ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

34

V.G.Dubrovsky, B.G.Konopelchenko

BACKLUND-CALOGERO GROUP FOR THE GENERAL DIFFERENTIAL SPECTRAL PROBLEM OF AN ARBITRARY ORDER

ПРЕПРИНТ 83—57

НОВОСИБИРСК

BACKLUND-CALOGERO GROUP FOR THE GENERAL
DIFFERENTIAL SPECTRAL PROBLEM OF
AN ARBITRARY ORDER

V.G.Dubrovsky, B.G.Konopelchenko

Institute of Nuclear Physics,
630090, Novosibirsk 90, USSR

A b s t r a c t

Infinitedimensional abelian group of general Backlund-Calogero transformations for the evolution equations integrable by Gelfand-Dikij-Zakharov-Shabat spectral problem of an arbitrary order is constructed. Structure of the recursion operator and transformation properties of the Backlund-Calogero transformations under the gauge group are considered.

I. Introduction

The inverse scattering transform (IST) method is a powerfull tool for the investigation of nonlinear differential equations (see e.g. [1-4]). Numerous partial differential equations have been integrated by this method.

One of the main problems of the IST method is to describe effectively the class of nonlinear equations to which this method is applicable and analyse their group-theoretical structure. There exist different approaches to this problem. A very convenient and simple method of description of the nonlinear equations integrable by Zakharov-Shabat spectral problem was proposed in the paper [5]. The method suggested in [5] (AKNS method) has been generalized to a number of different spectral problems [6-15]. The advantage of AKNS-method in comparison with the other versions of IST method consists in that it allows to find the general form of nonlinear equations connected with given spectral problem in a compact and convenient form and to calculate the infinitedimensional group of general Backlund transformations for these equations. The so called recursion operator plays a central role in the AKNS-method.

In the present paper we consider the general Gelfand-Dikij-Zakharov-Shabat spectral problem, i.e. the general scalar N-th order spectral problem

$$\left(\partial^N + V_{N-1}(x,t)\partial^{N-1} + \ldots + V_1(x,t)\partial + V_0(x,t)\right)\psi = \lambda^N\psi, \quad (1.1)$$

where $\partial \stackrel{def}{=} \partial/\partial x$, λ is a spectral parameter and $V_0(x,t), \ldots, V_{N-1}(x,t)$ are scalar functions such that $V_k(x,t) \xrightarrow[|x|\to\infty]{} 0$, $(k=0,1,\ldots,N-1)$, in the framework of AKNS-method. In the frames of the IST method the spectral problem (1.1) has been considered by Zakharov and Shabat [16] for the first time. This problem and associated evolution equations were investigated by another technique by Gelfand and Dikij [17].

In the present paper we construct the infinite-dimensional group of general Backlund transformations of the potentials for the spectral problem (1.1) - so called Backlund-Calogero (BC) group. In order to define the action of this BC-group on the manifold of potentials $\{V(x,t); V \stackrel{def}{=} (V_0, V_1, \ldots, V_{N-1})^T\}$ one must calculate the recursion operator. The principal equation for the calculation of the recursion operator is of the form $\lambda^N \mathcal{G}\chi(\lambda) = \mathcal{F}\chi(\lambda)$, where $\mathcal{G}$ and $\mathcal{F}$ are certain N x N matrix differential operators and χ is a column with N components. The main feature of this equation is that the rank of matrix $\mathcal{G}$ is N-1 and therefore the equation $\lambda^N \mathcal{G}\chi = \mathcal{F}\chi$ contains a constraint $\sum_{k=1}^{N} \ell_k \chi_k = 0$.

The standard way to deal with the constraint $\sum_{k=1}^{N} \ell_k \chi_k = 0$ is to solve it, for example, with respect to χ_N and to introduce the quantity $\chi_{(N)} \stackrel{def}{=} (\chi_1, \ldots, \chi_{N-1}, 0)^T$ which contains only independent variables. As a result, one obtains $(N-1)\times(N-1)$ matrix recursion operator Λ_N which acts on the space of independent variables $\chi_{(N)}$: $\Lambda_N \chi_{(N)} = \lambda^N \chi_{(N)}$. The case $V_{N-1} = 0$ was considered in [14].

The second way of dealing with the constraint $\sum_{k=1}^{N} \ell_k \chi_k = 0$ is not to solve it at all and define an action of the recursion operator Λ on the whole N-dimensional space of all components $\chi_1, \ldots, \chi_N$: $\Lambda\chi = \lambda^N \chi$. One can introduce such a recursion operator but it is not defined uniquely. The uncertainty which appears in the calculation of such recursion operator can be effectively described. With the use of this recursion operator Λ the action $V \to V'$ of BC-group on the manifold of potentials is given by the relation

$$\sum_{k=0}^{N-1} B_k(\Lambda^+, t)(\mathcal{K}_k^+ V' - \mathcal{M}_k^+ V) - f(\Lambda^+, t)\ell^+ \phi = 0 , \tag{1.2}$$

where $B_k(\Lambda^+, t)$, $f(\Lambda^+, t)$ are arbitrary functions entire on Λ^+; $\mathcal{K}_k^+, \mathcal{M}_k^+, \ell^+$ - are certain operators and $\phi(x,t)$ is an arbitrary scalar function. Infinitedimensional abelian group of transformations (1.2) plays an important role in the analysis of the group theoretical properties of nonlinear evolution equations integrable by the problem (1.1).

In the paper the transformation properties of (1.2) under the gauge transformations which conserve (1.1) are considered. It is shown that the whole uncertainty which appears in the construction of transformations (1.2) is of the pure gauge nature. A manifestly gauge invariant form of transformations

(1.2) is also given.

The paper is organized as follows. In the second section a group of the gauge transformations which preserve (1.1) is considered. The gauge invariants are calculated. In the third section a direct scattering problem for (1.1) is discussed and some important relations are obtained. In section 4 the recursion operator is calculated. The BC-group is constructed in section 5. In section 6 the transformation properties of (1.2) under gauge transformations are considered and manifestly gauge invariant part of (1.2) is obtained. The general form of nonlinear equations integrable by the problem (1.1) is calculated in section 7. The examples of transformations (1.2) for the case N = 2 are given in section 8.

II. Gauge group

The spectral problem (1.1), as it is easy to see, is invariant under the transformations

$$\psi(x,t,\lambda) \longrightarrow \tilde{\psi}(x,t,\lambda) = g(x,t)\psi(x,t,\lambda) , \tag{2.1}$$

$$V_k(x,t) \longrightarrow \tilde{V}_k(x,t) = g(x,t)\sum_{n=0}^{N-k} C_{k+n}^{k} V_{k+n}(x,t)\partial^n(1/g(x,t)),$$

where $g(x,t)$ is any differentiable function such that $g(x,t) \xrightarrow[|x|\to\infty]{} 1$ and $C_n^k = \frac{n!}{(n-k)!k!}$. The transformations (2.1) form an infinite-dimensional abelian group of gauge transformations for the problem(1.1). This group is the subgroup of the general gauge transformations group which was discussed in [18, 19].

It is clear that there exist N-1 independent functions $W_0(V_0, \ldots, V_{N-1})$, $W_1(V_0, \ldots, V_{N-1})$, ..., $W_{N-2}(V_0, \ldots, V_{N-1})$ which are invariant under the gauge transformations (2.1), i.e. the functions such that $W_k(V_0', \ldots, V_{N-1}') = W_k(V_0, \ldots, V_{N-1})$ $(k = 0, 1, \ldots, N-2)$. An explicit form of the invariants W_0, $W_1, \ldots, W_{N-2}$ can be found directly from (2.1) by excluding the function $g(x,t)$. For our purpose the following set of the invariants is convenient [20,24]:

$$W_k - V_k - \frac{1}{N}\sum_{n=1}^{N-k} C^k_{k+n} V_{k+n}\left(\partial - \frac{1}{N}V_{N-1}\right)^{n-1} V_{N-1}, (k=0,\ldots,N-2) \quad (2.2)$$

The gauge invariance of the problem (1.1) allows us to impose additional constraints (gauge conditions) on the potentials $V_0, V_1, \ldots, V_{N-1}$. For example, one can transform any linear superposition $\sum_{k=0}^{N-1}\alpha_k V_k(x,t)$ into zero and, in particular, any (but only one) potential V_k into zero by an appropriate gauge transformation. We will shortly refer to the gauge condition as the gauge. The transition from one gauge to another one is performed by a certain gauge transformation.

