20

ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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NUMERICAL CALCULATIONS OF RELATIVISTIC PROPAGATORS VIA FEYNMAN PATHS IN PROPER TIME FORMALISM

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НОВОСИБИРСК

ABSTRACT

Economical numerical method for evaluation of relativistic quark propagators in arbitrary external gauge field is developed. It is based on Fock—Schwinger proper time formalism and Feynman path integral representation. The method is tested in the case of abelian homogeneous external field by comparison with Schwinger analytical solution.

1. INTRODUCTION

Evaluation of hadron properties (masses, decay constants, etc.) by the use of corresponding vacuum current-current correlators is now widely used (see, e.g., the QCD sum rules [1, 2] and numerical estimates on the lattice [3]). An important ingredient of such studies is the calculation of quark/gluon propagation in some rather nontrivial vacuum fields.

The main problem here is, of course, the long-distance structure of vacuum fields, including that responsible for the confinement mechanism. There are known many attempts to attack it, basing on phenomenological treatment [4—6] and direct lattice simulation [7, 8]. In particular, numerical study of the lattice QCD demonstrates explicitly the existence of the confinement phenomenon [7] and confirms the possibility to reproduce main features of hadronic spectrum [3], starting directly from the first principles of the theory. However, these attempts are only first steps towards the real understanding.

This paper is devoted to another, more technical problem of calculation of the quark propagation in arbitrary external (vacuum) field. Existing analytical estimates (see, e.g. [1, 2] and references therein) are reliable only at small enough distances, where both the perturbation theory and operator expansion technique are applicable due to the asymptotic freedom. Generally speaking, the knowledge of correlators at small distances is not sufficient, and in fact one needs to calculate them up to distances comparable with hadronic scale (order of 1 fermi). Since this problem has no explicit small parameter, the only known hopefull way to treat it remains its numerical simulation.

Here a new numerical method for the calculation of quark propagators in arbitrary external gauge field is developed. As a result, knowing the propagator one can calculate in the same external field many interesting related quantities, like current-current correlators, effective action [10, 11] (fermion determinant), etc. In some respects our method is a continuous analog of the lattice hoping parameter expansion [12], but instead of 4-dim. lattice in the space-time it uses only one discretized variable, being Fock—Schwiger proper time [13, 10]. In this methodical paper we concentrate on the derivation of the method, and study its main properties using exactly solvable problem of the relativistic particle propagation in homogeneous abelian external field, while its applications to QCD will be published elsewhere.

In sect.2 Fock—Schwinger proper time parametrization for quark propagator [10, 13] is used, which makes the relativistic problem similar to a well studied in the literature [14] nonrelativistic one. Corresponding Feynman path integral representation completes this section, while some useful detailes of numerical calculation of the path integral are collected in sect.3. Comparison of numerical results with the exact analytical solution known [10] for the case of homogeneous abelian field is given in sect.4.

2. FEYNMAN PATH REPRESENTATIONS FOR QUARK PROPAGATOR AND EFFECTIVE ACTION

Let us start with some notations. Below we deal with Euclidean quantities related with Minkowsky ones in a usual way:

$$x_m = x_m^{(M)}, \qquad \gamma_m = -i\gamma_m^{(M)} \qquad (m=1,\ 2,\ 3)$$

$$x_4 = i x_0^{(M)}, \qquad \gamma_4 = \gamma_0^{(M)} \tag{2.1}$$

and the commutation relation between momentum and coordinate operators:

$$[\hat{p}_\mu,\ \hat{x}_\nu] = i\delta_{\mu\nu} \tag{2.2}$$

are assumed. Following to Schwinger [10] we introduce the basis of eigenstates for coordinate operator x:

$$\hat{x}_\mu | x\rangle = x_\mu | x\rangle \tag{2.3}$$

and come to Euclidean version of the proper time parametrization of the spinor particle propagator:

$$G(x,y,A) = \langle x| \frac{1}{\not p + \not A - im} |y\rangle = \langle x|(\not p + \not A + im)\frac{1}{(\not p + \not A)^2 + m^2}|y\rangle =$$

$$= \left[\left(i\frac{\partial}{\partial x_\mu} + A_\mu\right)\gamma_\mu + im\right]\int_0^\infty ds\ \langle x|\exp\{-[(\not p + \not A)^2 + m^2]s\}|y\rangle \tag{2.4}$$

Thus the whole problem is reduced to calculation of the matrix

$$U(x,y,s;A) = \langle x|\exp\{-[(\not p + \not A)^2 + m^2]s\}|y\rangle \tag{2.5}$$

which can be treated as a transition amplitude from point y to point x during the interval of (imaginary) «time» s, describing the propagation of some fictive particle in four-dimensional space. The corresponding Hamiltonian is

$$H = (\not p + \not A)^2 + m^2 = (p_\mu + A_\mu^a t^a)^2 + \frac{1}{2}\sigma_{\mu\nu} G_{\mu\nu}^a t^a \tag{2.6}$$

where

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c \tag{2.7}$$

so one can see the analogy with the nonrelativistic problem of propagation of a particle having a mass equal to 1/2 and moving in 4-dim. space in the external time-independent (nonabelian) field $A(x)$. Using this analogy the amplitude $U(x,\ y,\ s,\ A)$ can be easily expressed through the known [14] Feynman path integral, which can be calculated at least numerically using a well elaborated Monte Carlo methods.

In order to obtain a Feynman path representation for the matrix $U(y,\ x,\ s,\ A)$ let us consider eq. (2.5) for small time slice $s \to 0$. Expanding the exponent up to terms of second order in A, and sandwiching the operators after their rearrangement by the full set of momentum eigenstates $\sum_p |p\rangle\langle p| = 1$ we have:

$$U(x,y,s;A) = \exp(-m^2 s)\cdot\{1 -$$

$$-\sum_p [-s\cdot\langle x|\partial_\mu A_\mu + A_\mu^2|p\rangle\langle p|\exp(-p^2 s)|y\rangle -$$

$$-2s\langle x|A_\mu|p\rangle\langle p|p_\mu \exp(-p^2 s)|y\rangle +$$

$$+2s^2\langle x|A_\mu A_\nu + i\partial_\mu A_\nu|p\rangle\langle p|p_\mu p_\nu \exp(-p^2 s)|y\rangle]\} \tag{2.8}$$

