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44

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ON THE RELAXATION OF N-LEVEL SYSTEM UNDER GAUSS NOISE

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НОВОСИБИРСК

ABSTRACT

In the evolution of an N-level system interacting with an external noise there is in general a specific phase transition from damped oscillations to an aperiodic motion when the intensity of the external noise increases. But in some cases the region of aperiodic motion can be absent. On tne other hand, several such regions can exist. Under some conditions a strong interaction with environment can stabilize a nonstationary quantum state of the system.

In Refs [1—8] the influence of an external noise on quantum beats in a two-level system was considered. The motion of a particle in a double bottomed potential well was studied there for the situation when the noise cannot knock the particle over the barrier. This simple model is interesting since it describes such various physical phenomena as the oscillations of optical activity of molecules, the behaviour of the magnetic flux in a superconducting ring with a Josephson junction and of a system of vortices in a droplet of superfluid helium.

A curious peculiarity of the mentioned model is the qualitative change of the character of the motion when the noise intensity increases—from damped oscillations to an aperiodic motion. This change can be considered as a specific phase transition [5—7]. It is important here that in the oscillating regime the relaxation of a state localized at $t=0$ in one of the wells is accelerated with the increase of the noise intensity, but in the aperiodic regime the relaxation becomes slower as the noise increases. In other words, an extremely strong noise stabilizes such a state. The existence of the mentioned phase transition in this situation leads to the hypothesis [7] that it is impossible in principle to observe quantum-mechanical beats in a system strongly interacting with its environment. Another aspect of the same problem is the stabilization of a quantum state by measurements [3,4,8].

The natural question arises: to what extent are these conclusions independent of the specific structure of the quantum system and of nature of the noise. The question is especially appropriate since even for a two-level system one can choose the interaction with environment in such a way that the character of the relaxation does not change qualitatively at any intensity of the noise (see below).

Let us consider an N-level quantum system interacting with its environment. It is convenient for us to present the density matrix of the system as

$$\varrho=\frac{1}{N}(I+\sum_i P_i\lambda_i) \qquad (1)$$

Here λ_i are N^2-1 traceless Hermitean matrices of the dimension NxN which are the generators of the SU(N) group. Evidently $N-1$ parameters P_i, the components of «the polarization vector», determine completely the density matrix of the N-level system. In terms of the same matrices λ_i one can expand both the Hamiltonian of the considered quantum system

$$H=\bar{\Omega}I+\sum_i \Omega_i\lambda_i \qquad (2)$$

and its interaction with a medium

$$V(t)=\bar{V}(t)I+\sum_i V_i(t)\lambda_i . \qquad (3)$$

From the equation $i\dot{\varrho}=[H+V(t),\varrho]$ we find

$$\dot{P}_i=\sum_{k,j} f_{ikj}\,[\Omega_k+V_k(t)\,]\,P_j \qquad (4)$$

Here f_{ikj} are the structure constants of the SU(N) group:

$$[\lambda_i\lambda_k]=i\sum_j f_{ikj}\lambda_j$$

They are antisymmetric in any pair of the indices.

As to the functions $V_k(t)$, we shall assume that they are random pulses, both their duration τ and the intervals between them being much smaller than the characteristic periods Ω^{-1} of the unperturbed motion. It allows one to neglect the free motion of the system when averaging over the interaction with the medium. The formal solution of the eq. (4) for the intervals $t\ll\Omega^{-1}$ can be written as

$$P_i(t)=\sum_{n=0}^{\infty}\int_0^t dt_1\int_0^{t_1} dt_2\ldots\int_0^{t_{n-1}} dt_n\sum_{k,j} f_{ik_1j_1}V_{k_1}(t_1)f_{j_1k_2j_2}V_{k_2}(t_2)\times\ldots$$
$$\ldots\times f_{j_{n-1}k_nj_n}V_{k_n}(t_n)\,P_{j_n}(0)\,. \qquad (5)$$

Assume further that the characteristic amplitude V of the pulses satisfies the condition $V\tau\ll1$. One can check that it allows to restrict, when averaging the expression (5) over the pulses, to the pair correlators

$$\langle V_m(t)\,V_n(t')\rangle=2v_{mn}\delta\,(t-t'). \qquad (6)$$

It can be easily shown then that for the intervals $t\ll\Omega^{-1}$ the average value of the polarization vector $p_i(t)=\langle P_i(t)\rangle$ satisfies the equation $\dot{p}_i=-\sum_j\eta_{ij}p_j$ where

$$\eta_{ij}=\sum_{k,m,n} f_{imk}f_{jnk}v_{mn}\,. \qquad (7)$$

For arbitrary time intervals one should restore the term with Ω, coming after it to the equation

$$\dot{p}_i=\sum_j(\omega_{ij}-\eta_{ij})\,p_j\,,\quad \omega_{ij}=\sum_k f_{ikj}\Omega_k\,. \qquad (8)$$

The matrix ω_{ij} is antisymmetric, and η_{ij} as well as v_{mn} is symmetric and nonnegative.