For the further purposes it is convenient to represent the spectral problem (1.1) in the well-known matrix Frobenius form

$$\frac{\partial F}{\partial x} = (A + P(x,t))F, \quad (2.3)$$

where $F = (\psi, \partial\psi, \ldots, \partial^{N-1}\psi)^T$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ \lambda^N & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 0 \\ -V_0, & -V_1, & \ldots, & -V_{N-1} \end{pmatrix}. \quad (2.4)$$

The gauge transformations (2.1) have now the form

$$F \to \tilde{F} = GF, \; P \to \tilde{P} = G(A+P)G^{-1} - A + (\partial G)\cdot G^{-1}, \quad (2.5)$$

where $G_{ik} = C^{k-1}_{i-1}\partial^{i-k} g(x,t), i \geq k; \; G_{ik} = 0, \; i<k$. Introducing N-component column $V \overset{def}{=} (V_0, V_1, \ldots, V_{N-1})^T$, one can represent the gauge transformation (2.5) in the form

$$V \to \tilde{V} = \tau(g)V + \mathcal{V}(g) \quad (2.6)$$

where $\tau(g) = g(G^T)^{-1}$ and $\mathcal{V}_k(g) = C^k_N g\partial^{N-k}(1/g), (k=0,\ldots,N-1)$.

Using the explicit form of $\tau(g)$ and $\mathcal{V}(g)$, it is not difficult to show that

$$\tau(g_2)\tau(g_1) = \tau(g_2 g_1), \quad \mathcal{V}(g_2 g_1) = \tau(g_2)\mathcal{V}(g_1) + \mathcal{V}(g_2), \quad (2.7)$$

i.e. that the transformations (2.6) indeed form a group.

The form (2.6) of the gauge transformations (2.1) is useful for many purposes. For example, the invariants W_k can be written in the form

$$W = \tau(\tilde{\rho}^{-1})V + \mathcal{V}(\tilde{\rho}^{-1}), \quad (2.8)$$

where $W \overset{def}{=} (W_0, \ldots, W_{N-2}, 0)^T$ and $\tilde{\rho}(x,t) = exp(-\frac{1}{N}\int^x dx' V_{N-1}(x',t))$. Then the potentials V_k can be represented as the functions on invariants W_k and "gauge" variables $\rho(x,t)$

$$V(x,t) = \tau(\rho)W + \mathcal{V}(\rho). \quad (2.9)$$

III. Direct scattering problem and some important relations

We will study the problem (1.1) in the form (2.3). We assume that $V_k(x,t) \longrightarrow 0$ at $|x| \longrightarrow \infty$ so fast that all integrals which will appear in our calculations will exist.

We introduce, in a standard manner [1-3], the fundamental matrices-solutions $F^+(x,t,\lambda)$, $F^-(x,t,\lambda)$ of the problem (2.3) given by their asymptotic behaviour

$$F^+(x,t,\lambda) \xrightarrow[x\to\infty]{} E(x,\lambda), \; F^-(x,t,\lambda) \xrightarrow[x\to-\infty]{} E(x,\lambda), \quad (3.1)$$

where $E(x,\lambda) = D(\lambda)exp(\bar{A}x)$ - the fundamental matrix-solution of the equation $\partial E/\partial x = AE$, where $\bar{A}$ is diagonal matrix: $\bar{A}_{ik} = \lambda q^{i-1}\delta_{ik}$, $D_{ik} = \frac{1}{\sqrt{N}}(\lambda q^{k-1})^{i-1}$, $(i,k=1,\ldots,N)$ and $q = exp(2\pi i/N)$. Here and below δ_{ik} is Kronecer symbol ($\delta_{ik} = \{ {1, \; i=k \atop 0; \; i\neq k}$). Let us note that λq^{i-1}, $(i=1,\ldots,N)$ are eigenvalues of matrix A, by definition $\lambda > 0$ and $A = D\bar{A}D^{-1}$.

In a standard manner we introduce the scattering matrix $S(\lambda,t)$: $F^+(x,t,\lambda) = F^-(x,t,\lambda)S(\lambda,t)$.

Let $P(x,t)$ and $P'(x,t)$ be two different potentials and $F^+, F^{+\prime}, S, S'$ be corresponding solutions and scattering matrices for (2.3). One can show that

$$S'(\lambda,t)-S(\lambda,t)=-\int_{-\infty}^{+\infty}dx\,(F^{+}(x,t,\lambda))^{-1}(P'(x,t)-P(x,t))(F^{+}(x,t,\lambda))'. \quad (3.2)$$

Formula (3.2) which relates a variation of the potential to those of the scattering matrix plays a fundamental role in the AKNS method.

The mapping $P \to S(\lambda,t)$ given by the spectral problem (2.3) establish a correspondence between the transformations $P \xrightarrow{B} P'$ on the manifold of potentials $\{P(x,t),\ P(x,t)\xrightarrow[|x|\to\infty]{} 0\}$ and the transformations $S \xrightarrow{B} S'$ on the manifold of the scattering matrices $\{S(\lambda,t)\}$.

We will consider only such transformations B that

$$S(\lambda,t)\longrightarrow S'(\lambda,t)=\bar{B}^{-1}(\lambda,t)S(\lambda,t)\bar{C}(\lambda,t) \quad (3.3)$$

where $\bar{B}(\lambda,t)$ and $\bar{C}(\lambda,t)$ are arbitrary diagonal matrices (i.e. $\bar{B}_{ik}=B_i(\lambda,t)\delta_{ik}$, $\bar{C}_{ik}=C_i(\lambda,t)\delta_{ik}$, $i,k=1,\dots,N$). We confine ourselves by the transformations of the form (3.3) by two reasons: 1) the linearity of the transformation law (and, therefore, its readily integrability) of the scattering matrix is a main idea of the inverse scattering transform method (see e.g. [1-4]) and 2) the generalized AKNS-technique allows us to construct in an explicit form the transformations of the potential $P\to P'$ which correspond to the transformations of the scattering matrix of the form (3.3).

Let us rewrite the transformation law (3.3) in the form $S'-S=(1-\bar{B})S'-S(1-C)$. From the comparison of it with (3.2) we find

$$(S^{-1}(1-\bar{B})S)_F=-\int_{-\infty}^{+\infty}dx((F^{+})^{-1}(P'-P)(F^{+})')_F \quad (3.4)$$

where for arbitrary matrix φ we denote by φ_F the off-diagonal part of matrix φ : $(\varphi_F)_{ik}=\varphi_{ik}-\varphi_{ii}\delta_{ik}$, $(i,k=1,\dots,N)$. Further, it is not difficult to justified that the following identity holds

$$(S^{-1}(\lambda,t)(1-\bar{B}(\lambda,t))S'(\lambda,t))_F= \quad (3.5)$$

$$=\int_{-\infty}^{+\infty}dx\{(F^{+}(x,t,\lambda))^{-1}(P(x,t)(1-B(\lambda^N,t))-(1-B(\lambda^N,t))P'(x,t)(F^{+}(x,t,\lambda))'\}_F$$

where $B(\lambda^N,t)=D\bar{B}(\lambda,t)D^{-1}$. Equalizing the left-hand and right-hand sides of the equations (3.4) and (3.5) we obtain

$$\int_{-\infty}^{+\infty}dx\,tr(B(\lambda^N,t)P'(x,t)-P(x,t)B(\lambda^N,t))\,\overset{++}{\Phi}{}^{(F)}(x,t,\lambda))=0 \quad (3.6)$$

where tr denotes the matrix trace. The quantity $\overset{++}{\Phi}$ is the tensor product $(F^{+})'$ and F^{+}: $(\overset{++}{\Phi}{}^{(ik)})_{em}\overset{def}{=}(F^{+})'_{ek}(F^{+})^{-1}_{im}$, $(i,k,\ell,m=1,\dots,N)$.

Since all elements of matrix $\bar{A}$ are different, matrices B and $\bar{B}=D^{-1}BD$ can be represented (see [21]) in the forms:

$$B(\lambda^N,t)=\sum_{k=0}^{N-1}B_k(\lambda^N,t)A^k\ ,\quad \bar{B}(\lambda,t)=\sum_{k=0}^{N-1}B_k(\lambda^N,t)\bar{A}^k$$

where $B_k(\lambda^N,t)$ are scalar functions. Using these expressions one can rewrite (3.6) in the form

$$\left\langle\sum_{k=0}^{N-1}(A^k(\lambda^N)P'-PA^k(\lambda^N))B_k(\lambda^N,t)\overset{++}{\Phi}{}^{(F)}\right\rangle=0 \quad (3.7)$$

where $\langle\varphi\rangle\overset{def}{=}\int_{-\infty}^{+\infty}dx\,tr(\varphi(x))$.

The equality (3.7) is the fundamental relation between $P(x,t)$, $P'(x,t)$ and $F^{+}(x,t,\lambda)$, $F^{+\prime}(x,t,\lambda)$ under transformations (3.3) of the scattering matrix. This equality contains the quantities $A^k(\lambda^N)$, $B_k(\lambda^N,t)$, $(k=0,\dots,N-1)$ which explicitly depends on spectral parameter λ^N. Next step (which is standard for AKNS-technique) consists in the converting of the relation (3.7) into the form which does not contain explicit dependence on λ^N. In order to do this one must calculate so-called recursion operator.