Making some calculations and exponentiating the result, for small s we obtain:

$$U(x,y,s;A) =$$

$$= \frac{1}{(4\pi s)^2}\exp\left\{-sm^2 - \frac{(x-y)^2}{4s} + iA_\mu^a t^a (x-y)_\mu - \frac{s}{2}\sigma_{\mu\nu} G_{\mu\nu}^a t^a\right\} \tag{2.9}$$

where the fields are taken at the central point:

$$A_\mu,\ G_{\mu\nu} = A_\mu,\ G_{\mu\nu}|_{\bar{x} = {}^1/_2(x+y)} \tag{2.10}$$

Now consider the case of large number N of subsequent small steps Δs in time such that $\Delta s \cdot N = s_0$. In the limit $N \to \infty$ but fixed time interval s_0 we come to a required Feynman path integral:

$$U(x,y,s;A) = C(s_0)\int_{z(0)=y}^{z(s_0)=x} Dz[s]\exp\{-\int_0^{s_0} ds\ L(z,\dot z)\} \tag{2.11}$$

where matrix «lagrangian» is as follows

$$L(z,\dot z)=\frac{\dot z^2}{4}-iA_\mu\dot z_\mu+\frac{1}{2}\sigma_{\mu\nu}G_{\mu\nu}+m^2 \tag{2.12}$$

and $C(s_0)$ is some (infinite) normalization factor. For any finite N:

$$C_N(s_0)=\left(\frac{N}{4\pi s_0}\right)^{2N} \tag{2.13}$$

Substituting (2.11) into (2.4) we have for spinor propagator:

$$G(x,y;A)=\left[\left(i\frac{\partial}{\partial x_\mu}+A_\mu(x)\right)\gamma_\mu+im\right]\times$$

$$\times\int_0^\infty ds_0\,C(s_0)\int_{z(0)=y}^{z(s_0)=x}Dz[s]\exp\{-\int_0^{s_0}ds\,L(\dot z,z)\} \tag{2.14}$$

In the case of scalar particle the propagator takes a more simple form:

$$D(x,y;A)=\int_0^\infty ds_0\,C(s_0)\int_{z(0)=y}^{z(s_0)=x}Dz[s]\exp\{-\int_0^{s_0}ds\,L_{sc}(z,\dot z)\} \tag{2.15}$$

where

$$L_{sc}(z,\dot z)=\frac{\dot z^2}{4}-iA_\mu\dot z_\mu+m^2 \tag{2.16}$$

coincides exactly with a lagrange function for some fictive scalar particle moving in 5-dimensional space-time.

Now let us write down the path integral representation for the effective action coming from the interaction of the polarized quark vacuum with the external field [10, 11]:

$$\delta S_{eff}=-\langle j_\mu\rangle\cdot\delta A_\mu \tag{2.17}$$

which is also related to the functional determinant of the operators

$$K^{(sc)}=(i\partial+A)^2+m^2$$

$$K^{(sp)}=(i\not\partial+\not A)-im \tag{2.18}$$

where $S^{(sc)}_{eff}={}^1/_2\ln\det K^{(sc)}$ and $S^{(sp)}_{eff}=-\ln\det K^{(sp)}$ for the scalar and

spinor case respectively. Following to Schwinger [10] we see, that the effective action is given by corresponding integration of the trace of the same evolution matrix $U(x,x,s,A)$:

$$S_{eff}(m,A)=\frac{1}{2}\int d^4x\int\frac{ds}{s}\,\mathrm{tr}\,U(x,x,s;A) \tag{2.19}$$

and with eq. (2.11) we come to a needed path representation. In order to subtract infinities we calculate below a regularized quantity:

$$S^R_{eff}=[S_{eff}(m,A)-S_{eff}(m,0)]-[S_{eff}(M,A)-S_{eff}(M,0)] \tag{2.20}$$

where M is the Pauli—Villars mass regulator.

3. NUMERICAL EVALUATION OF PATH INTEGRALS

Let us outline now our technique on a particular example of a propagator evaluation because the evaluation of other quantities (current-current correlators, effective actions, etc.) is quite analogous.

For evaluation of eqs (2.14—15) we need to sum over all the paths connecting the points x, y and corresponding to different proper time intervals s_0. In order to do this we simulate an ensemble of paths $\{x[s]|_{s_0}\}$ with weights corresponding to free $(A=0)$ propagation of scalar particle:

$$W(x[s]|_{s_0})=\frac{C(s_0)\exp\{-\int_0^{s_0}ds\,L_0(z,\dot z)\}}{D(x,y;A=0)} \tag{3.1}$$

where the normalization factor $C(s_0)$ coinsides with that in eq. (2.11) and

$$L_0(z,\dot z)=\frac{1}{4}\dot z^2+m^2 \tag{3.2}$$

With this path ensemble the propagator (2.14) can be calculated as an average

$$G(x,y;A)=\frac{1}{N_{path}}\sum_{\{paths\}}\left(\left[i\frac{\partial}{\partial x_\mu}+A_\mu(x)\right]\gamma_\mu+im\right)\times$$

$$\times\{\exp[-\int_0^{s_0} ds\,(L-L_0)]\cdot D(x,y;A=0)\} \qquad (3.3)$$

However such a choice of the path ensemble seems to be a not optimal one to saturate the path integral (2.11), in practice this loss is well compensated by a very high speed of simulation of paths with a particular weight (3.1).

Let us describe the simulation procedure in more detailes. First of all, the time interval s_0 is split into N steps $\Delta s = s_0/N$ and at each step eq. (2.10) is applied. Each path is now characterized by a set of numbers $(z_1, ..., z_{N+1})$ with $z_1=y$, and $z_{N+1}=x$, and by the proper time value s_0. The corresponding weight (3.1) takes the form:

$$W(z_1,\cdots,z_{N+1};s_0)\propto\left(\frac{N}{4\pi s_0}\right)^{2N}\exp\left\{-\frac{N}{4s_0}\sum_{i=1}^{N}(z_{i+1}-z_i)^2-m^2s_0\right\} \qquad (3.4)$$

Now a transformation to new variables

$$\xi_i=[z_i-y-\frac{(i-1)}{N}(x-y)]\cdot s^{-1/2} \qquad (3.5)$$

with distribution

$$W(\xi_1,\cdots,\xi_{N+1};s_0)=\frac{\partial(z_2,\cdots,z_N;s_0)}{\partial(\xi_2,\cdots,\xi_N;s_0)}W(z_1,\cdots,z_{N+1};s_0)\propto$$

$$\propto(s_0)^{N-1}\cdot\left(\frac{N}{4\pi s_0}\right)^{2N}\exp\left\{-\frac{(x-y)}{4s_0}-m^2s_0\right\}\times$$

$$\times\exp\left\{-\frac{N}{4}\sum_{i=1}^{N}(\xi_{i+1}-\xi_i)^2\right\} \qquad (3.6)$$

allows to factorize the distributions for ξ_i and s_0. Further Fourier transformation

$$\xi_i=\frac{a_0}{\sqrt{N}}+\sqrt{\frac{2}{N}}\sum_{k=1}a_k\cos[q_k(i-1)]+b_k\sin[q_k(i-1)]$$

$$q_k=\frac{2\pi k}{N} \qquad (3.7)$$

factorizes the distributions over all the random variables completely:

$$W(a_1,\cdots;b_1,\cdots;s_0)=W_s(s_0)\cdot\prod_{k=1}\prod_{\mu=1}^{4}W_k(a_{\mu,k})\cdot W_k(b_{\mu,k}) \qquad (3.8)$$

where

$$W_s(s_0)\propto\frac{1}{s_0}\exp\left\{-\frac{(x-y)^2}{4s_0}-m^2s_0\right\} \qquad (3.9)$$

and

$$W_k(a)\propto\exp\left\{-N\sin^2\left(\frac{\pi k}{N}\right)\cdot a^2\right\} \qquad (3.10)$$

so the whole random path can be simulated directly. Let us remind, that more general but indirect iteration methods, like a popular Metropolis algorithm [15], require sometimes a rather large number of «empty» iterations in order to obtain a new independent path. Obviously, our direct way of path simulation takes essentially smaller computer time.