The evolution of the polarization vector p_i depends both on the quantitative relation between the matrices ω_{ij} and η_{ij} and on the de-

tailed properties of the latter. In the limit $\eta\ll\omega$ the motion of the (N^2-1)-dimensional vector p_i is sufficiently evident: it rotates with slowly decreasing absolute magnitude.

In the general case the decrements γ_k of the system (8) are eigenvalues of the matrix $\eta_{ij}-\omega_{ij}$. Consider the limit $\eta\gg\omega$. It is convenient here to diagonalize the matrix η by means of an orthogonal transformation. It leaves the matrix ω_{ij} antisymmetric. If the noise is completely arbitrary, the most natural situation corresponds to nondegenerate and nonvanishing eigenvalues η_k of the dissipative matrix. In this case we find by means of elementary perturbation theory

$$\gamma_k=\eta_k-\sum_n\frac{\omega_{kn}^2}{\eta_k-\eta_n}+\cdots \qquad (9)$$

(we retain here the previous notation ω for the transformed matrix). All the terms in the expansion (9) are real. On the other hand, as it was mentioned above, at $\eta\to0$ the eigenvalues γ become imaginary. It means that the series (9) in ω has a finite radius of convergence. In other words, at some finite value of ω/η the qualitative change in the character of the evolution of the system takes place, from an aperiodic motion to damped oscillations. Therefore, in the general case of an N-level system the same phase transition takes place as in the two-level example considered in Refs [5—7].

The case when some eigenvalues $\bar{\eta}$ of the matrix η_{ij} are degenerate, requires a special consideration. In the lowest order in ω we restrict to the subspace of degenerate eigenvectors. We perform in it the unitary transformation diagonalizing the corresponding submatrix ω. Its eigenvalues are evidently imaginary. The submatrix η proportional to the unit one does not change under this transformation. Therefore, the eigenvalues γ_k degenerate in the zeroth approximation in ω are in the first order

$$\gamma_k=\bar{\eta}-i\omega_k. \qquad (10)$$

If all eigenvalues of the matrix η are degenerate, formula (10) is evidently the exact solution of the problem, and there is no phase transition. In a more general case in the next orders of perturbation theory in ω an imaginary part appears in other eigenvalues as well, due to the term $-i\omega_k$ in (10). Therefore, damped oscillations take place here not only at large frequencies, but also at small ones. As to the intermediate region, damped oscillations may survive here as well, so that there is no phase transition. However, it is possible that the regions of aperiodic motion arise here. Then the number of phase transitions is even. Moreover, even in the nondegenerate case at intermediate frequencies not one but several transitions between periodic and aperiodic regimes are possible.

As an illustration to this assertions we shall use a two-level system with somewhat more general than in Refs [1—8] interaction with a medium. We take $H=\frac{1}{2}\omega\sigma_x$, $V(t)=\frac{1}{2}[V_1(t)\sigma_x+V_3(t)\sigma_z]$. Then the secular equation is

$$\begin{vmatrix} \eta_{11}-\gamma & 0 & \eta_{13} \\ 0 & \eta_{11}+\eta_{33}-\gamma & -\omega \\ \eta_{13} & \omega & \eta_{33}-\gamma \end{vmatrix}=0 \qquad (11)$$

Nonnegativity of the matrix η_{ij} means that

$$\eta_{11}\geqslant 0,\ \eta_{33}\geqslant 0,\ \eta_{11}\cdot\eta_{33}\geqslant\eta_{13}^2.$$

In the special case $V_1(t)=0$, considered in the Refs [1—8], $\eta_{33}=\eta_{13}=0$ and along with $\gamma=\eta_{11}$ we get

$$\gamma=\frac{1}{2}[\eta_{11}\pm\sqrt{\eta_{11}^2-4\omega^2}] \qquad (12)$$

(Although the eigenvalues of the matrix η are doubly degenerate, γ has no imaginary part at small ω, since the matrix elements ω_{ij} in the subspace of degenerate eigenvectors vanish.) From the expression (12) the phase transition at $\omega=\frac{1}{2}\eta_{11}$ is seen.