IV. Recursion operator

So it is necessary to be able to exclude the explicit dependence on λ^N in the expressions of the form $(A^k(\lambda^N)P' - PA^k(\lambda^N))B_k(\lambda^N,t)\overset{++}{\Phi}^{(F)}$, $(k=0,1,\dots,N-1)$ in (3.7). It can be done with the use of recursion operator. Let us calculate it. Using equation (2.3) and equation $\partial F^{-1}/\partial x = -F^{-1}(A+P)$, one can show that $\overset{++}{\Phi}^{(in)}$ satisfy the equation

$$\frac{\partial \overset{++}{\Phi}^{(in)}(x,t,\lambda)}{\partial x} = [A, \overset{++}{\Phi}^{(in)}] + P'\overset{++}{\Phi}^{(in)} - \overset{++}{\Phi}^{(in)}P\,. \qquad (4.1)$$

In virtue of the special forms of the matrices A and $P(x,t)$, all matrix elements of $\overset{++}{\Phi}^{(in)}$ can be expressed through N matrix elements $(\overset{++}{\Phi}^{(in)})_{kN}$, $(k=1,\dots,N)$ [14, 20].

Let us introduce the operation Δ of projection onto the last column of the matrix: $(\Phi_\Delta)_{ik} = \Phi_{iN}\delta_{kN}, (i,k=1,\dots,N)$. With the use of (4.1) and explicit forms of A and P one gets [14, 20]:

$$\sum_{m=0}^{N} \mathcal{P}^m(\overset{++}{\Phi}_\Delta \circ V_m) = \lambda^N \overset{++}{\Phi}_\Delta \qquad (4.2)$$

where $\mathcal{P} = \partial - A + P'(x,t)$, $(\Phi \circ V_m)_{ik} \overset{def}{=} \Phi_{ik}V_m$. Then it is not difficult to show that the operator $\mathcal{P}^m$ is linear on λ^N

$$\mathcal{P}^m = \lambda^N r_m + S_m, \quad m = 0,1,\dots,N. \qquad (4.3)$$

Substituting (4.3) into (4.2), we obtain

$$\lambda^N \mathcal{Y}\, \overset{++}{\Phi}^{(in)}_\Delta(\lambda) = \mathcal{F}\, \overset{++}{\Phi}^{(in)}_\Delta(\lambda) \qquad (4.4)$$

where

$$\mathcal{Y} = \sum_{m=0}^{N} r_m V_m - 1, \quad \mathcal{F} = -\sum_{m=0}^{N} S_m V_m \qquad (4.5)$$

where $V_N \equiv 1$.

With the use of equations (2.4), (4.3) and (4.5), one can show that the matrix operator $\mathcal{Y}$ is a lowertriangular one:

$$\mathcal{Y} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{Y}_{21} & 0 & 0 & \dots & 0 & 0 \\ \mathcal{Y}_{31} & \mathcal{Y}_{32} & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathcal{Y}_{N1} & \mathcal{Y}_{N2} & \mathcal{Y}_{N3} & \dots & \mathcal{Y}_{NN-1} & 0 \end{pmatrix} \qquad (4.6)$$

where $\mathcal{Y}_{i\,i-1} = -N\partial + V_{N-1} - V'_{N-1}$, $(i=1,2,\dots,N)$; $\mathcal{Y}_{ik} = 0$, $k > i$ and all the rest matrix elements of $\mathcal{Y}$ are more complicated. The elements of the first line of the matrix operators S_k are:

$$(S_k)_{1\ell} = C_k^{\ell-1}(-\partial)^{k+1-\ell}, \quad \ell = 1,\dots,k+1;$$
$$(S_k)_{1\ell} = 0, \; \ell = k+2,\dots,N; \quad k = 1,\dots,N-1. \qquad (4.7)$$
$$(S_N)_{1\ell} = C_N^{\ell-1}(-\partial)^{N+1-\ell} - V'_{\ell-1}, \quad \ell = 1,\dots,N.$$

Therefore the matrix operator $\mathcal{Y}$ is a degenerate one. As a result, the first equation (4.4) is a relation between $\overset{++}{\Phi}_{kN}$, $(k=1,\dots,N)$ which does not contain λ^N, i.e. the constraint. The expression for this constraint can be obtain with the use of (4.7):

$$\sum_{k=1}^{N} \ell_k \overset{++}{\Phi}^{(in)}_{kN} = 0 \qquad (4.8)$$

where

$$\ell_k = -\sum_{n=1}^{N-k+1} C^{k-1}_{k+n-1}(-\partial)^n V_{k+n-1} - V_{k-1} + V'_{k-1}\,. \qquad (4.9)$$

The degeneracy of the matrix operator $\mathcal{Y}$ (its rank is N-1) and the existence of the constraint (4.8) are the fundamental properties of equation (4.4) which serves for the calculation of the recursion operator. Such a situation is a typical one for AKNS method [5-15].

There are two ways to deal with the constraint (4.8):
1) the first way is to solve equation (4.8) with respect to the one of the components $\overset{++}{\Phi}_{kN}$, $(k=1,\dots,N)$ and to calculate the

recursion operator which acts on the space of (N-1)-independent variables; 2) the second way is do not solve the constraint (4.8) and to define an action of the recursion operator on the whole N-dimensional space of all components $\overset{++}{\varphi}_{kN}$, $(k=1,\dots,N)$.

Usually only the first (standard) way of solving the constraint was used in the framework of AKNS-method for different spectral problems [5-15] and also for the problem (1.1) [15]. In the present paper we will follow to the second way of dealing with constraint. Let us calculate the recursion operator which acts on the N-dimensional space of all components $\overset{++}{\varphi}_{kN}, (k=1,\dots,N)$.

Let us denote

$$(E_\alpha)_{ik} \overset{def}{=} \delta_{ik} - \delta_{i\alpha}\delta_{k\alpha}\,, \quad \varphi_{\Delta_\alpha} \overset{def}{=} E_\alpha \varphi_\Delta\,, \tag{4.10}$$
$$\chi = (\chi_1,\dots,\chi_N)^T \overset{def}{=} (\overset{++}{\varphi}_{1N},\dots,\overset{++}{\varphi}_{NN})^T, \quad \chi_{(\alpha)} \overset{def}{=} E_\alpha \chi$$

and introduce the operator M with the following matrix elements

$$M_{ik} = \delta_{ik} - \delta_{iN} \ell_N^{-1} \ell_k\,, \; (i,k=1,\dots,N)\,. \tag{4.11}$$

In virtue of (4.6), equation (4.4) is equivalent to the equation

$$\lambda^N \mathcal{G} \chi = E_1 \mathcal{F} \chi \tag{4.12}$$

supplemented by the constraint, which can be represented as follows

$$\chi = M\chi = M\chi_{(N)}\,. \tag{4.13}$$

The equivalence of two forms (4.8) and (4.13) of constraint follows from the fact that the operator ℓ_N has in this case trivial kernel.

Then we introduce the operator $\tilde{\mathcal{G}}$ such that $\tilde{\mathcal{G}}\mathcal{G} = E_N$. From (4.12) and (4.13) we have

$$\lambda^N \chi = M\tilde{\mathcal{G}}\mathcal{F}\chi \overset{def}{=} \Lambda_s \chi\,. \tag{4.14}$$

The operator $\Lambda_s = M\tilde{\mathcal{G}}\mathcal{F}$ is just the recursion operator which acts in the whole N-dimensional space $(\chi_1,\dots,\chi_N)$. Equation (4.14) is compatible with (4.13). However, Λ_s is not the most

general recursion operator which can be defined on the whole N-dimensional space.

The general form of the recursion operator which acts on the whole N-dimensional space $\chi = (\chi_1,\dots,\chi_N)^T$, $\lambda^N \chi(\lambda) = \Lambda\chi(\lambda)$ is

$$\Lambda = \Lambda_s + Q \otimes \ell \tag{4.15}$$

where $\ell \overset{def}{=} (\ell_1,\dots,\ell_N)$, $Q \overset{def}{=} (Q_1,\dots,Q_N)^T$ where $Q_1,\dots, Q_N$ are arbitrary operators and $\otimes$ denotes a tensor product.

Indeed the difference $\Lambda - \Lambda_s = \Delta$ should satisfy the condition $\Delta\chi = 0$. Since χ has N-1 independent components, the rank of the matrix Δ is equal to 1. As a result, taking into account (4.8), we have $\Delta_{ik} = Q_i \ell_k$ where Q_i are arbitrary operators. (4.15) is proved.

Taking into account that $\ell \cdot M = 0$, we see that the operator Λ^n has the structure analogous to (4.15), i.e.

$$\Lambda^n = \Lambda_s^n + Q_{(n)} \otimes \ell \tag{4.16}$$

where $Q_{(n)}$ are certain operators.

So there exist a certain freedom in the construction of the recursion operator which acts on the whole N-dimensional space.