Completing this section let us concern some specifics in the practical calculation of spinor propagator (2.4), (2.14), which can be written as

$$G(x,y;A)=\int_0^{\infty}ds_0\,PU(x,y,s_0;A) \qquad (3.11)$$

where the projection operator P having a form

$$P\equiv(\not p+\not A+im)\equiv(i\gamma_\mu\partial_\mu+A^a_\mu\gamma_\mu t^a+im) \qquad (3.12)$$

contains a differentiation over the end point x of the path. For our N-point path

$$U(x,y,s_0;A)=\sum_{\{paths\}}U(x,z_2;\Delta s)\,U(z_2,z_3;\Delta s)\cdots U(z_N,y;\Delta s) \qquad (3.13)$$

Acting by P on U gives

$$PU=\sum_{\{paths\}}\left[-\frac{i(x-z_2)_\mu\gamma_\mu}{2\Delta s}+im+O(N^{-1/2})\right]U \qquad (3.14)$$

Note that for each given path the first term in the brackets describes the endpoint velocity and diverges at large-N limit as $N^{1/2}$ since

$\Delta s = s_0/N$ and $|x - z_2| \sim N^{-1/2} \cdot s_0^{1/2}$ for a random walk in z_2. Obviously, a simple averaging over paths cancels this divergency but requires a rather large ensemble of paths growing with N at least proportionally to N. However, the requirement on the statistics can be essentially reduced, if one averages the velocity along the path. In fact

$$PU = UP \tag{3.15}$$

in continium limit, and one should take

$$G(x, y, A) = \sum_{\{paths\}} \left[\frac{1}{N} \sum_{i=2}^{N} U_1 \cdots U_{i-1} P U_i \cdots U_N + \right.$$

$$\left. + \frac{1}{2N} (PU_1 \cdots U_N + U_1 \cdots U_N P) \right] \tag{3.16}$$

to solve this problem completely.

4. NUMERICAL STUDY OF A CASE OF HOMOGENEOUS ABELIAN FIELD

In order to understand how well the method works, where and why it fails, let us test it in the well known [10, 16] case of homogeneous abelian field. Except for the plane wave case, it is the only known one which allows an exact analytical solution [10]. Below, starting with a brief but useful discussion of its properties, we concern some problem of our numerical method and complete the section by the comparison of our numerical results with the exact analytical ones.

4.1. SOME PROPERTIES OF THE EXACT ANALYTICAL SOLUTION

The general expression for the evolution matrix defined by eqs (2.5), (2.11) is known to be [10]:

$$U = \frac{1}{(4\pi s)^2} \exp\left\{ -sm^2 - \frac{1}{4}(x-y)_\mu [F \operatorname{cth}(Fs)]_{\mu\nu} (x-y)_\nu - \frac{s\sigma_{\mu\nu}F_{\mu\nu}}{2} \right\} \times$$

$$\times \exp\left\{ -\frac{1}{2} \operatorname{tr} \ln [(Fs)^{-1} \operatorname{sh}(Fs)] \right\} \exp\left\{ -i \int_y^x dz_\mu A_\mu(z) \right\} \tag{4.1}$$

where $F_{\mu\nu}$ is the field stress tensor and integral over z is taken along the straight line connecting the points x and y.

For simplicity we consider below only the case of constant Euclidean electric field $A = (0, 0, 0, E \cdot x_1)$ and calculate the propagator for $(x - y) = (0, 0, 0, t)$. Substituting (4.1) into (2.14) we come to the following expression for a spinor particle case:

$$G(t, E) = \int_0^\infty ds\, d(t, s, E) \cdot W(t, s, E) \tag{4.2}$$

where

$$W(t, s, E) = \frac{1}{(4\pi s)^2} \exp\left\{ -m^2 s - \frac{t^2}{4s} \frac{Es \cdot \operatorname{ch}(Es)}{\operatorname{sh}(Es)} \right\} \frac{Es}{\operatorname{sh}(Es)} \tag{4.3}$$

$$d(t, s, E) = i \left[m \operatorname{ch}(Es) - \gamma_4 \frac{tE}{2} \frac{1}{\operatorname{sh}(Es)} + i\gamma_1\gamma_4 m \operatorname{sh}(Es) \right] \tag{4.4}$$

In the case of scalar particle one has $d = 1$.

Let us start the discussion with some well known limits of this solution. In case of weak enough field the main contribution in eq. (4.2) comes from $Es \ll 1$. In this limit eqs (4.3), (4.4) take a more simple form:

$$W(t, s, E) \simeq \exp\left\{ -m^2 s - \frac{t^2}{4s} \left(1 + \frac{E^2 s^2}{3} \right) \right\} \tag{4.5}$$

$$d(t, s, E) \simeq im \cdot \left\{ 1 - \frac{\gamma_4 t}{ms} \right\} \tag{4.6}$$

and the integral in eq. (4.3) can be taken analytically and expressed via Bessel functions. We consider however only its further limits:

A. The nonrelativistic limit $(mt \gg 1)$. Taking the integral by a saddle point method and keeping the leading contribution only, one comes in the scalar particle case to the well known result [14]:

$$D(t, E) = \frac{m^{1/2}}{2(2\pi t)^{3/2}} \exp\left(-mt - \frac{E^2 t^3}{24m} \right) \tag{4.7}$$

while in the spinor particle case one has an extra factor $im(1 - \gamma_4)$:

$$G(t, E) = \frac{im^{3/2}(1 - \gamma_4)}{2(2\pi t)^{3/2}} \exp\left(-mt - \frac{E^2 t^3}{24m} \right) \tag{4.7a}$$

The condition $Es \ll 1$ implies here that $Et/2m \ll 1$. Note, that the main contribution comes from $s \sim t/2m$, so the fifth time s (the «proper time») is simply proportional to the usual time variable: the dominating paths are very close to the classical one.

B. The ultrarelativistic limit ($mt \ll 1$). The dominant contribution comes from paths with $s \sim t^2$, which are in fact nothing but some random walk from point $x=0$ to $x=t$ inside a region of size t. For the scalar particle propagation we have:

$$D(t, E) = \frac{2}{\pi^{3/2}(te)^2} \exp\left\{-\frac{t^2}{8}\left(m^2 + \frac{E^2 t^2}{12}\right)\right\} \quad (4.8)$$

while the spinor propagator has a slightly different behaviour:

$$G(t, E) = -2i\left(\frac{3}{2\pi}\right)^{3/2} \frac{\gamma_4}{(te)^3} \exp\left\{-\frac{t^2}{12}\left(m^2 + \frac{E^2 t^2}{12}\right)\right\} \quad (4.9)$$

The above condition $Es \ll 1$ implies here that $Et^2 \ll 1$.

