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Abstract

Using a simple example we show that the distribution for the energy levels of integrable systems is not the uncorrelated Poisson as is commonly believed. In particular, the spectrum was found to be rather rigid. We conjecture that these are typical properties of the integrable quantum systems.

Energy level statistics is an important property of many physical systems such as complex atoms and molecules, heavy nuclei etc. This problem has attracted recently much attention (see e.g. (1-18) among physicists, chemists and mathematicians. In particular the level statistics provides an indication of the type of motion of a quantum system. It is commonly believed that the "level repulsion", i.e. the Wigner statistics for level spacings, is related to the nonintegrable, chaotic motion, while the lack of repulsion, i.e. the Poisson statistics, corresponds to integrable motion. Actually, this is no absolute rule; indeed, the energy level repulsion in nonintegrable systems, as confirmed by several numerical computations, is due to some real interaction among the unperturbed states that leads to a formation of eigenstates which are superposition of many unperturbed states. However, a repulsion which may be said to have a kinematical rather than dynamical origin may take place in integrable systems. The simplest example of such a kinematical repulsion is given by the one-degree-of-freedom conservative system. From the viewpoint of level statistics one could say that there is a strong repulsion in this case since the spacings are equal to the frequency of the classical motion which is typically different from zero. A more interesting example of kinematical repulsion has been given by Berry and Tabor[(2)] in a two-degrees-of-freedom harmonic oscillator.

The distinction between integrable and non-integrable systems becomes much less clear when higher-order statistics, i.e. correlations between many levels, are taken into account. In the search for distinctive properties, spectral sequences of simple model systems have been subjected to various statistical tests. For example, Bohigas et al.[(16)] were able to establish a definite similarity of fluctuation properties between the spectral sequence of Sinai's billiard and strings of eigenvalues of random matrices in the Gaussian Orthogonal Ensemble.

In this letter we communicate the results obtained by statistical processing a string of 10^5 eigenvalues of the rectangular incommensurate billiard, calculated by reordering the double sequence

$$E_{mn} = \alpha m^2 + n^2 \qquad (1)$$

with α an irrational number.

In a previous paper[11], we discussed the algorithmic properties of the same sequence, and we showed that it is not a truly random one. Being not based on specific tests, our argument may leave some doubt that, nevertheless, the sequence may appear "random" to empirical tests. Thus, we have performed the following tests.

1) The distribution of the level spacings. This is shown in Fig. 1 and it looks fairly close to the Poisson distribution. Nevertheless, for small spacings we found statistically reliable deviations from Poisson's uncorrelated statistics. In fact, for the first interval in Fig. 1 the deviation of the number of spacings from the expected value is approximately 17 times larger than the standard deviation.

The distribution inside this interval is also shown in Fig. 1. Again the first interval of the latter distribution shows the biggest fluctuation with the actual number of spacings now larger than expected, approximately 18 times the standard deviation.

The χ^2 value for all the 10 sub-intervals is approximately 626 and even if we exclude the first sub-interval it is still 297 which corresponds to negligible confidence level. Apart from the whole interval (0,0.1) the agreement with Poisson's law seems to be rather good judging by Fig. 1. However, the calculated χ^2 value for 21 intervals is again too large: 69.8, corresponding to a confidence level $\sim 10^{-7}$. This is another indication that the sequence is not completely random: in fact, it exhibits too large fluctuations* for a random sequence which are expecially clear in the distribution of deviations from the Poisson law (Fig. 2). Not only there are substantial distortions of the Gaussian shape but what is more important the width of the distribution is about 3.3 times larger than the expected one. This implies that the whole distribution is definitely different from uncorrelated Poisson statistics. In terms of χ^2 test the value about 10^4 was obtained for 900 intervals which correspond to a completely negligible confidence level.

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*) A similar observation was made in Ref. (17).

2) The Δ_3 statistics of Dyson and Mehta[19] which characterizes long-term correlations between levels, or the so called "rigidity" of the spectrum. Specifically, for a given number L of levels we computed an average $\bar{\Delta}_3(L)$ in two different ways:

a) by averaging $\Delta_3(E_m,L)$, computed along a segment of L levels starting from level E_m, over a string $E_{n_1} \leq E_m \leq E_{n_2}$ ("spectral average") with $\alpha = \pi/3$;

b) by averaging Δ_3 over a number of different values of α chosen at random in a given interval ("ensemble average").