On the other hand, if $V_3(t)=0$, then $\eta_{11}=\eta_{13}=0$ and the roots of eq. (11) look as follows:

$$\gamma_0=0\ ,\ \gamma_\pm=\eta_{33}\pm i\omega.$$

The absence of the phase transition is evident.

Consider now the eq. (11) without additional simplifications. It is a cubic equation for γ and the character of its roots depends on the sign of the discriminant

$$D=\frac{1}{27}[D_0(\eta)+D_1(\eta)\omega^2+D_2(\eta)\omega^4+\omega^6]. \qquad (13)$$

Here

$$D_0(\eta)=-\frac{1}{4}(\eta_{11}\eta_{33}-\eta_{13}^2)^2[(\eta_{11}-\eta_{33})^2+4\eta_{13}^2]\leqslant 0.$$

In the nondegenerate case $\eta_{11}\eta_{33}-\eta_{13}^2>0$, $D_0<0$ and D is negative at small ω together with D_0. It means that here all the roots of the secular equation are real and the motion is aperiodic. On the other hand at large ω $D=\frac{\omega^6}{27}>0$, two roots are complex and damped oscillations take place, Therefore, at intermediate frequencies, where $D=0$, the phase transition takes place. It can be easily seen that at $D_1<0$ or $D_1>0$, $D_2>0$ there is only one such transition (see curves *a*,*b*, at Fig. 1). However, if $D_1>0$, $D_2<0$, then under some relations among D_i three phase transitions, i. e. two regions of aperiodic motion, are possible (see Fig. 2).

In the degenerate case $\eta_{11}\eta_{33}-\eta_{13}^2=0$ the coefficient $D_0=0$ and $D_1=\eta_{33}\ (\eta_{11}+\eta_{33})^2>0$. It means that at small ω there are two complex roots $\gamma_\pm$. If $\eta_{11}<8\eta_{33}$, the regime of oscillations exists at all ω (see curve *a* at Fig. 3) . In the opposite case $\eta_{11}>8\eta_{33}$ at intermediate frequencies there is a region of aperiodic motion (curve *b* at Fig. 3), i. e., the phase transition takes place twice. In a remarkable way, the aperiodic regime arises here not at the most strong damping.

It is seen from the expression (12) that at $\eta_{11}\gg\omega$ one of the decrements $\gamma\approx\omega^2/\eta_{11}$ is anomalously small and falls off with the increase of η_{11}. It means that in the used representation for the problem of a double bottomed potential well, when the states localized in the left or right well are taken as basis ones, such a state is stabilized by a noise that does not knock the particle over the barrier [1—8]. This state is an eigenstate for the noise operator σ_z. A natural generalization on the N-level case is the problem of the relaxation of the eigenstate of the operator λ_N under the noise $V_\nu(t)\lambda_N$. In this case the dissipation matrix (7) looks as

$$\eta_{ij}=\sum_k f_{ikN}\,f_{jkN}\,v_{NN}$$

Its elements turn to zero if at least one of the indices i or j equals to N or refers to the matrix commuting with λ_N. The total number of commuting λ-matrices is $N-1$. Therefore, the matrix η_{ij} may be presented in a box form:

$$
\begin{array}{c c}
 & \begin{array}{cc} k & \alpha \end{array} \\
\begin{array}{c} k \\ \\ \alpha \end{array} & \left(\begin{array}{cc} \eta & 0 \\ & \\ 0 & 0 \end{array} \right) \begin{array}{l} N(N-1) \\ \\ (N-1) \end{array} \\
 & \begin{array}{cc} N(N-1) & N-1 \end{array}
\end{array}
\qquad (14)
$$

We reduce again the upper left box in (14) by orthogonal transformation to the diagonal form $\eta_k \delta_{kl}$ (here and below $k,l=1,2,\ldots, NN-1)$). In the matrix ω_{ij} we reduce the right lower box $\omega_{\alpha\beta}$ $(\alpha,\beta=1,2,\ldots, N-1\equiv n)$ to the diagonal form $i\omega_\alpha \delta_{\alpha\beta}$ by means of an unitary transformation $U_{\alpha\beta}$. Then the corresponding eigenvalues γ are

$$\gamma_\alpha = -i\omega_\alpha + \sum_{k\alpha_1} \frac{1}{\eta_k} |\omega_{k\alpha_1} U_{\alpha_1 \alpha}|^2 + \cdots$$

Therefore, in this, more general, case as well the decrement is small and falls off as η_k increases.