Now we consider the case of strong field limit ($Es \gtrsim 1$). Let us stress, that although the integration over s in eq. (4.2) is not so trivial in this limit, the integrand is a well behaved function, and the result can be easily computed at least numerically. Nevertheless, in this limit some rather unexpected problem arises in the spinor case, when we calculate the original path integral (2.11) by direct numerical summation over the paths. Let us discuss it in more detail. Indeed, one can see from eqs (2.11—12) that for large s the individual contribution of each path can be estimated as

$$\lim_{s \to \infty} \frac{1}{(4\pi s)^2} \exp\left\{-m^2 s - \frac{s\sigma_{\mu\nu}F_{\mu\nu}}{2} - \int_0^s d\tilde{s}\left(\frac{\dot{x}^2}{4} - iA_\mu \dot{x}_\mu\right)\right\} \sim$$

$$\sim \frac{1}{s^2} \exp(-m^2 s + Es) \,, \quad (4.10)$$

so for $E > m^2$ it grows exponentially with s. Although the phase of this contribution changes rapidly from path to path and the sum over the paths at any fixed large s must be in fact finite and small, the statistics needed to reach it is enormous. Moreover, the larger s is, the larger statistics is needed, while the net contribution of corresponding paths into the final answer becomes quite unessential. Let us stress, that this problem is inherent to the spinor case, while in the scalar case all the individual path contributions are always order of unity.

In order to overcome this problem we suppress «too lengthy» paths (corresponding to too large s) introducing by hand a cut-off s_c, so the summation goes over the paths with $s < s_c$. Now consider the conditions to ensure the dependence on the cut-off value to be small.

The integrand in eq. (4.2) has in the spinor case two pieces with

rather different behaviour in the large-s limit:

$$d \cdot W \underset{s \to \infty}{\sim} -\frac{i\gamma_4 t E^2}{(4\pi)^2 s} \exp\left\{-\frac{t^2 E}{4} - (m^2 + 2E)s\right\} +$$

$$+ i(1 + i\gamma_1\gamma_4)\frac{mE}{(4\pi)^2 s} \exp\left(-\frac{t^2 E}{4} - m^2 s\right) \quad (4.11)$$

one of them falls exponentially with a slope $(m^2 + 2E)$, while the slope of another piece is only m^2. The main contribution in integration of the former piece comes from

$$Es \sim \text{arc sh}\,(Et/2(m^2 + 2E)^{1/2}) .$$

The requirement, that corresponding individual path contributions eq.(4.10) must be order of unity in small mass limit implies that $Et^2/8 \lesssim 1$.

The latter piece is potentially dangerous, since its integration involves $s \sim 1/m^2$, which in the small mass case correspond to large individual path contributions because of $Es \gg 1$. Introducing a cut-off in s, we suppress as well the contribution of this piece. Obviously, the uncertainty introduced by cut-off is maximal for $m^2 \sim E$ and is again inessential in the massless limit for those propagator components and for all the related quantities, which remain nonzero in this limit.

Analytical answer for the effective action can be obtained by substituting eq. (4.1) into eq. (2.19). The regularized expression (see eq.(2.20)) has a simple form:

$$S^R_{eff} = \frac{1}{2(4\pi)^2} \int_0^\infty \frac{ds}{s^3}\left(e^{-m^2 s} - e^{-M^2 s}\right)\left(\frac{Es}{\text{sh}\,(Es)} - 1\right) \quad \text{(scalar)} \quad (4.12)$$

$$S^R_{eff} = \frac{2}{(4\pi)^2} \int_0^\infty \frac{ds}{s^3}\left(e^{-m^2 s} - e^{-M^2 s}\right)\left(\frac{\text{ch}\,(Es)Es}{\text{sh}\,(Es)} - 1\right) \quad \text{(spinor)} \quad (4.13)$$

Subtracting the charge renormalization [10] one comes to the Heisenberg—Euler lagrangian:

$$L_{H.-E.} = \frac{1}{2(4\pi)^2} \int \frac{ds}{s^3}\left(e^{-m^2 s} - e^{-M^2 s}\right)\left(\frac{Es}{\text{sh}\,(Es)} - 1 + \frac{E^2 s^2}{6}\right) \quad \text{(scalar)} \quad (4.14)$$

$$L_{H.-E.} = \frac{2}{(4\pi)^2} \int \frac{ds}{s^3}\left(e^{-m^2 s} - e^{-M^2 s}\right)\left(\frac{Es\,\text{ch}\,(Es)}{\text{sh}\,(Es)} - 1 - \frac{E^2 s^2}{3}\right) \quad \text{(spinor)} \quad (4.15)$$

Concerning here the above considered problem of exponentially large «too lengthy» path contributions, which arises again in the spinor particle case in the strong field limit, we see, that the effective action integral (4.13) is saturated by $s \sim 1/E$, and a reasonable cut-off of order $1/E$ is possible. Moreover, an additional interesting consequence of this observation is, that in the limit $E/m^2 \gg 1$ the result is insensitive to particle mass m, since $m^2 s \ll 1$, and up to terms of order $O(m^2/E)$ it coincides with the massless limit. Thus the calculation in the massless particle case is possible as well. This observation is valid also in the scalar particle case.

Unfortunately, it is not the case for Heisenberg—Euler lagrangian (4.15), which is essentially contributed by $s \sim 1/m^2 \gg 1/E$, or by paths with individual contributions being too large. In the strong field limit a reasonable choice of cut-off is impossible, and our method fails.

4.2. NUMERICAL RESULTS

The gauge invariant quantities we have studied numerically are two-point current-current correlators, which are often consider in the context of QCD sum rules [1, 2]. Below we compute the quantities:

$$K_S = i\,\langle j_{sc}(t)\, j_{sc}^{+}(0)\rangle_E = |D(t, E)|^2 \quad (4.16)$$

(scalar currents, scalar particles), and

$$K_V = -i\,\langle j_\mu(t)\, j_\mu^{+}(0)\rangle_E = \mathrm{Sp}\,\{\gamma_\mu G(t, E)\gamma_\mu G(-t, E)\} \quad (4.17)$$

(vector current, spinor particles). To exclude their uninteresting strong but trivial variation in many orders of magnitude, below we consider the ratios of of correlators to the corresponding free ones, taken with absence of external field:

$$R_{S,V} = K_{S,V}(t, E)/K_{S,V}(t, E=0).$$

In order to make the situation being closer to reality, we take the field strength $E = 0.5$ GeV^2, which implies

$$\sum_{\mu,\nu} |F_{\mu\nu}|^2 = 2E^2 = 0.5 \text{ GeV}^4 \quad (4.18)$$

in correspondence with the known phenomenological estimate for the gluon vacuum condensate [2]:

$$\sum_{\mu,\nu,a} \langle (gG^a_{\mu\nu})^2\rangle \simeq 0.5 \text{ GeV}^4 \quad (4.19)$$

The dependence of results on the number N of steps in proper time s is shown in Fig.1, where the correlators are calculated at the typical in the hadron world distance $t = 3$ GeV^{-1} or 0.6 fm and the mass corresponds to that of charmed quarks. It is seen that a good agreement is provided already by $N = 8$—10.