The results so obtained for $\bar{\Delta}_3(L)$ are plotted in Fig. 3. The straight line in this figure $\bar{\Delta}_3 = L/15$ corresponds to the behaviour of $\bar{\Delta}_3(L)$ for the uncorrelated Poisson statistics of the level spacings. For small L, $\bar{\Delta}_3(L)$ is close to this line, but then a kind of saturation occurs. Henceforth, $\bar{\Delta}_3(L)$ becomes a very slowly increasing function, such as one would expect for a rather regular sequence. On the other hand, if one looks at the set of eigenstates on the (n,m) plane which form a perfectly regular lattice (Fig. 5) one is led, indeed, to expect $\bar{\Delta}_3(L) \approx$ const or, at most, a very slowly increasing function.

The apparent controversy in these results can be explained as follows.

Consider a ring $E_1 \leq E \leq E_2$ inside which there are L levels, with $L \approx \frac{\pi}{4}(E_2 - E_1)$, and boundary layers of width ε along each of the two borders of the ring (Fig. 5). Due to incommensurability of curve E = const and the integer lattice the number of levels in the layer fluctuates. Actually, provided ε is small ($\varepsilon \ll 1$), we may assume that those levels come roughly as if at random. Hence, $\Delta_3(E,L)$ computed over a string of L such levels, starting from level E, would behave as L/15.

On the other hand, due to the regularity of the lattice, we can not expect the same for too long strings. In the latter case $\bar{\Delta}_3(L)$ will be approximately 1/15 of the "effective nonrigid length" of the string, corresponding to some critical value of $\varepsilon = \varepsilon_{cr} \sim 1$ which can be determined by numerical experiments. The total number of those "random" levels lying within the two boundary layers near E_1 and E_2 is approximately $\frac{\pi}{2}\varepsilon_{cr}\left[\sqrt{E_1} + \sqrt{E_2}\right]$

for $\alpha \approx 1$, so that we expect

$$\Delta_3(E,L) \approx \frac{1}{15}\,\frac{\pi}{2}\,\varepsilon_{cr}\left[\sqrt{E}+\sqrt{E+L\frac{4}{\pi}}\right] \qquad \text{for} \quad L \gtrsim \pi\varepsilon_{cr}\sqrt{E} \tag{2}$$

Instead, if $L \lesssim \pi\varepsilon_{cr}\sqrt{E}$, we expect $\Delta_3(E,L) \approx L/15$.

By averaging these expressions of Δ_3 in the two ways a) and b) described above, we obtain analitycal estimates to be compared with the numerical data in Fig. 3 and 4. In particular, from the data in Fig. 3 we obtain $\varepsilon_{cr} = 0.40 \div 0.42$.

For an accurate check of the square root dependence on L it is convenient to take E = 0 in Eq. (2), and compute the ensemble average which gives

$$\Delta_3(L) = \frac{\varepsilon_{cr}\sqrt{\pi L}}{15} \tag{3}$$

Fig. 4 shows $\overline{\Delta}_3(L)$ averaged over 20 values of α within the interval (0.9, 1.2). It is seen that the square-root dependence is verified with a quite good accuracy. Moreover, fitting Eq. (3) to numerical results gives $\varepsilon_{cr} = 0.53$ which is close to the value obtained from spectral averaging over different segments of levels with the same α .

In conclusion it appears that, at least for the integrable systems discussed here, the level sequence is overall rather rigid but behaves as a random one over small energy intervals. In particular, the latter explains the irregular behaviour of spacings leading roughly to the Poisson distribution.

The argument presented here for the two-dimensional case may be easily generalized to N dimensions and gives

$$\overline{\Delta}_3(L) \approx \begin{cases} \frac{\sqrt{N}}{15} L^{\frac{N-1}{N}} \; ; & L \gtrsim N^{N/2} \\ L/15 \; ; & L \lesssim N^{N/2} \end{cases} \tag{4}$$

assuming $E = \sum_{i=1}^{N} \alpha_i n_i^2$; $\alpha_i \approx 1$ and $N \gg 1$.