In our further calculations we will also need the operator Λ^+ adjoint to the operator Λ with respect to the bilinear form

$$\langle\!\langle \chi, \Phi \rangle\!\rangle = \sum_{i=1}^{N} \int dx\, \chi_i(x)\, \Phi_i(x)\,. \tag{4.17}$$

The operator Λ^+ is

$$\Lambda^+ = \Lambda_s^+ + \ell^+ \otimes Q^+ \qquad \text{where} \quad \Lambda_s^+ = \mathcal{F}^+ \tilde{\mathcal{G}}^+ M^+\,. \tag{4.18}$$

One also has

$$(\Lambda^+)^n = (\Lambda_s^+)^n + \ell^+ \otimes Q_{(n)}^+\,. \tag{4.19}$$

The operator M^+ adjoint to the operator M (4.11) has the following properties:

$$E_N M^+ = M^+,\quad M^+ E_N = E_N,\quad (M^+)^2 = M^+. \qquad (4.20)$$

The operators $\mathcal{Y}^+$, $\mathcal{F}^+$, r_m^+ and S_m^+ are calculated by formulas

$$\mathcal{Y}^+ = \sum_{n=0}^{N} V_n r_n^+ - 1,\ \mathcal{F}^+ = -\sum_{n=0}^{N} V_n s_n^+;\ (\mathcal{P}^+)^m = \lambda^N r_m^+ + S_m^+, \qquad (4.21)$$
$$(m=0,1,\dots,N).$$

With the use of (4.12) and (4.13) it is not difficult to obtain the operator $\Lambda_N = \tilde{\mathcal{Y}}\mathcal{F}M$ which acts on the subspace of (N-1)-independent variables $\chi_{(N)} \overset{def}{=} (\chi_1,\dots,\chi_{N-1},0)^T$: $\lambda^N\chi_{(N)} = \Lambda_N\chi_{(N)}$. The operator Λ_N^+ adjoint to the operator Λ_N has the form

$$\Lambda_N^+ = M^+\mathcal{F}^+\tilde{\mathcal{Y}}^+. \qquad (4.22)$$

Note that the recursion operator Λ_N is defined uniquelly.

V. Bäcklund-Calogero group

Here we obtain the nonlinear transformations $V \to V'$ which corresponds to the transformations (3.3) of the scattering matrix. For this we must exclude the explicit dependence on λ^N which is contained in (3.7).

Firstly we see that A^k is the linear function on λ^N:

$$A^k(\lambda^N) = \lambda^N(R^T)^{N-k} + R^k,\ k=0,1,\dots,N \qquad (5.1)$$

where $R_{ik} = \delta_{ik+1}$, $(i,k=1,\dots,N)$. For the quantities $\langle A^kP'\overset{++}{\Phi} - A^k\overset{++}{\Phi}P\rangle$, $(k=0,1,\dots,N-1)$ in (3.7) we have:

$$\langle A^k(\lambda^N)P'\overset{++}{\Phi} - A^k(\lambda^N)\overset{++}{\Phi}P\rangle = \langle A^k(\lambda^N)P'\overset{++}{\Phi}_{\Delta_{N-k}} - A^k(\lambda^N)\overset{++}{\Phi}_{\Delta_N}P\rangle. \qquad (5.2)$$

Then one can show from (4.1) that [14]:

$$\overset{++}{\Phi}_{\Delta_{N-k}} = \sum_{m=0}^{k} \mathcal{P}^{k-m}(\overset{++}{\Phi}_\Delta \circ V_{N-m})A^{-k} = (\lambda^N\mathcal{Y}_{(k)} + \mathcal{F}_{(k)})\overset{++}{\Phi}_\Delta A^{-k}, \qquad (5.3)$$
$$(k=0,1,\dots,N-1)$$

where

$$\mathcal{Y}_{(k)} \overset{def}{=} \sum_{m=0}^{k} r_{k-m}V_{N-m},\quad \mathcal{F}_{(k)} \overset{def}{=} \sum_{m=0}^{k} s_{k-m}V_{N-m}, \qquad (5.4)$$
$$(k=0,1,\dots,N-1).$$

Let us introduce the column with N-components $V(x,t) \overset{def}{=} (V_0(x,t),\dots,V_{N-1}(x,t))^T$ and proceed according to (4.10) from $\overset{++}{\Phi}_\Delta$ to $\chi \overset{def}{=} (\chi_1,\dots,\chi_N)^T$. By using the relations $\lambda^N\chi = \Lambda\chi$, $B_k(\lambda^N,t)\chi = B_k(\Lambda,t)\chi$, $(k=0,\dots,N-1)$ and also (4.17), (4.18), (5.1-5.3) we obtain for (3.7):

$$-\langle\sum_{k=0}^{N-1} B_k(\lambda^N,t)(A^k(\lambda^N)P' - PA^k(\lambda^N))\overset{++}{\Phi}^{(F)}\rangle = \qquad (5.5)$$
$$= \langle\langle\chi(\lambda)\{\sum_{k=0}^{N-1} B_k(\Lambda^+,t)(\mathcal{K}_k^+V' - \mathcal{M}_k^+V)\}\rangle\rangle = 0.$$

The freedom analogous to that of Λ^+ (4.18) and $(\Lambda^+)^n$ (4.19) appears in the calculation of the operators $\mathcal{K}_k^+$ and $\mathcal{M}_k^+$ too. These are of the form:

$$\mathcal{K}_k^+ \overset{def}{=} \Lambda^+\mathcal{Y}_{(k)}^+ + \mathcal{F}_{(k)}^+ + \ell^+\otimes\tilde{Q}_{(k)}^+,\quad \mathcal{M}_k^+ \overset{def}{=} \Lambda^+R^{N-k} + (R^T)^k + \ell^+\otimes\tilde{\tilde{Q}}_{(k)}^+, \qquad (5.6)$$
$$(k=0,\dots,N-1).$$

where $\tilde{Q}_{(k)} \overset{def}{=} (\tilde{Q}_{1(k)},\dots,\tilde{Q}_{N(k)})$, $\tilde{\tilde{Q}}_{(k)} \overset{def}{=} (\tilde{\tilde{Q}}_{1(k)},\dots,\tilde{\tilde{Q}}_{N(k)})$ where $\tilde{Q}_{1(k)},\dots,\tilde{Q}_{N(k)}$; $\tilde{\tilde{Q}}_{1(k)},\dots,\tilde{\tilde{Q}}_{N(k)}$ - are arbitrary operators and

$$\mathcal{Y}_{(k)}^+ = \sum_{m=0}^{k} V_{N-m}r_{k-m}^+,\quad \mathcal{F}_{(k)}^+ = -\sum_{m=0}^{k} V_{N-m}s_{k-m}^+. \qquad (5.7)$$

The variables $\chi_1,\dots,\chi_N$ in (5.5) are not independent and obey the constraint (4.8). As a result, from the equality (5.5) it follows

$$\sum_{k=0}^{N-1} B_k(\Lambda^+,t)(\mathcal{K}_k^+V' - \mathcal{M}_k^+V) - f(\Lambda^+,t)\ell^+\phi = 0 \qquad (5.8)$$

where $B_k(\Lambda^+,t)$, $(k=0,1,\dots,N-1)$ and $f(\Lambda^+,t)$ are arbitrary functions entire on Λ^+ and Λ^+, $\mathcal{K}_k^+$, $\mathcal{M}_k^+$ are any operators of the form (4.18), (5.5) and $\phi(x,t)$ is an arbitrary scalar function.

Indeed, it follows from (5.5) that $\sum_{k=0}^{N-1} B_k(\Lambda^+,t)\cdot(\mathcal{K}_k^+V'-\mathcal{M}_k^+V)=Z^+$ where Z^+ is any column for which its adjoint Z obeys the condition $Z\chi=0$. It is not difficult to see that the general form of Z is $Z_k = f(\lambda^N,t)\phi(x,t)\ell_k$, $(k=1,\dots,N)$ where $\phi(x,t)$ and $f(\lambda^N,t)$ are arbitrary scalar functions. Using $\lambda^N\chi=\Lambda\chi$, we have $Z_k\chi_k(\lambda)=\phi(x,t)\ell_k(f(\Lambda^+,t)\chi)_k$. Therefore, $Z_k^+=(f(\Lambda^+,t)\ell^+)_k\phi$ and hence (5.8) is proved.

The relation (5.8) is equivalent to the following

$$\sum_{k=0}^{N-1} B_k(\Lambda_S^+,t)(\mathcal{K}_{(S)k}^+V'-\mathcal{M}_{(S)k}^+V)-\ell^+\phi=0 \qquad (5.9)$$

where $\Lambda_S^+=\mathcal{F}^+\mathcal{Y}^+M^+$ and $\phi(x,t)$ is arbitrary scalar function and

$$\mathcal{K}^+_{(S)k}\stackrel{def}{=}\Lambda_S^+\mathcal{Y}^+_{(k)}+\mathcal{F}^+_{(k)},\quad \mathcal{M}^+_{(S)k}\stackrel{def}{=}\Lambda_S^+R^{N-k}+(R^T)^k, \qquad (5.10)$$
$$(k=0,1,\dots,N-1).$$

Indeed substituting the expression (4.18) for Λ^+ into (5.8), using (4.19) and the fact that $M^+\ell^+=0$, we obtain (5.9).