Comparison of numerical results with the exact analytical ones obtained by substituting eqs (4.2—4) into eqs (4.16—17) is given in Fig.2. Typical statistics is here about 500 and 350 paths per point in scalar and spinor case respectively, which corresponds to CPU time order of a few minutes at the middle power computer (about $5 \cdot 10^5$ op/sec). All the results correspond to $N = 10$. An interesting feature is seen, that the nonrelativistic estimate (given by eq. (4.7)) is almost coinciding with the relativistic one in the scalar case but deviates considerably from it in the spinor case.

The results for lighter quarks are shown in Fig.3. The dependence of correlator ratios on the mass of quark is well reproduced up to zero mass for both the scalar and spinor particles. Some small but systematic deviation from the analytical curve, which is seen in the scalar case at $N = 10$, disappears if one takes $N = 20$. In the calculations with spinor particles we use the cut-off values $s_c = 4$, 7 GeV^{-2}. To study the sensitivity to s_c we plot the analytical curves with the same cut-offs, as well as that with $s_c = \infty$. We see that at $s_c > 7$ GeV^{-2} the results are insensitive to s_c, and corresponding uncertainty does not exceed several per cent.

Now let us turn to the results of the effective action calculations (Figs 4, 5). A reasonable accuracy for effective action and Heisenberg—Euler lagrangian is reached at $N = 15$—20. Typical number of paths is taken about 1000 (scalar case) and 500 (spinor case). Nice results are obtained in the scalar case (Fig.4), which has in principle no problems in the strong field limit $E/m^2 \gg 1$ too. In the spinor case (Fig.5) there is no problem as well in calculation of the effective action, if one takes the corresponding cut-off value inside some «stability region»:

$$s_c = (2\text{—}8)/E.$$

For larger values of s_c the method fails, while for lower ones the results are strongly dependent on s_c (Fig.6). In contrast, calculations of the Heisenberg—Euler lagrangian (in spinor case) appear to be not so successful, since the corresponding stability region for this quantity is more narrow, and in the strong field limit ($E/m^2 > 10$—20) it disappears at all (see Fig.6).

5. SUMMARY AND DISCUSSION

Let us summarize the main points of the paper. The method, developed here allows to calculate propagators (and other related quantities) for scalar and spinor particles in arbitrary external gauge field. It based on the path integral representation and contains only one discretized variable, Fock—Schwinger «proper time». Simple algorithm of path simulation takes CPU time which is negligible in comparison with the evaluation of an individual path contribution.

The method shows high efficiency, especially for the scalar case. In the spinor case an additional suppression of too lengthy (in proper time) paths is needed, for their individual contributions are divergent. A reasonable cut-off is possible in computations of many quantities of interest, but in some cases, where «too lengthy» paths appears to be essential (e.g., in Heisenberg—Euler lagrangian calculations in the strong field limit), the method fails.

Our method allows also a straightforward generalization for gluon case. This work is now in progress, and its applications to related problems of vacuum structure (instanton-instanton interaction, etc.) are under study.

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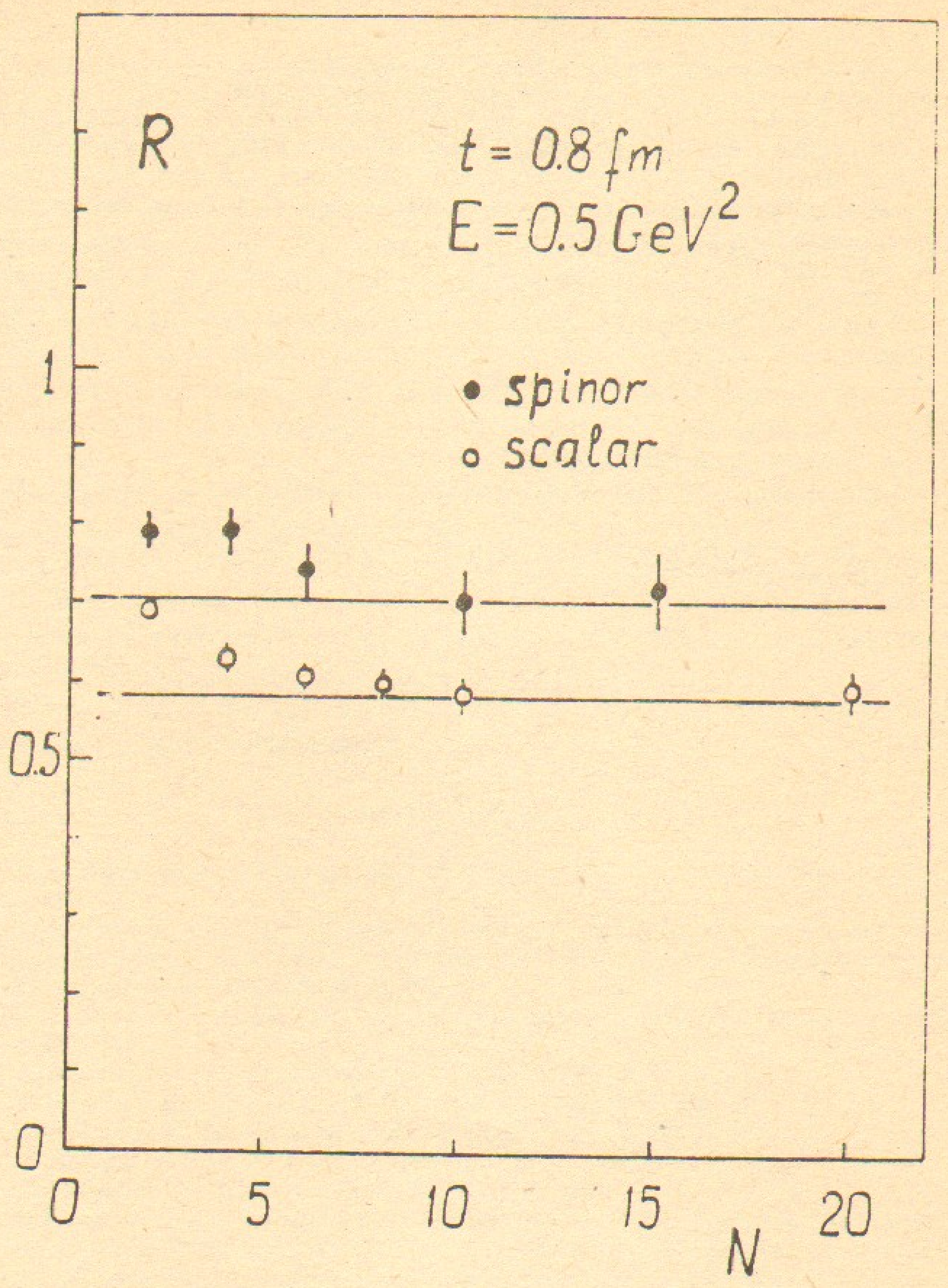

Fig.1. The dependence of the ratio $R=K(t, E)/ K(t, 0)$ on the number of steps in proper time. The lines show the exact analytical results. The open and closed points correspond to scalar and spinor cases respectively.