Let us choose as an initial state of the system the eigenstate of the interaction operator λ_N. It means that the initial density matrix is expanded only in the matrices λ_α that can be diagonalized together with λ_N. The corresponding polarization vector p_α lies in the subspace α. From the expression (15) it follows that it precesses in this subspace, its absolute magnitude slowly damping.

The considered case corresponds to the rank N^2-N of the matrix η_{ij} when there is only one independent correlator. If the zero eigenvalue of this matrix is unique, i.e., its rank is N^2-2, slow damping of the corresponding mode follows trivially from the general formula (9).

The obtained result, stabilization by a strong external action of the state that is the eigenstate of the corresponding operator, is quite natural from the physical point of view. The random nature of the action is of no special importance here.

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REFERENCES

1. *M. Simonius*, Phys. Rev. Lett. 40 (1978) 980.
2. *R.A. Harris and L. Stodolsky*, Phys. Lett. 78B (1978) 313.
3. *R.A. Harris and L. Stodolsky*, J. Chem. Phys. 74 (1981) 2145.
4. *R.A. Harris and L. Stodolsky*, Phys. Lett. 116B (1982) 464.
5. *S. Chakravarti*, Phys. Rev. Lett. 49 (1982) 681.
6. *A.T. Bray and M.A. Moore*, Phys. Rev. Lett. 49 (1982) 1545.
7. *S. Putterman*, Phys. Lett. 98A (1983) 324.
8. *R.A. Harris and R. Silbey*, J. Chem. Phys. 78 (1983) 7330.