Even though the derived expressions for the level statistics are related to the particular model (1) we conjecture that the qualitative structure of the spectrum would be the same for a typical integrable many-dimensional system. This view is supported, particularly, by the results of Ref.(17) where a similar behaviour has been observed in a different model.

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References

1. T.A.Brody, J.Flores, J.B.French, P.A.Mello, A.Pandey, S.S.M.Wong. Rev. Mod. Phys. 53 (1981), 385.
2. M.V.Berry, M.Tabor. Proc. Roy. Soc. London, 356 (1977), 375.
3. S.W.McDonald, A.N.Kaufman. Phys. Rev. Lett., 42 (1979), 1189.
4. G.Casati, F.Valz-Gris, I.Guarneri. Lett. Nuov. Cim., 28 (1980) 279.
5. G.M.Zaslavsky. Phys. Rep. 80 (1981), 157.
6. V.Buch, R.B.Gerber, M.A.Rather. J. Chem. Phys., 76 (1982) 5397.
7. H.S.Camarda, P.D.Georgopulos. Phys. Rev. Lett., 50 (1983), 492.
8. E.Haller, H.Koppel, L.S.Cederbaum. Chem. Phys. Lett., 101 (1983), 215. Phys. Rev. Lett., 52 (1984), 1665.
9. S.S.Wong, in "Chaotic Behaviour in Quantum Systems", edited by G.Casati, (Plenum, New York). In press.
10. M.Robnik. J.Phys. A.: Math. Gen. (to appear).
11. G.Casati, I.Guarneri, F.Valz-Gris. Phys. Rev. A. Rap. Comm. 30 (1984) 1586.
12. G.Casati, I.Guarneri, G.Mantica, in "Chaotic Behaviour in Quantum Systems" edited by G.Casati, (Plenum, New York). In press.
13. G.Casati, "Energy Levels Distribution and Chaos in Quantum Mechanics". Proceedings of the XII Int. Conf. on Differential Geometric Methods in Theoretical Physics, Claustral, 1983.
14. F.M.Izrailev. "Quasi-Energy Level Spacing Distribution for Classically Stochastic Quantum Systems". Preprint 84-63, Novosibirsk (1984).
15. T.Ishikawa, T.Yukawa. "Transition from Regular to Irregular Spectra in the Quantum Billiards". Preprint KEK-TH85 (1984).
16. O.Bohigas, M.S.Giannoni, C.Schmit. Phys. Rev. Lett., 52 (1984), 1.
17. T.H.Seligman, J.J.M.Verbaarschot, M.R.Zirnbauer. Phys. Rev Lett., 53 (1984), 215.
18. P.Pechukas. Phys. Rev. Lett., 51 (1983), 943.
19. M.L.Mehta. Random Matrices, Academic, New York, 1967.

Figure captions

Fig. 1. Level spacing distribution obtained from the first 100,000 levels (1) with $\alpha = \pi/3$. The dotted line is the Poisson distribution $P(s) = e^{-s}$.

Fig. 2. Histogram of the distribution for deviations $m_i = \frac{n_i^{ob} - n^{ex}}{\sqrt{n^{ex}}}$ of the observed number of spacings n_i^{ob} from the expected n^{ex} in the i-th interval. The intervals are so chosen that $n^{ex} = 90$ for each. The full line shows the Gaussian distribution of width $\sigma = 1$ corresponding to the uncorrelated Poisson statistics; the actual observed r.m.s. width $\sigma = 3.3$ (dashed line).

Fig. 3. The Δ_3 - statistics computed for model (1): spectral average $\bar{\Delta}_3$ over the first 2,850 (+) and over 10,000(•) levels with the same $\alpha = \pi/3$; ensemble average $\bar{\Delta}_3$ over several values of α for $10{,}000 \leq \frac{\pi}{4}E + L \leq 11{,}000$ (°) and $20{,}000 \leq \frac{\pi}{4}E + L \leq 21{,}000$ (Δ). The straight line: $\bar{\Delta}_3 = L/15$.