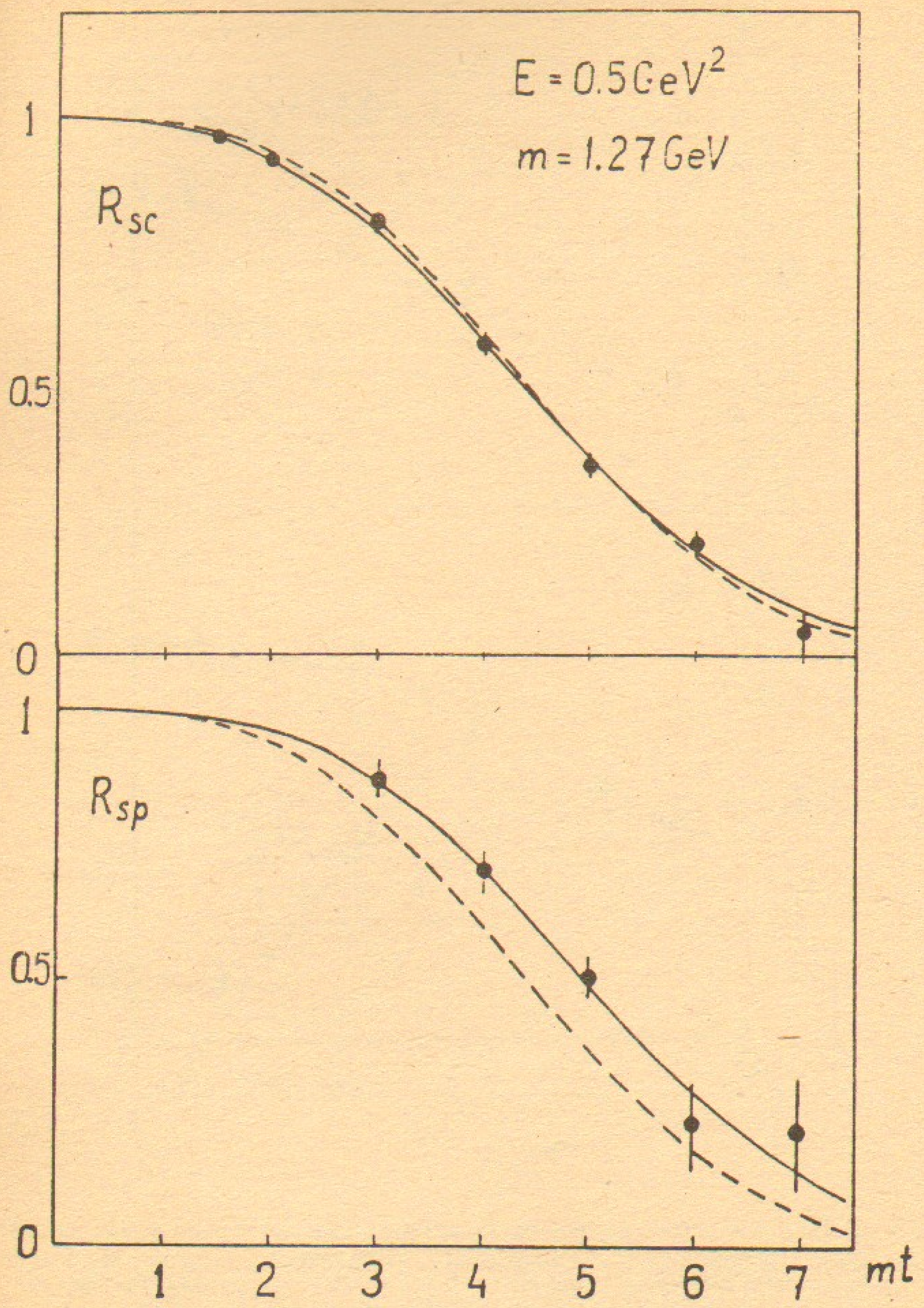

Fig.2. *a*) The scalar current-current correlator ratio versus *t*. The solid line corresponds to exact analytical result, while the dotted one is a nonrelatistic estimate.
b) The same, but in spinor case.

