - V_\infty)$, as it follows from (10.2), do not contain $V_{0\infty}$ and $V_{1\infty}$. The recursion operator L_3 for V_2 = 0 is

$$(L_3^+)_{13} = (L_3^+)_{23} = (L_3^+)_{33} = (L_3^+)_{31} = (L_3^+)_{32} = 0,$$

$$(L_3^+)_{11} = \frac{1}{3}\partial^3 + \frac{1}{3}V_1\partial + V_0 + \frac{1}{3}(\partial V_0)\partial^{-1},$$

$$(L_3^+)_{21} = \frac{2}{3}\partial^2 + \frac{2}{3}V_1 + \frac{1}{3}(\partial V_1)\partial^{-1}, \qquad (10.10)$$

$$(L_3^+)_{12} = -\frac{2}{9}\partial^4 - \frac{2}{9}V_1\partial^2 - \frac{2}{9}\partial^2(V_1\cdot) + \frac{2}{3}(\partial V_0) - \frac{2}{9}V_1^2 +$$

$$+ \frac{1}{3}(\partial^2 V_0)\partial^{-1} - \frac{2}{9}V_1(\partial V_1)\partial^{-1} - \frac{2}{9}\partial^2((\partial V_1)\partial^{-1}\cdot),$$

$$(L_3^+)_{22} = -\frac{1}{3}\partial^3 - \frac{1}{3}\partial(V_1\cdot) + \frac{1}{3}V_0 - \frac{1}{3}\partial((\partial V_1)\partial^{-1}\cdot) + \frac{2}{3}\partial(V_0\partial^{-1}\cdot).$$

In this case, equation (4.40) for $\Omega_1 = \omega_{10} + \omega_{11}L_3^+$, $\Omega_2 = \omega_{20} + \omega_{21}L_3^+$ is the following system:

$$\frac{\partial V_0}{\partial t} = \Big(\omega_{10} - \frac{1}{3}\omega_{11}V_{0\infty} + \frac{1}{9}\omega_{21}V_{1\infty}^2\Big)\partial V_0 + \Big(\omega_{20} - \frac{1}{3}\omega_{11}V_{1\infty} - \frac{2}{3}\omega_{21}V_{0\infty}\Big)\times$$

$$\times\Big(\partial^2 V_0 - \frac{2}{3}V_1\partial V_1 - \frac{2}{3}\partial^3 V_1\Big) + \omega_{11}\Big\{-\frac{2}{9}\partial^5 V_1 + \frac{1}{3}\partial^4 V_0 + \frac{2}{3}\partial(V_1\partial V_0) +$$

$$+ \frac{2}{3}\partial(V_0^2) - \frac{2}{3}V_1\partial^3 V_1 - \frac{4}{3}(\partial V_1)(\partial^2 V_1) - \frac{4}{9}V_1^2\partial V_1\Big\} + \omega_{21}\Big\{-\frac{1}{9}\partial^5 V_0 -$$

$$- \frac{5}{9}\partial^3(V_1V_0) - \frac{5}{9}\partial(V_0\partial^2 V_1) + \frac{5}{3}\partial(V_0\partial V_0) - \frac{5}{9}\partial(V_0V_1^2)\Big\}, \qquad (10.11)$$

$$\frac{\partial V_1}{\partial t} = \Big(\omega_{10} - \frac{1}{3}\omega_{11}V_{0\infty} + \frac{1}{9}\omega_{21}V_{1\infty}^2\Big)\partial V_1 + \Big(\omega_{20} - \frac{1}{3}\omega_{11}V_{1\infty} - \frac{2}{3}\omega_{21}V_{0\infty}\Big)\times$$

$$\times(2\partial V_0 - \partial^2 V_1) + \omega_{11}\Big\{-\frac{1}{3}\partial^4 V_1 + \frac{2}{3}\partial^3 V_0 + \frac{4}{3}\partial(V_0V_1) - \frac{2}{3}\partial(V_1\partial V_1)\Big\} +$$

$$+ \omega_{21}\Big\{-\frac{1}{9}\partial^5 V_1 - \frac{5}{9}\partial(V_1\partial^2 V_1) - \frac{5}{3}\partial(V_0\partial V_1) - \frac{5}{9}V_1^2\partial V_1 + \frac{5}{3}\partial(V_0^2)\Big\}.$$