So (5.9) gives the general form of nonlinear transformations $V\to V'$ which corresponds to the transformations (3.3) of the scattering matrix. Now the operators Λ_S^+, $\mathcal{K}^+_{(S)k}$, $\mathcal{M}^+_{(S)k}$ in (5.9) are defined uniquelly and all uncertainties connecting with the existence of the constraint (4.8) are contained in the term $\ell^+\phi$ only.

Multiplying the left-hand side of (5.9) by M^+ and using the relations $M^+\ell^+=0$, $M^+(\Lambda_S^+)^n=(\Lambda_N^+)^nM^+$, $M^+\mathcal{K}^+_{(S)k}=\mathcal{K}^+_{k(N)}$, $M^+\mathcal{M}^+_{(S)k}=\mathcal{M}^+_{k(N)}$ we obtain

$$\sum_{k=0}^{N-1} B_k(\Lambda_N^+,t)(\mathcal{K}^+_{k(N)}V'-\mathcal{M}^+_{k(N)}V)=0 \qquad (5.11)$$

where $\mathcal{K}^+_{k(N)}\stackrel{def}{=}\Lambda_N^+\mathcal{Y}^+_{(k)}+M^+\mathcal{F}^+_{(k)}$, $\mathcal{M}^+_{k(N)}\stackrel{def}{=}\Lambda_N^+R^{N-k}+M^+(R^T)^k$.

Emphasize that the whole uncertainty disappear after proceeding from (5.9) to (5.11). The relation (5.11) is just the same relation between V and V' which can be obtained from (3.7) by excluding the explicit dependence on λ^N with the use of recursion operator Λ_N (4.22) which acts in the space of

(N-1)-independent variables $\chi_{(N)}\stackrel{def}{=}(\chi_1,\dots,\chi_{N-1},\ 0)^T$.

The system of equations (5.11) due to the special forms of operators Λ_N^+, $\mathcal{K}^+_{k(N)}$ and $\mathcal{M}^+_{k(N)}$ contains N-1 nontrivial equations. The system (5.9), in contrast, contains N nontrivial equations.

It is easy to see that the transformations (3.3), (4.31) or (3.3), (5.11) form an abelian infinitedimensional group. We will refer this group as Backlund-Calogero group (BC-group). BC-group acts on the manifold of the scattering matrices $\{S(\lambda,t)\}$ by the formula (3.3) and on the manifold of the potentials $\{V(x,t)\}$ by the formulas (5.9) or (5.11).

Backlund was the first who considered concrete transformation of the type (5.11) (see e.g. [22]). Calogero constructed the general transformations of the form (5.11) (for the case N = 2 in the gauge $V_1=0$) for the first time [23,4].

VI. Gauge invariance and manifestly gauge-invariant formulation

Let us consider the transformation properties of nonlinear transformations $V\to V'$ (5.9) and (5.11) under the gauge transformations (2.1). Let the quantities F, V and F', V' are transformed independently with the different gauge functions $g_1(x,t)$ and $g_2(x,t)$:

$$F\xrightarrow{g_1}\tilde{F}=G_1F,\quad F'\xrightarrow{g_2}\tilde{F}'=G_2F',$$
$$V\xrightarrow{g_1}\tilde{V}=\tau(g_1)V+\mathcal{V}(g_1),\quad V'\xrightarrow{g_2}\tilde{V}'=\tau(g_2)V+\mathcal{V}(g_2) \qquad (6.1)$$

where $G_1=G(g_1)$, $G_2=G(g_2)$ and $G(g)$, $\tau(g)$, $\mathcal{V}(g)$ are defined in section II.

Let us obtain the transformation laws of the quantities which have appeared in the previous section. From the definition $\overset{++}{\Phi}(x,t,\lambda)$ and (6.1) it follows that

$$\overset{++}{\Phi}{}^{(in)}(x,t,\lambda)\xrightarrow{(g_1,g_2)}\tilde{\overset{++}{\Phi}}{}^{(in)}(x,t,\lambda)=G_2(x,t)\,\overset{++}{\Phi}{}^{(in)}(x,t,\lambda)\,G_1^{-1}(x,t) \qquad (6.2)$$

For the columns $\chi\stackrel{def}{=}(\overset{++}{\Phi}_{1N},\dots,\overset{++}{\Phi}_{NN})^T$ and

$\chi_{(N)} \overset{def}{=} (\overset{++}{\tilde{\Phi}}_{2N},\dots,\overset{++}{\tilde{\Phi}}_{N-1N},\,0)^T$ the law (6.2) gives

$$\chi \xrightarrow{(g_1,g_2)} \tilde{\chi} = \frac{1}{g_1} G(g_2)\chi \overset{def}{=} \pi(g_1,g_2)\chi;\ \chi_{(N)} \xrightarrow{(g_1,g_2)} \tilde{\chi}_{(N)} = E_N \pi(g_1,g_2)\chi_{(N)}. \quad (6.3)$$

Using the explicit form of the operators ℓ_k (4.9), transformation properties V, V' and $\pi(g_1,g_2) \overset{def}{=} \frac{1}{g_1}G(g_2)$ we get

$$\sum_{n=1}^{N} \tilde{\ell}_n \pi_{nm}(g_1,g_2) = \frac{g_2}{g_1}\ell_m,\ (m=1,\dots,N) \quad (6.4)$$

where $\tilde{\ell}_k = \ell_k(\tilde{V},\tilde{V}')$, $(k=1,\dots,N)$. In particular, for $m=N$ from (6.4) we have $\tilde{\ell}_N \frac{g_2}{g_1} = \frac{g_2}{g_1}\ell_N$.

It follows from (6.3) and (6.4) that the constraint (4.8) is the gauge invariant one:

$$\sum_{k=1}^{N} \tilde{\ell}_k \tilde{\chi}_k = \frac{g_2}{g_1}\sum_{k=1}^{N} \ell_k \chi_k. \quad (6.5)$$

Then the relations $\tilde{\chi} = \tilde{M}\tilde{\chi} = \tilde{M}\pi\chi_{(N)} = \pi\chi = \pi M\chi_{(N)}$ give the transformation law of the operator M (see (4.11) and (4.13)):

$$M \xrightarrow{(g_1,g_2)} \tilde{M} = \pi(g_1,g_2) M \pi^{-1}(g_1,g_2) \quad (6.6)$$

where $\tilde{M} \overset{def}{=} M(\tilde{V},\tilde{V}')$.

The transformation law of the recursion operator Λ_s^+ under the gauge transformations is the following

$$\Lambda_s^+ \xrightarrow{(g_1,g_2)} \tilde{\Lambda}_s^+ = (\pi^+(g_1,g_2))^{-1}\Lambda_s^+\pi^+(g_1,g_2) + \tilde{\ell}^+ \otimes Q^+ \quad (6.7)$$

where $\tilde{\Lambda}_s^+ \overset{def}{=} \Lambda_s^+(\tilde{V},\tilde{V}')$, $Q^+ = (Q_1^+,\dots,Q_N^+)$; $Q_1^+,\dots,Q_N^+$ are certain operators uniquelly defined by the gauge transformation.

Recursion operator Λ_N^+ on the contrary to Λ_s^+ has homogeneous transformation law

$$\tilde{\Lambda}_N^+ = (\pi^+(g_1,g_2))^{-1}\Lambda_N^+\pi^+(g_1,g_2) \quad (6.8)$$

where $\tilde{\Lambda}_N^+ \overset{def}{=} \Lambda_N^+(\tilde{V},\tilde{V}')$. The relation (6.8) can be proved by using the transformation law (6.3) and the fact that

Λ_N^+ acts on the subspace of N-1 independent variables $\chi_{(N)} = (\overset{++}{\tilde{\Phi}}_{1N},\dots,\overset{++}{\tilde{\Phi}}_{N-1N},\,0)^T$.

From (6.8) and relation $M^+\ell^+ = 0$ it follows that

$$(\tilde{\Lambda}_s^+)^n = (\pi^+(g_1,g_2))^{-1}(\Lambda_s^+)^n\pi^+(g_1,g_2) + \tilde{\ell}^+ \otimes Q_{(n)}^+ \quad (6.9)$$

where $Q_{(n)}^+ = (Q_{1(n)}^+,\dots,Q_{N(n)}^+)$ and $Q_{1(n)}^+,\dots,Q_{N(n)}^+$ are certain operators which are uniquelly defined by the gauge transformation.