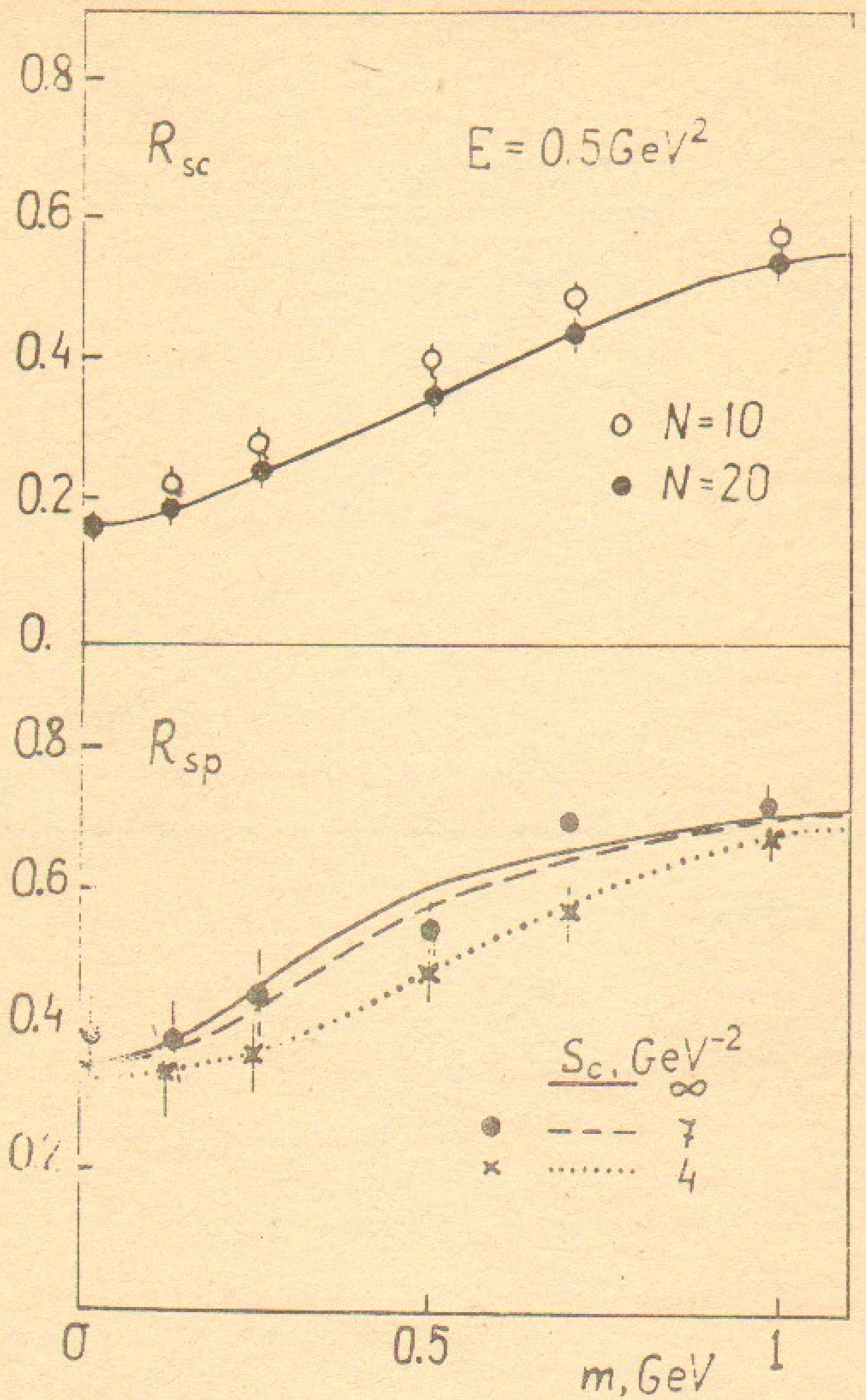

Fig.3. *a*) The scalar current-current correlator ratio versus quark mass *m*. The solid line corresponds to exact analytical result. *b*) The same, but in spinor case. Points and crosses correspond to cut-off $s_c = 7$ and 4 GeV^{-2} respectively. The lines correspond to the analytical results with the indicated cut-off.

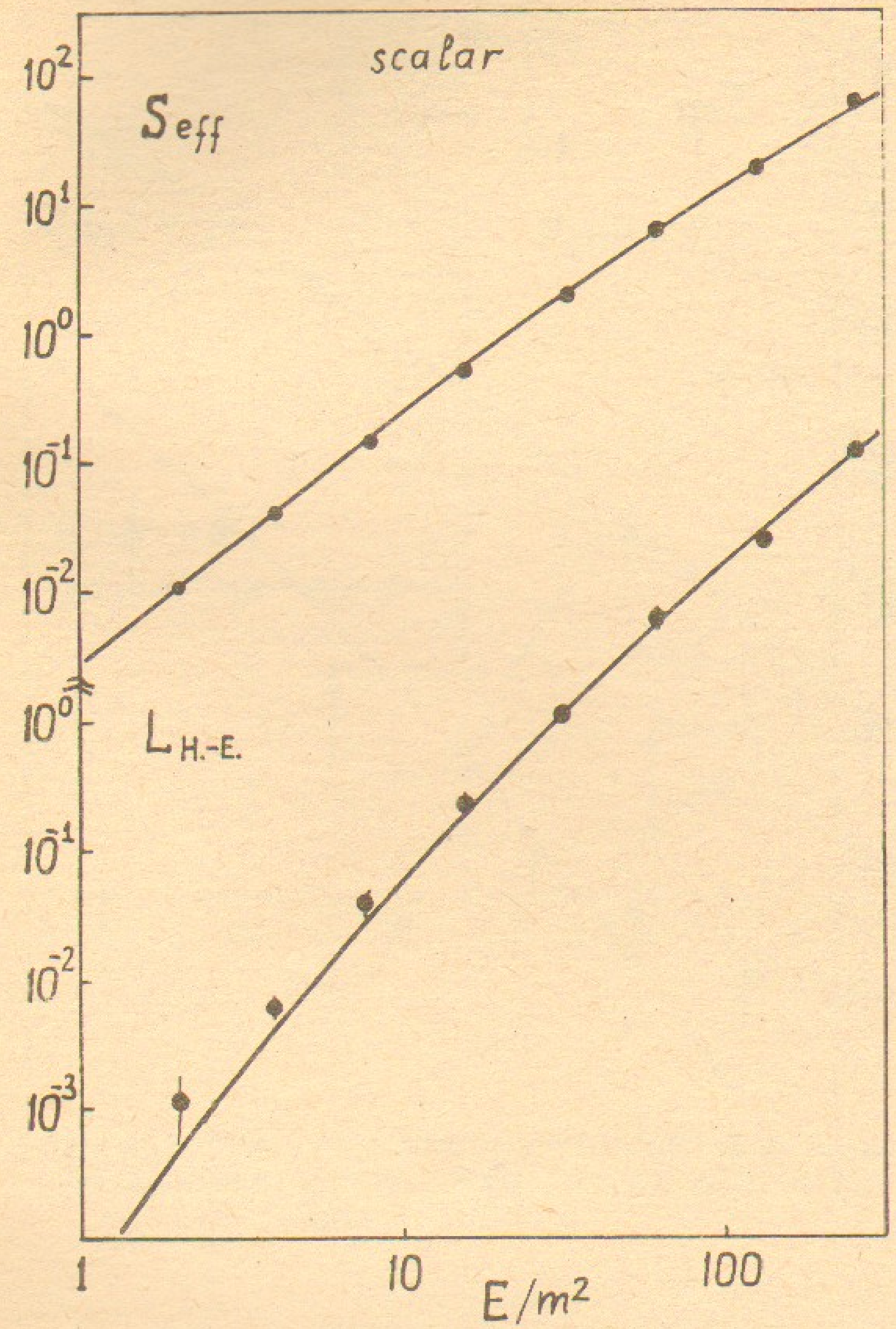

Fig.4. The effective action S^R_{eff} and the Euler—Heisenberg lagrangian $L_{H\text{-}E.}$ as a function of ratio E/m^2 (scalar case).