Fig. 4. Graph of ensemble average $\bar{\Delta}_3(L)$ for $L \leq 1{,}000$ showing the square-root dependence on L, $\alpha \approx 1$. The straight line fits Eq. (3) to the numerical data with $\varepsilon_{cr} = 0.53$.

Fig. 5. The set of eigenstates for model (1).

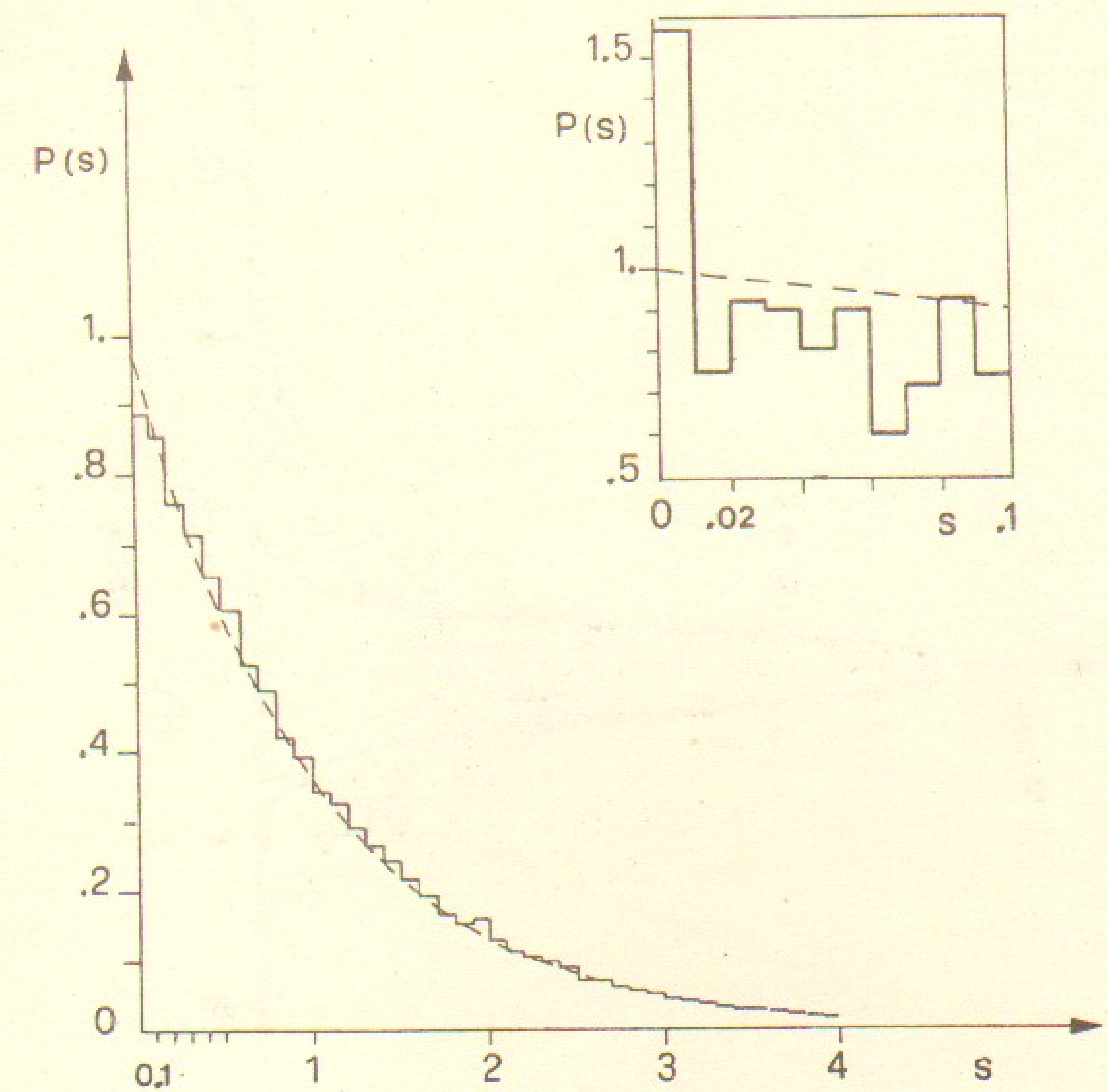
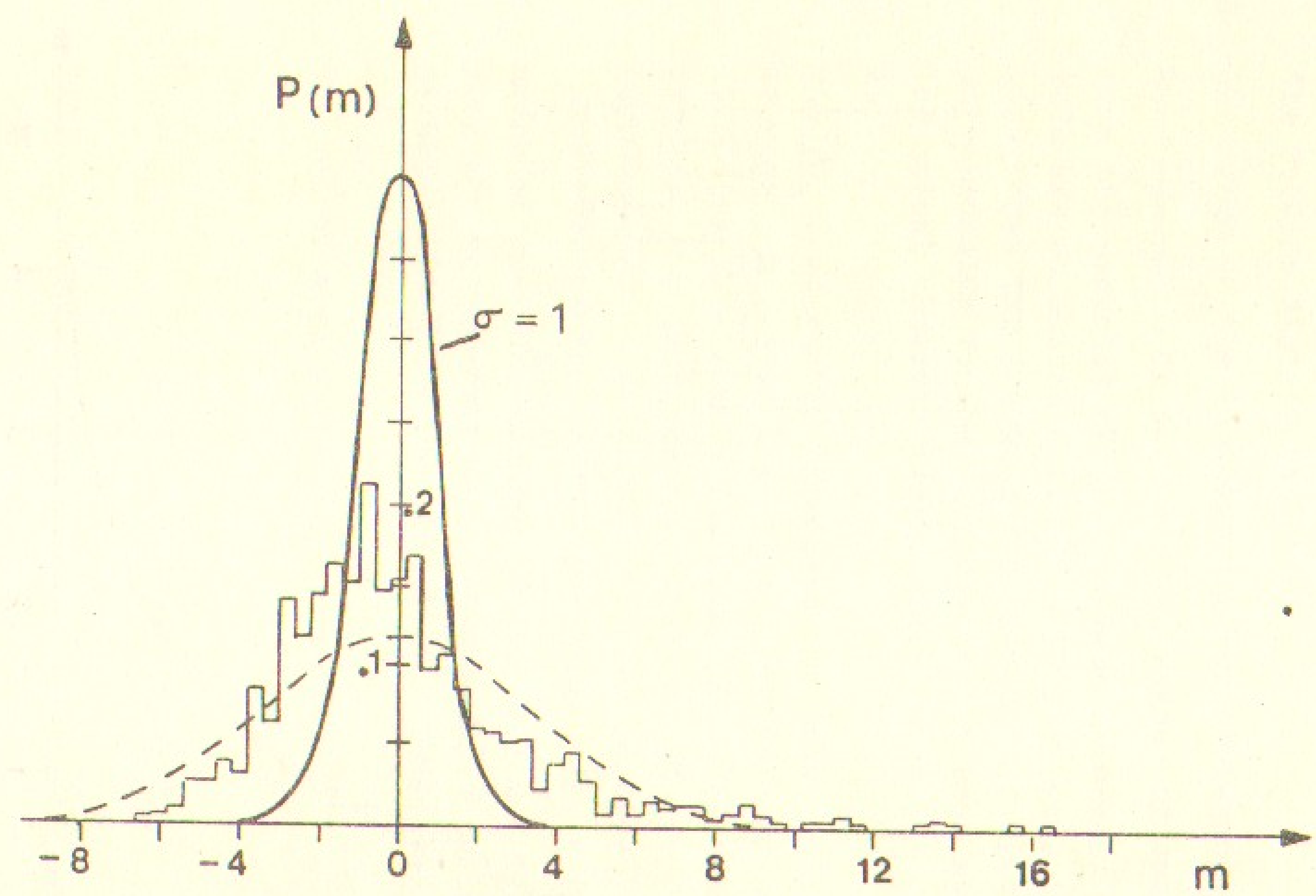
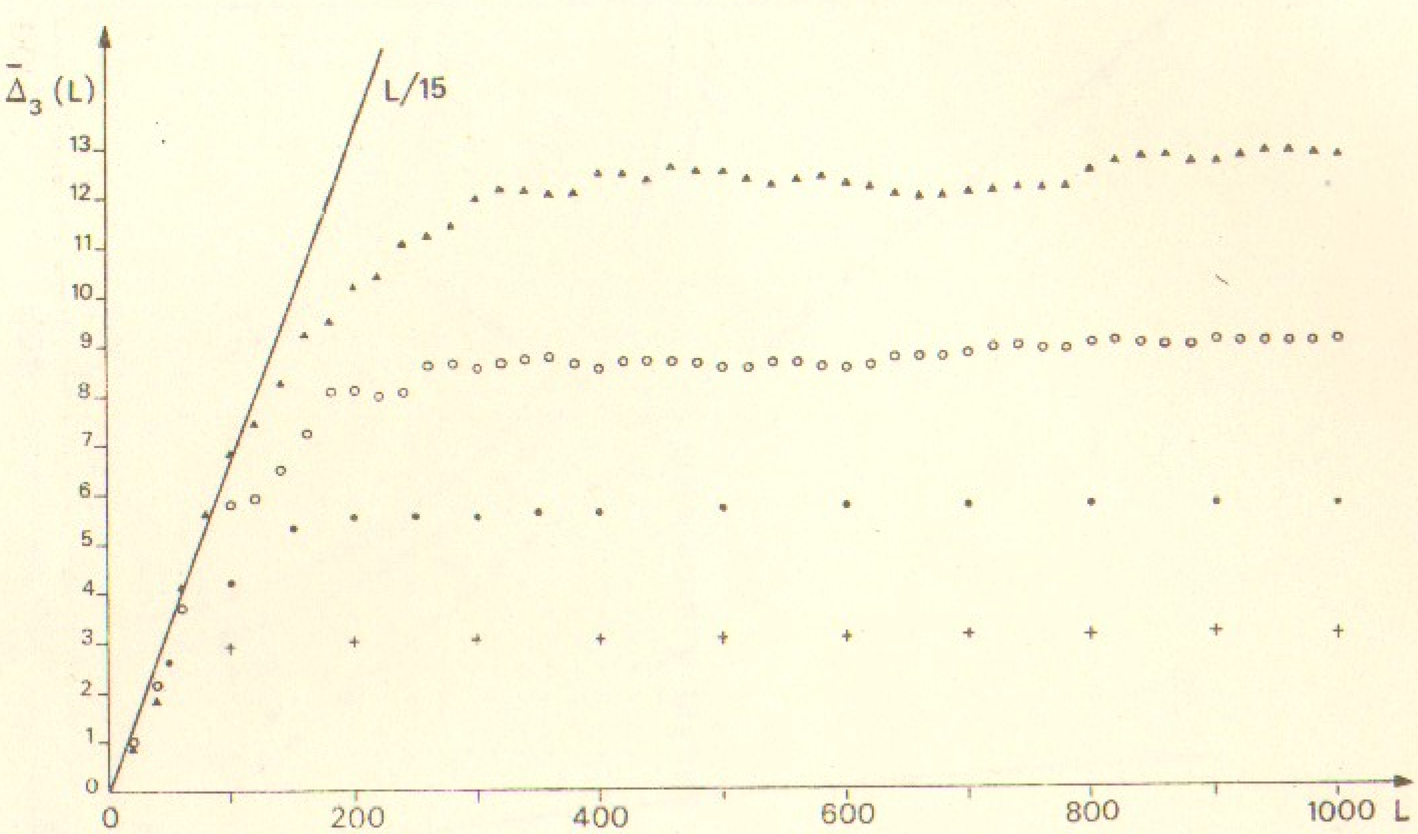