The system of equations (10.11) contains some well known equations as the particular cases. For $\omega_{10} = \omega_{11} = \omega_{21} = 0$ the system (10.11) is

$$\frac{\partial V_0}{\partial t} = \omega_{20}\Big(\partial^2 V_0 - \frac{2}{3}\partial^3 V_1 - \frac{2}{3}V_1\partial V_1\Big),$$

$$\frac{\partial V_1}{\partial t} = \omega_{20}(2\partial V_0 - \partial^2 V_1) .$$

This system, as it easy to see, is equivalent, for $\omega_{20} = i\frac{\sqrt{3}}{2}$, to the Boussinesq equation

$$\frac{\partial^2 V_1}{\partial t^2} = \frac{1}{2}\partial^4 V_1 + \partial^2(V_1^2) . \qquad (10.12)$$

Here $V_{1\infty} \neq 0$. If one introduces the variable $q(x,t)$ by $V_1(x,t) = q(x,t) + V_{1\infty}$, then for $q(x,t)$ we obtain the equation of nonlinear string

$$\frac{\partial^2 q}{\partial t^2} - 2V_{1\infty}\frac{\partial^2 q}{\partial x^2} = \frac{1}{2}\frac{\partial^4 q}{\partial x^4} + \frac{\partial^2(q^2)}{\partial x^2} . \qquad (10.13)$$

The applicability of the inverse scattering transform method to equation (10.13) was demonstrated in [32]. The other interesting special case of equations (10.11) is $\omega_{11}=0,\ \omega_{10}=-\frac{1}{9}V_{1\infty}^2\omega_{21}$, $\omega_{20}=\frac{2}{3}\omega_{21}V_{0\infty}$, i.e. the system

$$\frac{\partial V_0}{\partial t}=\omega_{21}\{-\frac{1}{9}\partial^5 V_0-\frac{5}{9}\partial^3(V_1V_0)-\frac{5}{9}\partial(V_0\partial^2V_1)+\frac{5}{3}\partial(V_0\partial V_0)-\frac{5}{9}\partial(V_0V_1^2),$$

$$\frac{\partial V_1}{\partial t}=\omega_{21}\{-\frac{1}{9}\partial^5V_1-\frac{5}{9}\partial(V_1\partial^2V_1)-\frac{5}{3}\partial(V_0\partial V_1)-\frac{5}{9}V_1^2\partial V_1+\frac{5}{3}\partial(V_0^2)\}. \tag{10.14}$$

For $\omega_{21}=-9,\ V_{0\infty}=0$ and under the reduction $V_0=\frac{1}{2}\partial V_1$ the system (10.14) is reduced to the equation

$$\frac{\partial V_1}{\partial t}=\partial^5V_1+5V_1\partial^3V_1+\frac{25}{2}(\partial V_1)(\partial^2V_1)+5V_1^2\partial V_1 \tag{10.15}$$

which was consider earlier in [8,33]. For $\omega_{21}=-9$ and $V_0=0$ the second equation (10.14) is

$$\frac{\partial V_1}{\partial t}=\partial^5V_1+5V_1\partial^3V_1+5(\partial V_1)(\partial^2V_1)+5V_1^2\partial V_1 \ . \tag{10.16}$$

This equation has been considered in [34,35]. Let us note that, for equations (10.14)-(10.16), $V_{1\infty}\neq 0$.

At last, for $\varphi=0,\ L^+=L_S^+,\ \Omega_2=const$ and $\Omega_1=0$, equation (8.31) is the Burgers equation.

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ОБЩАЯ ДИФФЕРЕНЦИАЛЬНАЯ СПЕКТРАЛЬНАЯ ЗАДАЧА ПОРЯДКА N: ОБЩАЯ СТРУКТУРА ИНТЕГРИРУЕМЫХ УРАВНЕНИЙ, НЕОДНОЗНАЧНОСТЬ РЕКУРСИОННОГО ОПЕРАТОРА И КАЛИБРОВОЧНАЯ ИНВАРИАНТНОСТЬ

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