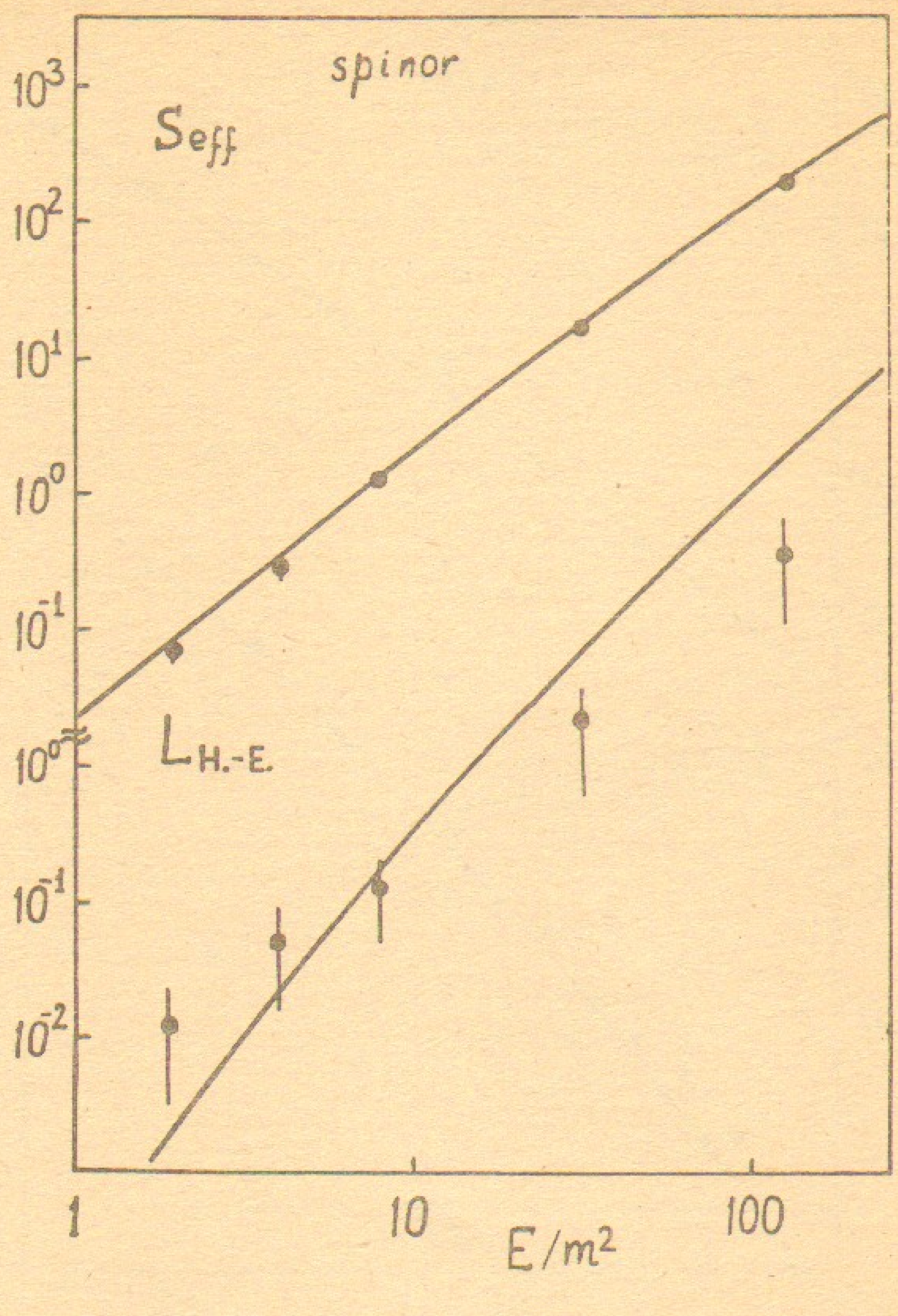

Fig.5. The same as in Fig.4, but in the spinor case.

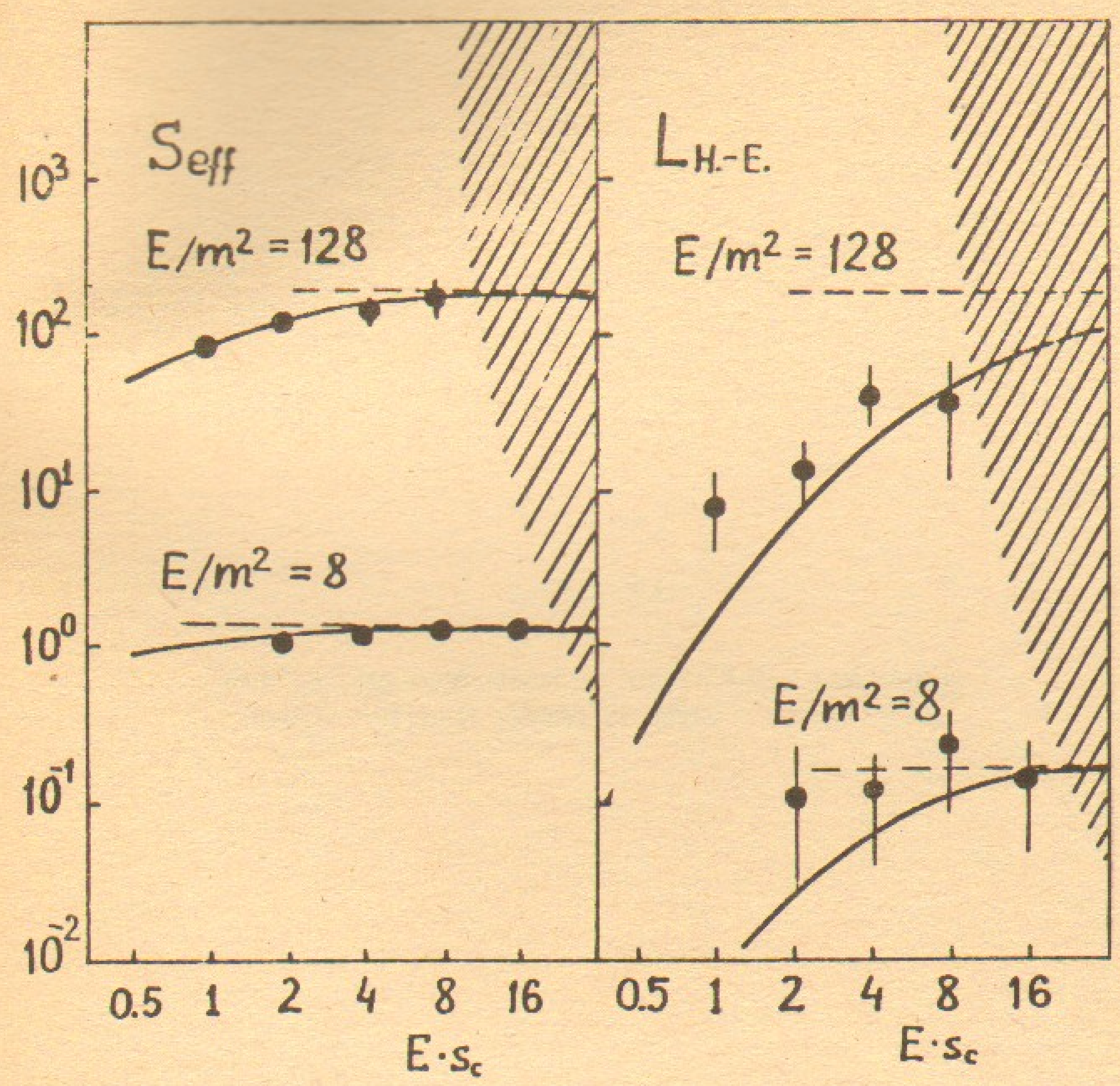

Fig.6. The dependence of the effective action S^R_{eff} and the Euler—Heisenberg lagrangian $L_{H.-E.}$ on the choice of s_c. The shaded area shows the region, where the method fails.

O.V. Zhirov

Numerical calculations of relativistic propagators via Feynman paths in proper time formalism

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