Fig. 1



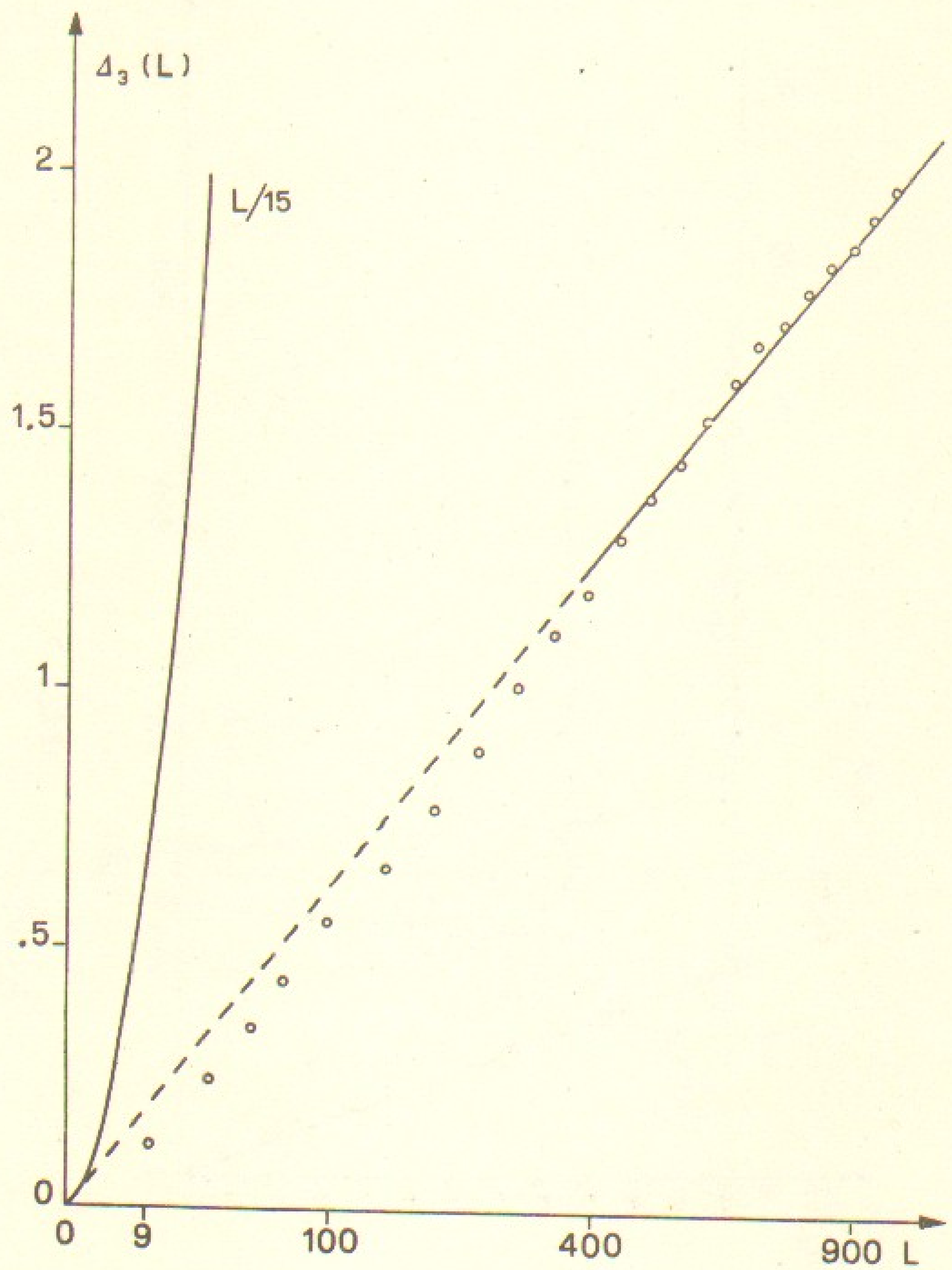
P(m)
σ = 1
.2
.1
-8
-4
0
4
8
12
16
m

Fig. 2



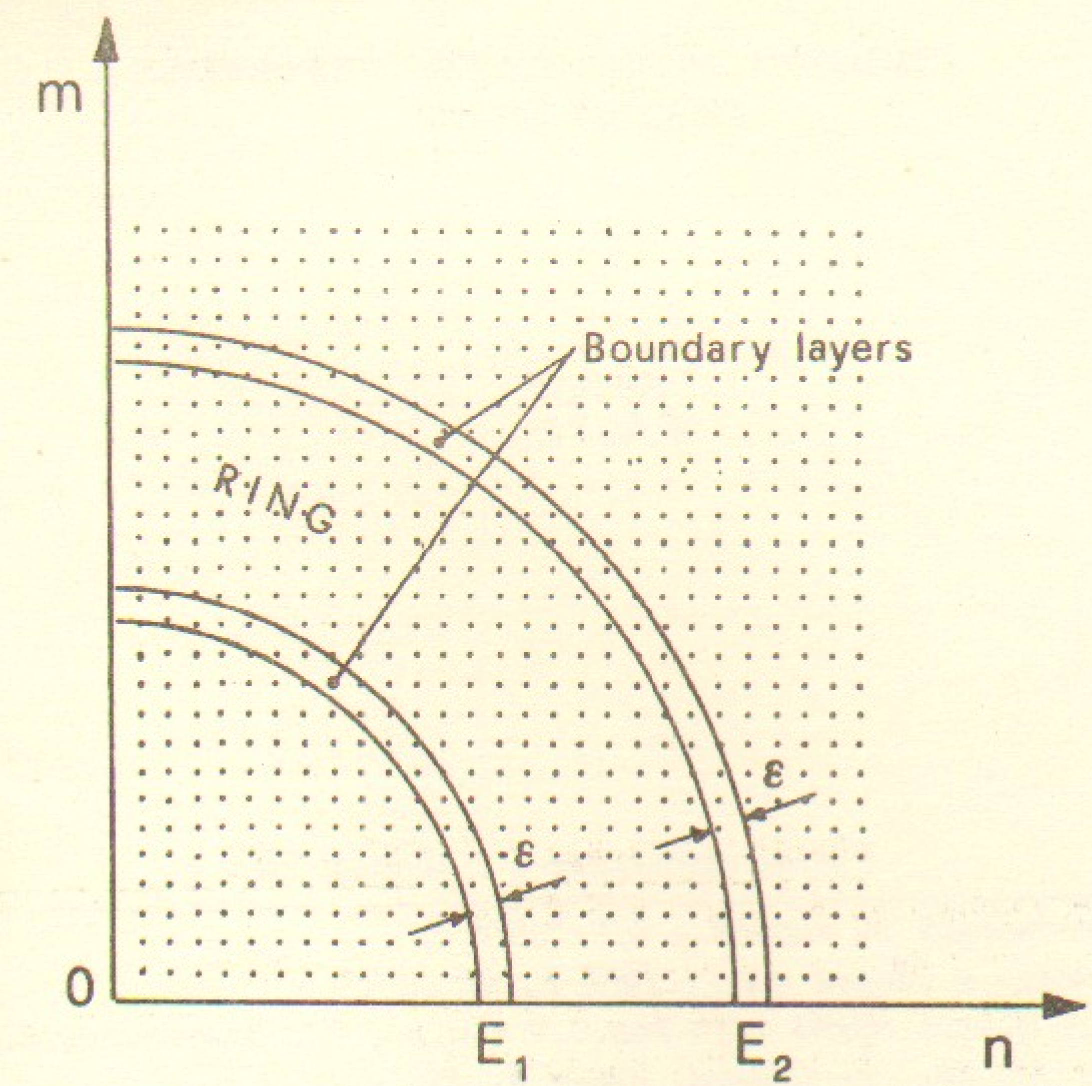
$\bar{\Delta}_3$ (L)
L/15
13
12
11
10
9
8
7
6
5
4
3
2
1
0
0
200
400
600
800
1000
L

Fig. 3



$\Delta_3(L)$
2
1.5
1
.5
0
L/15
0
9
100
400
900 L

Fig. 4



m
Boundary layers
RING
ε
ε
0
E_1
E_2
n

Fig. 5

15

И.Гварнери, Дж.Казати, Б.В.Чириков

СТАТИСТИКА УРОВНЕЙ ЭНЕРГИИ ИНТЕГРИРУЕМЫХ КВАНТОВЫХ СИСТЕМ

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