Using the gauge invariance of the relation $-\langle (A^kP' - PA^k)\overset{++}{\tilde{\Phi}}\rangle = \langle\langle \chi(\mathcal{K}_{(s)k}^+V' - \mathcal{M}_{(s)k}^+V)\rangle\rangle$ it is easy to obtain the transformation laws for the quantities $\mathcal{K}_{(s)k}^+V' - \mathcal{M}_{(s)k}^+V$ and $\mathcal{K}_{k(N)}^+V' - \mathcal{M}_{k(N)}^+V$:

$$\mathcal{K}_{(s)k}^+V' - \mathcal{M}_{(s)k}^+V = \pi^+(g_1,g_2)(\tilde{\mathcal{K}}_{(s)k}^+\tilde{V}' - \tilde{\mathcal{M}}_{(s)k}^+\tilde{V}) + \ell^+\varphi_k, \quad (6.10)$$

$$\mathcal{K}_{k(N)}^+V' - \mathcal{M}_{k(N)}^+V = \pi^+(g_1,g_2)(\tilde{\mathcal{K}}_{k(N)}^+\tilde{V}' - \tilde{\mathcal{M}}_{k(N)}^+\tilde{V}),\ (k=0,\dots,N-1)$$

where $\tilde{\mathcal{K}}_{(s)k}^+ \overset{def}{=} \mathcal{K}_{(s)k}^+(\tilde{V},\tilde{V}')$, $\tilde{\mathcal{M}}_{(s)k}^+ \overset{def}{=} \mathcal{M}_{(s)k}^+(\tilde{V},\tilde{V}')$ and so on; $\varphi_k(x,t)$, $(k=0,\dots,N-1)$ are certain scalar functions uniquelly defined by the gauge transformation (6.1).

From (6.9) and (6.10) we obtain the transformation laws of the nonlinear transformations $V \to V'$ (5.9) and (5.11) under the gauge transformations (6.1):

$$\sum_{k=0}^{N-1} B_k(\Lambda_s^+,t)(\mathcal{K}_{(s)k}^+V' - \mathcal{M}_{(s)k}^+V) - \ell^+\phi =$$
$$= \pi^+(g_1,g_2)\Big(\sum_{k=0}^{N-1} B_k(\tilde{\Lambda}_s^+,t)(\tilde{\mathcal{K}}_{(s)k}^+\tilde{V}' - \tilde{\mathcal{M}}_{(s)k}^+\tilde{V}) - \tilde{\ell}^+\tilde{\phi}\Big), \quad (6.11)$$

$$\sum_{k=0}^{N-1} B_k(\Lambda_N^+,t)(\mathcal{K}_{k(N)}^+V' - \mathcal{M}_{k(N)}^+V) =$$
$$= \pi^+(g_1,g_2)\Big(\sum_{k=0}^{N-1} B_k(\tilde{\Lambda}_N^+,t)(\tilde{\mathcal{K}}_{k(N)}^+\tilde{V}' - \tilde{\mathcal{M}}_{k(N)}^+\tilde{V}\Big). \quad (6.12)$$

where the function $\tilde{\phi}$ can be expressed through ϕ, Q^+, B_k and φ_k (k = 0, 1, ..., N-1). In the case N = 2, B_0 = const and B_1 = const, this expression is given in section 8.

From the form of the relation connecting $\tilde{\phi}$ with ϕ, Q^+, B_k

B_k and φ_k $(k=0,\dots,N-1)$ it follows that for a given ϕ it is always possible to find such a gauge functions $g_1(x,t)$, $g_2(x,t)$ to obtain any function $\tilde{\phi}$ given in advance. In particular, one can always convert any ϕ into $\tilde{\phi}=0$. Therefore the transformations $V \to V'$ of the form (5.9) with the same functions B_k, $(k=0,1,\dots,N-1)$ and different functions $\tilde{\phi}$ are gauge-equivalent to each other. Thus, the whole freedom which appears in transformations $V \to V'$ (5.9) is of the pure gauge nature.

Gauge-invariant formulation of the nonlinear transformations $V \to V'$, which correspond to the transformation law (3.5) of the scattering matrix, one can obtain from (5.11) by using the special gauge transformation from the potentials V, V' to the invariants W, W'. Or equivalently, one can to pick out from (5.9) its gauge-invariant part, this can be done by multiplying (5.9) by M^+.

Indeed, multiplying (5.9) by M^+, we get (5.11). Then we make the special gauge transformations (6.1) from the potentials V, V' to the invariants W, W':

$$W=\tau(\beta_1^{-1})V+\mathcal{V}(\beta_1^{-1}),\quad W'=\tau(\beta_2^{-1})V+\mathcal{V}(\beta_2^{-1}) \tag{6.13}$$

where $g_1(x,t)=\beta_1^{-1}=\exp(\frac{1}{N}\int^x dx' V_{N-1}(x',t))$; $g_2(x,t)=\beta_2^{-1}=\exp(\frac{1}{N}\int_{-\infty}^{x} dx' V'_{N-1}(x',t))$. Using (6.13) we have

$$\sum_{k=0}^{N-1} B_k(\Lambda_W^+,t)(\mathcal{K}^+_{k(W)}W'-\mathcal{M}^+_{k(W)}W)= \tag{6.14}$$
$$=(\pi^+(g_1,g_2))^{-1}\Big(\sum_{k=0}^{N-1} B_k(\Lambda_N^+,t)(\mathcal{K}^+_{k(N)}V'-\mathcal{M}^+_{k(N)}V)$$

where

$$\Lambda_W^+ \overset{def}{=} \Lambda_N^+(W,W'),\ \mathcal{K}^+_{k(W)} \overset{def}{=} \mathcal{K}_{k(N)}(W,W'),\ \mathcal{M}^+_{k(W)} \overset{def}{=} \mathcal{M}^+_{k(N)}(W,W'). \tag{6.15}$$

So the nonlinear transformations of the BC-group (5.11) can be represented in the manifestly gauge-invariant form:

$$\sum_{k=0}^{N-1} B_k(\Lambda_W^+,t)(\mathcal{K}_{k(W)}W'-\mathcal{M}^+_{k(W)}W)=0. \tag{6.16}$$

Now let us pay attention to the fact that the general Backlund-Calogero group, which was constructed in the previous section, contains as the subgroup the group of gauge transformations.

Indeed, let us consider, for example, the transformation (5.9) with $B_0=1$, $B_k=0$, $(k=1,\dots,N-1)$. It has the form $V'=V+e^t\phi$ or, in components,

$$V'_k=(1-\phi(x,t))^{-1}\sum_{l=0}^{N-k} C^k_{k+l}V_{k+l}\,\partial^l(1-\phi(x,t)).$$

This is the gauge transformation (2.1) with a gauge function $g(x,t)=(1-\phi(x,t))^{-1}$.

Emphasize that the potentials V, V' in general transformations (5.9) and (5.11) of Backlund-calogero group are transformed under the gauge transformations independently with the different gauge functions $g_1(x,t)$ and $g_2(x,t)$.

VII. General form of nonlinear equations

BC-group constructed in the section V contains the transformations of various types. Let us consider its one-parameter subgroup given by the matrices

$$\bar{B}(\lambda,t)=\bar{C}(\lambda,t)=\sum_{k=0}^{N-1}\exp\Big(-\int_t^{t'} ds\,\Omega_k(\lambda^N,s)\Big)\cdot\bar{A}^k. \tag{7.1}$$

It is easy to show that the transformation (3.5) with the matrices $\bar{B}$ and $\bar{C}$ of the form (7.1) is a displacement in time t: $S'(\lambda,t)=S(\lambda,t')$. The corresponding transformations (5.9) and (5.11) give, in the explicit form, the time evolution of the potential V: $V(x,t)\to V(x,t')$. Different evolution laws correspond to different functions $\Omega_k(\lambda^N,t)$. An identity transformation is given by the functions $B_0=1$, $B_1=\dots=B_{N-1}=0$.

Here we obtain from the transformations (5.9) the corresponding nonlinear equations. Let us consider the infinitesimal displacement in time: $t \to t' = t + \varepsilon$ where $\varepsilon \to 0$. In this case

$$V(x,t') = V(x,t+\varepsilon) = V(x,t) + \varepsilon \frac{\partial V(x,t)}{\partial t},$$
$$B_k(\lambda^N, t) = \delta_{k0} - \varepsilon \Omega_k(\lambda^N, t), \quad k = 0, 1, \ldots, N-1; \tag{7.2}$$
$$\phi(x,t) = \varepsilon \varphi(x,t)$$

where $\varphi(x,t)$ is arbitrary scalar function. Substituting (7.2) into (5.9) we obtain

$$\frac{\partial V(x,t)}{\partial t} - \sum_{k=0}^{N-1} \Omega_k(L_s^+, t) \mathcal{Z}^+_{(s)k} V - \ell^+ \varphi = 0 \tag{7.3}$$

where $L_s^+ \stackrel{def}{=} \Lambda_s^+|_{V=V'}$, $\mathcal{Z}^+_{(s)k} = (\mathcal{K}^+_{(s)k} - \mathcal{M}^+_{(s)k})|_{V=V'}$ and the operators Λ_s^+, $\mathcal{K}^+_{(s)k}$, $\mathcal{M}^+_{(s)k}$ are given by the formulas (4.18), (5.10).

The system of N equations (7.3) is just the general form of the evolution equations integrable by the problem (1.1) via the inverse scattering transform method. The transformations (5.9) are the general Backlund-Calogero transformations for the equations (7.3). In the case $\partial B_k / \partial t = 0, (k = 0, \ldots, N-1)$ the transformations (5.9) are the general auto-Backlund transformations for the equations (7.3). Infinite-dimensional group of auto-Backlund transformations also contains as a subgroup an abelian infinite-dimensional symmetry group of the equations (7.3).