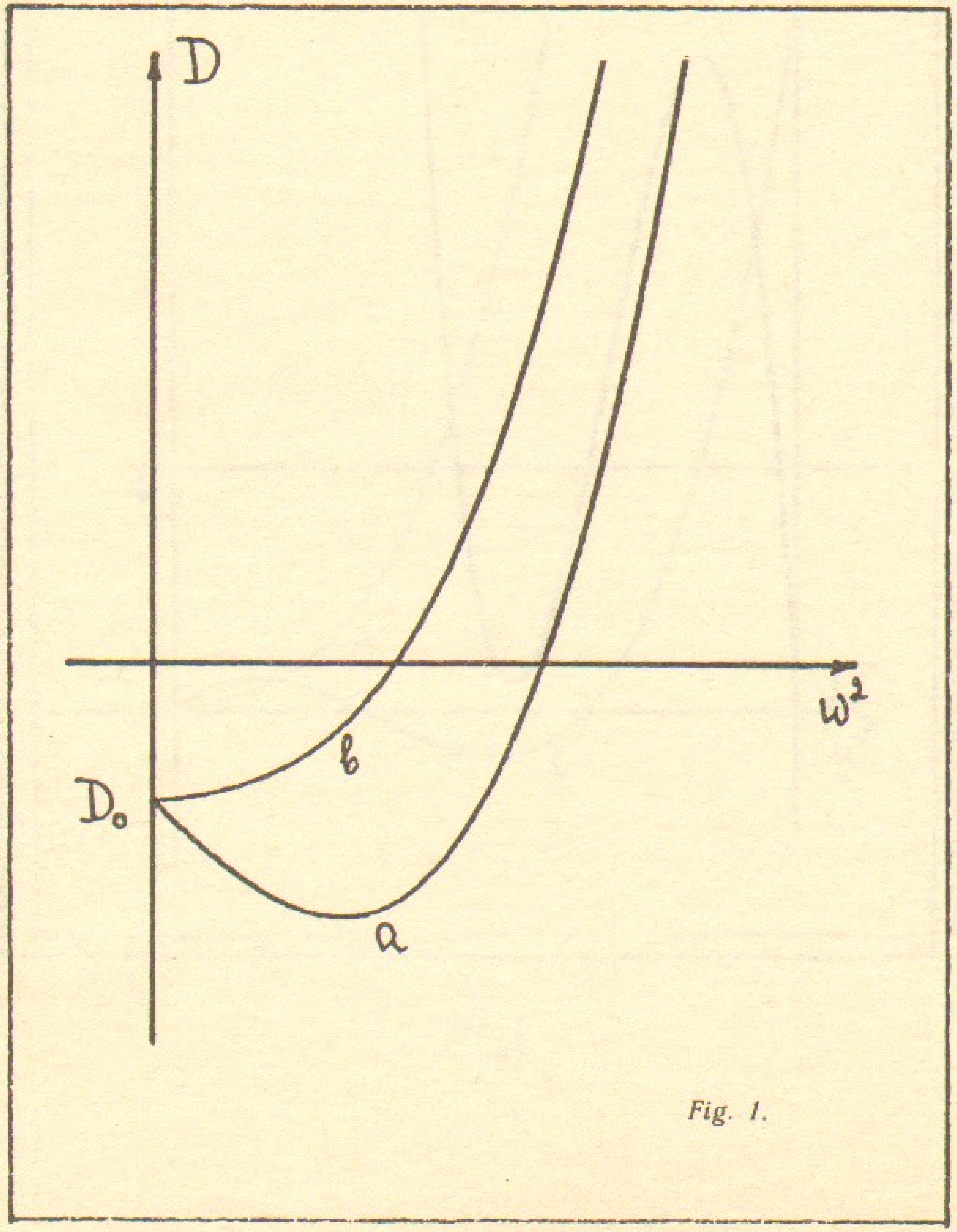

Fig. 1.

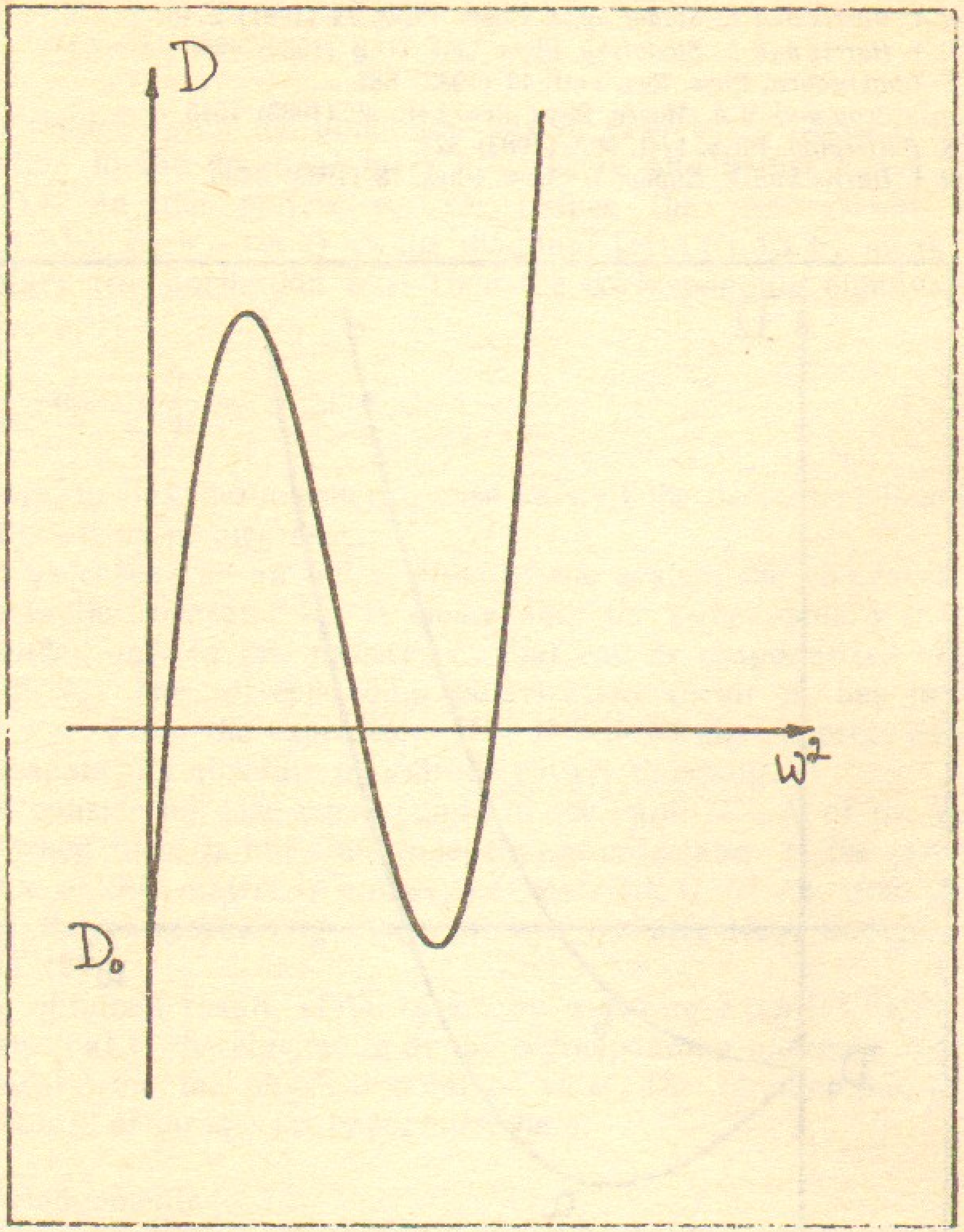

Fig. 2.