In more details, the properties of evolution equations (7.3) have been considered in [20].

VIII. The examples: N = 2

The general formulas (4.9), (4.11) give

$$\ell_1^+ = -\partial^2 - V_1 \partial + V_0' - V_0, \quad \ell_2^+ = -2\partial + V_1' - V_1,$$

$$M^+ = \begin{pmatrix} 1, & -\ell_1^+(\ell_2^+)^{-1} \\ 0, & 0 \end{pmatrix}, \quad (\ell_2^+)^{-1} = \frac{1}{2} \exp\left(\frac{1}{2}\int_x^{+\infty}(V_1 - V_1')\right) \int_{-\infty}^{x} dy \exp\left(-\frac{1}{2}\int_y^{+\infty}(V_1 - V_1')\right). \tag{8.1}$$

From the formulas (4.18) and (4.21) we have for the operator Λ_s^+:

$$\Lambda_s^+ = \begin{pmatrix} V_0' - (\partial V_0')(\ell_2^+)^{-1}, & ((\partial V_0') - V_0' \ell_2^+)(\ell_2^+)^{-1} \ell_1^+ (\ell_2^+)^{-1} \\ V_1' - (\partial V_1')(\ell_2^+)^{-1} - \ell_1^+(\ell_2^+)^{-1}, & (\ell_1^+ + (\partial V_1') - V_1' \ell_2^+)(\ell_2^+)^{-1} \ell_1^+ (\ell_2^+)^{-1} \end{pmatrix}. \tag{8.2}$$

For $\mathcal{K}^+_{(s)k} V' - \mathcal{M}^+_{(s)k} V$, $(k = 0, 1)$ from (5.7), (5.10) and (8.2) one can obtain the following expressions

$$\mathcal{K}^+_{(s)0} V' - \mathcal{M}^+_{(s)0} V = V' - V,$$

$$\mathcal{K}^+_{(s)1} V' - \mathcal{M}^+_{(s)1} V = \begin{pmatrix} (\partial V_0') \exp\left(-\frac{1}{2}\int_{-\infty}^{x}(V_1 - V_1')\right) \\ \ell_1^+ \exp\left(-\frac{1}{2}\int_{-\infty}^{x}(V_1 - V_1')\right) + (\partial V_1') \exp\left(-\frac{1}{2}\int_{-\infty}^{x}(V_1 - V_1')\right) \end{pmatrix}. \tag{8.3}$$

Let us write out the basic quantities and relations for the gauge transformations. The formulas (2.1) and (6.3) give

$$\pi^+(g_1, g_2) = \begin{pmatrix} \frac{g_2}{g_1}, & \frac{\partial g_2}{g_1} \\ 0, & \frac{g_2}{g_1} \end{pmatrix}, \qquad \begin{aligned} \widetilde{V}_0 &= V_0 + V_1 g \partial\left(\tfrac{1}{g}\right) + g \partial^2\left(\tfrac{1}{g}\right), \\ \widetilde{V}_1 &= V_1 + 2g\partial\left(\tfrac{1}{g}\right). \end{aligned} \tag{8.4}$$

In the case N = 2 there exists only one invariant W_0 which has the form

$$W_0 = V_0 - \frac{1}{2}\partial V_1 - \frac{1}{4} V_1^2. \tag{8.5}$$

It was shown in the paper [24] that the Miura and the

Gardner transformations [25] are the gauge transformations. Indeed, the Miura transformation

$$\tilde V_0 = -\frac{1}{2}\partial V_1 - \frac{1}{4}V_1^2 \qquad (8.6)$$

as it follows from (8.5) is the gauge transformation from the gauge $(V_0=0,\ V_1)$ to the gauge $(\tilde V_0,\ \tilde V_1=0)$. Let us consider the general linear gauge $\alpha_0 V_0 + \alpha_1 V_1 = 0$ where α_0 and α_1 are constants. One can introduce the function $u(x,t)$ such that $V_0=\beta_0 u$, $V_1=\beta_1 u$ where β_0 and β_1 are some constants $(\alpha_0\beta_0 + \alpha_1\beta_1 = 0)$. From (8.5) we obtain the Gardner transformation

$$\tilde u = \frac{\beta_0}{\beta_0'}u - \frac{1}{2}\frac{\beta_1}{\beta_0'}\partial u - \frac{1}{4}\frac{\beta_1^2}{\beta_0'}u^2 \qquad (8.7)$$

as the gauge transformation from one general linear gauge ($V_0=\beta_0 u$, $V_1=\beta_1 u$) to another one ($\tilde V_0 = \beta_0'\tilde u$, $\tilde V_1 = \beta_1'\tilde u = 0$) with $\beta_1' = 0$.

With the use of (8.3) and (8.4) by the direct calculations one can for the relations (6.12) the following expressions:

$$\mathcal{K}^+_{(s)k}V' - \mathcal{M}^+_{(s)k}V = \pi^+(g_1,g_2)(\tilde{\mathcal{K}}^+_{(s)k}\tilde V' - \tilde{\mathcal{M}}^+_{(s)k}\tilde V) + \ell^+\varphi_k,\ (k=0,1) \qquad (8.8)$$

where $\varphi_0 = 1 - g_2/g_1$, $\varphi_1 = g_2(\partial(1/g_2))\exp(-\frac{1}{2}\int_{-\infty}^{x}(V_1 - V_1'))$.

From (6.4), (8.4) and (8.6) one can show that (6.7) has the form

$$\tilde\Lambda_s^+ = (\pi^+(g_1,g_2))^{-1}\Lambda_s^+\pi^+(g_1,g_2) + \begin{pmatrix}\tilde\ell_1^+\\ \tilde\ell_2^+\end{pmatrix}\otimes(Q_1^+,\ Q_2^+) \qquad (8.9)$$

where $Q_1^+ = -g_2(\partial(1/g_2))(\tilde\ell_2^+)^{-1}$, $Q_2^+ = g_2(\partial(1/g_2))(\tilde\ell_2^+)^{-1}\tilde\ell_1^+(\tilde\ell_2^+)^{-1}$.

In the case $B_0(\Lambda_s^+,t) = B_0 = const$, $B_1(\Lambda_s^+,t) = B_1 = const$ the relation (6.13) is

$$\sum_{k=0,1}B_k(\mathcal{K}^+_{(s)k}V' - \mathcal{M}^+_{(s)k}V) - \ell^+\phi = \pi^+(g_1,g_2)\Big(\sum_{k=0,1}B_k(\tilde{\mathcal{K}}^+_{(s)k}\tilde V' - \tilde{\mathcal{M}}^+_{(s)k}\tilde V) - \tilde\ell^+\tilde\phi\Big) \qquad (8.10)$$

where $\tilde\phi = (\phi - B_0\varphi_0 - B_1\varphi_1)(g_1/g_2)$ and φ_0, φ_1 are given in (8.8).

The nonlinear transformations $V \to V'$ of the form (5.9) with the constant B_0 and B_1 are given by the relations

$$B_0(V_0'-V_0) + B_1(\partial V_0')\exp\Big(-\frac{1}{2}\int_{-\infty}^{x}(V_1-V_1')\Big) + \partial^2\phi + V_1\partial\phi - (V_0'-V_0)\phi = 0, \qquad (8.11a)$$

$$B_0(V_1'-V_1) + B_1\{V_0'-V_0 + \frac{1}{2}(\partial(V_1'+V_1)) + \frac{1}{4}(V_1^2 - V_1'^2)\}\times \qquad (8.11b)$$

$$\times\exp\Big(-\frac{1}{2}\int_{-\infty}^{x}(V_1-V_1')\Big) + 2\partial\phi - (V_1'-V_1)\phi = 0 .$$

Let us also present some nonlinear equations integrable by the problem (1.1). From the invariant part of the system of equations (7.3) we have

$$\frac{\partial W_0}{\partial t} + \partial^3 W_0 + 6W_0\partial W_0 = 0,\quad W_0 = V_0 - \frac{1}{2}\partial V_1 - \frac{1}{4}V_1^2 . \qquad (8.12)$$

In the gauges $(V_0,\ V_1=0)$, $(V_0=0,\ V_1)$ and $(V_0=\beta_0 u,\ V_1=\beta_1 u)$ from (8.12) one can obtain the KdV, mKdV and Gardner equations respectively:

$$\frac{\partial V_0}{\partial t} + \partial^3 V_0 + 6V_0\partial V_0 = 0, \qquad (8.13)$$

$$\frac{\partial V_1}{\partial t} + \partial^3 V_1 - \frac{3}{2}V_1^2\partial V_1 = 0, \qquad (8.14)$$

$$\frac{\partial u}{\partial t} + \partial^3 u + 6\beta_0 u\partial u - \frac{3}{2}\beta_1^2 u^2\partial u = 0. \qquad (8.15)$$

Let us return to the transformations (8.11). Putting $V_1 = V_1' = 0$ and excluding then the function ϕ one can obtain the well known Backlund transformation (BT) [22] for the KdV equation

$$2B_0(V_0'-V_0)+B_1\partial(V_0'+V_0)+B_1(V_0'-V_0)\int_{-\infty}^{x}(V_0'-V_0)=0$$

or

$$\partial(V_0'+V_0)+(V_0'-V_0)+\sqrt{\left(\frac{2B_0}{B_1}\right)^2-2(V_0'-V_0)}=0 . \qquad (8.16)$$

By performing an analogous calculations in the gauge $V_0=V_0'=0$ one can obtain from (8.11) the relation

$$\left\{\partial+\tfrac{1}{2}(V_1+V_1')\right\}\left\{\frac{B_1}{2}\partial(V_1'+V_1)+B_0(V_1'-V_1)-\right.$$
$$\left.-\frac{B_1}{8}(V_1'-V_1)\int_{-\infty}^{x}(V_1'^2-V_1^2)\right\}=0 .$$

From this relation we have the well known BT for the mKdV equation

$$2B_0(V_1'-V_1)+B_1\partial(V_1'+V_1)-\frac{B_1}{4}(V_1'-V_1)\int_{-\infty}^{x}(V_1'^2-V_1^2)=0 .$$

Integral term in the last equation can be easily excluded and the BT for the mKdV takes the form

$$\partial(V_1'+V_1)+\frac{1}{2}(V_1'-V_1)\sqrt{\left(\frac{4B_0}{B_1}\right)^2+(V_1'+V_1)^2}=0 . \qquad (8.17)$$

The function ϕ can be excluded from (8.11) before the fixing a gauge. Excluding ϕ after some calculations one can obtain from (8.11)

$$2B_0(W_0'-W_0)+B_1\partial(W_0'+W_0)+B_1(W_0'-W_0)\int_{-\infty}^{x}(W_0'-W_0)=0 \qquad (8.18)$$

e.g. the invariant part (6.15) of the transformations (5.9) of the BC-group.

Let us fix the gauge in (8.18) by the following way: $(V_0=\beta_0 u,\ V_1=\beta_1 u)$ and $(V_0'=\beta_0 u',\ V_1'=\beta_1 u')$. Then from (8.18) we have

$$B_0(W_0'-W_0)+\frac{B_1}{2}\partial(W_0'+W_0)+\frac{B_1}{2}(W_0'-W_0)\int_{-\infty}^{x}(W_0'-W_0)=$$
$$=\left[\beta_0-\tfrac{1}{2}\beta_1\partial-\tfrac{1}{4}\beta_1^2(u'+u)\right]\left\{B_0(u'-u)+\frac{B_1}{2}\partial(u'+u)+\right. \qquad (8.19)$$
$$\left.+\frac{B_1}{2}(u'-u)\int_{-\infty}^{x}\left[\beta_0(u'-u)-\frac{\beta_1^2}{4}(u'^2-u^2)\right]\right\}=0 .$$

From (8.19) we obtain the Backlund transformation for the Gardner equation (8.15)

$$2B_0(u'-u)+B_1\partial(u'+u)+$$
$$+B_1(u'-u)\int_{-\infty}^{x}\left[\beta_0(u'-u)-\frac{\beta_1^2}{4}(u'^2-u^2)\right]=0 .$$

The integral term in this equation can be excluded and, as a result, we get

$$\partial(u'+u)+(u'-u)\sqrt{\left(\frac{2B_0}{B_1}\right)^2-4\beta_0(u'+u)+\frac{\beta_1^2}{4}(u'+u)^2}=0 . \qquad (8.20)$$

In the mixed gauge $(V_0, V_1=0)$, $(V_0'=0, V_1')$ from (8.18) we obtain the following relation after excluding the integral term:

$$\partial\left(-\tfrac{1}{2}\partial V_1'-\tfrac{1}{4}V_1'^2+V_0\right)+\left(-\tfrac{1}{2}(\partial V_1')-\tfrac{1}{4}V_1'^2-V_0\right)\cdot \qquad (8.21)$$
$$\cdot\sqrt{\left(\frac{2B_0}{B_1}\right)^2-2\left(-\tfrac{1}{2}\partial V_1'-\tfrac{1}{4}V_1'^2+V_0\right)}=0 .$$

The transformation (8.21), as it is easy to see, is the product of the BT from V_0 to V_0' (BT (8.16) for the KdV) and Miura transformation $V_0=-\frac{1}{2}\partial V_1'-\frac{1}{4}V_1'^2$.

Analogously one can prove that the transformation (8.18) in the gauges $(V_0=0, V_1)$ and $(V_0', V_1'=0)$ is the product of the two transformations: BT (8.17) for the mKdV from V_1 to V_1' and Miura transformation $V_0'=-\frac{1}{2}\partial V_1-\frac{1}{4}V_1^2$ from V_1 to V_0.

References

1. Zakharov V.E., Manakov S.V., Novikov S.P., Pitaevski L.P., Theory of solitons. The method of inverse problem, Moscow, Nauka, 1980 (in Russian).
2. "Solitons", Topics in Current Physics, v. 17, Eds. Bullough R., Caudrey P., Springer-Verlag. Heidelberg, 1980.
3. Ablowitz M.J., Segur H., Solitons and the Inverse Scattering Transform, SIAM Philadelphia 1981.
4. Calogero F., Degasperis A., Spectral Transform and Solitons: Tools to solve and investigate nonlinear evolution equations, V. 1, North-Holland Publishing Company, Amsterdam, 1982.
5. Ablowitz M.J., Kaup D.J., Newell A.C., Segur H., Stud. Appl. Math., 53, 249-315 (1974).
6. Newell A.C., Proc. Roy. Soc. (London), A365, 283-311 (1979).
7. Kulish P.P., Notes of LOMI scientific seminars, 96, 105-112 (1980) (in Russian).
8. Gerdjikov V.S., Ivanov M.I., Kulish P.P., Teor. Mat. Fyz., 44, 342 (1980).
9. Konopelchenko B.G., Phys. Lett. 75A, 447-450 (1980); 79A, 39-43 (1980); 95B, 83-89 (1980); 100B, 254-260 (1981); 108B, 26-32 (1982).
10. Konopelchenko B.G., J. Phys. A: Math. Gen., 14, 1237-1259 (1981); 3125-3145 (1981).
11. Gerdjikov V.S., Kulish P.P., Physica 3D, 549-564 (1981).
12. Konopelchenko B.G., Mokhnachev V.G., J. Phys. A: Math. Gen., 14, 1849-1861 (1981); Konopelchenko B.G., Formusatic I.B., J. Phys. A: Math. Gen., 15, 2017-2040 (1982).
13. Konopelchenko B.G., Lett. Math. Phys., 6, 309-314 (1982); Funct. Analysis and its Appl., 16, No. 3, 63-65 (1982) (in Russian).
14. Dubrovsky V.G., Konopelchenko B.G., Fortschritte der Physik, 32, No. 4 (1984); Preprint Institute of Nuclear Physics, 82-009, Novosibirsk (1982).

15. Konopelchenko B.G., Fortschritte der Physik, 31, No. 4, (1983).
16. Zakharov V.E., Shabat A.B., Funct. Analysis and its Appl. No. 3, 43 (1974).
17. Gelfand I.M., Dikij L.A., Funct. Analysis and its Appl., 10, No. 1, 18 (1976); 10, No. 4, 13 (1976); 11, No. 2, 11-27 (1977); 12, No. 2, 8-23 (1978).
18. Zakharov V.E., Shabat A.B., Funct. Analysis and its Appl. 13, No. 3, 13 (1979).
19. Zakharov V.E., Mikhailov A.V., Z.E.T.F., 74, 1953 (1978) (in Russian).
20. Konopelchenko B.G., Dubrovsky V.G., Preprint Institute of Nuclear Physics, 83-026, Novosibirsk (1983).
21. Gantmacher F.R. The theory of matrices, Moskva, Nauka, 1967 (in Russian).
22. Bäcklund Transformations, the Inverse Scattering Method, Solitons and their Applications, Ed. R.Miura, Lecture Notes in Physics, 1976, v. 515.
23. Calogero F., Lett. Nuovo Cimento, 14, 537-543 (1975).
24. Konopelchenko B.G., Phys. Lett., 92A, 323-327 (1982).
25. Miura R.M., J. Math. Phys., 9, 1202 (1967).

В.Г.Дубровский, Б.Г.Конопельченко

ГРУППА ПРЕОБРАЗОВАНИЙ БЭКЛУНД-КАЛОДЖЕРО ДЛЯ ОБЩЕЙ ДИФФЕРЕНЦИАЛЬНОЙ СПЕКТРАЛЬНОЙ ЗАДАЧИ ПРОИЗВОЛЬНОГО ПОРЯДКА

Препринт
№ 83-57

Работа поступила - 8 апреля 1983 г.

Ответственный за выпуск - С.Г.Попов
Подписано к печати 16.04-1983 г. МН 17520
Формат бумаги 60х90 1/16 Усл.1,9 печ.л., 1,6 учетно-изд.л.
Тираж 290 экз. Бесплатно. Заказ № 57.

Ротапринт ИЯФ СО АН СССР, г.Новосибирск, 90