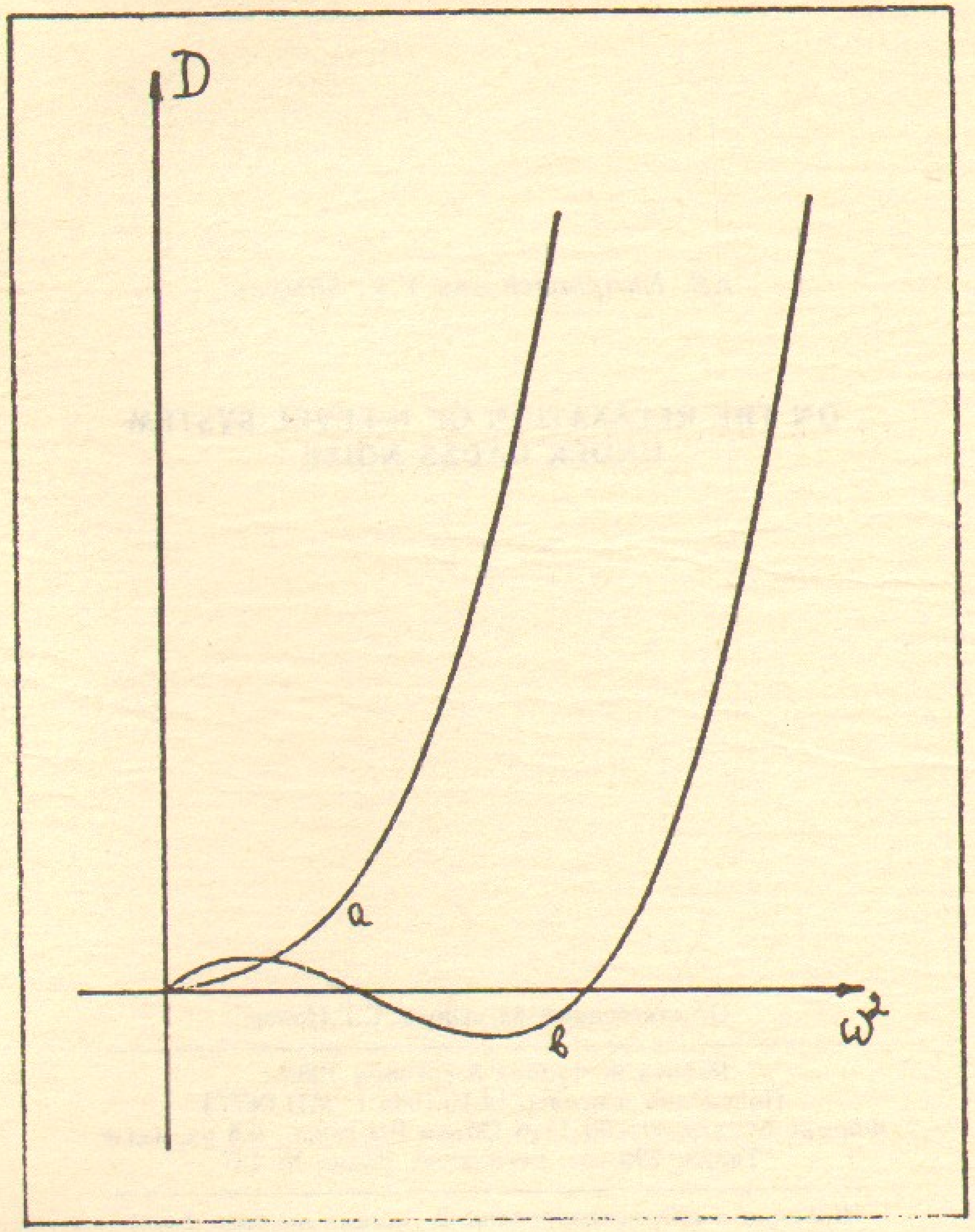

Fig. 3.

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