ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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ON THE THEORY OF MAGNETIC CORONAL HEATING

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ABSTRACT

Coronal heating connected with electric currents generated by cellular convective flows on the photosphere is discussed. These current systems represent a potential source of free energy which may be released by the complex reconnection processes. The latter may be treated phenomenologically (Heyvaerts and Priest, 1984) as the relaxation to the minimum energy state in a time scale τ_r. The relaxed state is determined by the constraint of global magnetic helicity conservation in highly conductive plasmas (Taylor, 1974). The model of an array of closely packed flux tubes (Parker, 1983); Browning et al., 1986) is considered, and the coronal heating rate is obtained for an arbitrary quasi-static photospheric motions. It is also shown that discontinuities in the magnetic field (current sheets) cannot arise under a fluent velocity field at the photospheric boundary. This suggests that the coronal field can simply adjust to the slowly changing photospheric displacements. Therefore, it gives rise to doubt about the «topological dissipation» of magnetic energy (Parker, 1972, 1983).

1. INTRODUCTION

In recent years it has become clear that the heating of the solar corona is related to the presence of magnetic fields, with the kinetic energy of photospheric pulsations as an energy source. Such a magnetic heating is provided by motions of the photospheric footpoints of the coronal field that permanently build up coronal electric currents. The present understanding of this process is described in reviews by, for example, Kuperus et al. (1981), Priest (1982).

The physical mechanisms of coronal magnetic heating are rather different for high-frequency and low-frequency driving photospheric pulsations. If the frequency of photospheric perturbations $\omega \gtrsim v_A/L$, where v_A is the Alfven velocity and L—the length-scale of magnetic structure, hydromagnetic waves are excited, whose damping may supply heat (Ionson, 1978; Priest, 1982; Heyvaerts and Priest, 1983). For slow enough motions, when $\omega \ll v_A/L$, the coronal configuration may be regarded as being in a state of quasistatic evolution through a series of equilibria. It is this latter case that we are concerned in this paper.

For solar coronal conditions the plasma thermal pressure is small compared with a magnetic one. Therefore during such a quasi-static evolution magnetic field remains to be almost force free, and generated electric currents flow along the coronal magnetic field lines from one photospheric footpoint to another. The main difficulty in the problem of coronal magnetic heating is that simple Ohmic dissipation, acting over global length-scales of the coronal field, is completely inadequate to balance the radiative and conductive losses of the plasma. This is due to the fact that the electrical conductivity of the coronal plasma is very high, and required dissipation can be attained only if the magnetic field varies on a scale that is much smaller than the length scales of granulation, which are responsible for inducing the currents (Rosner et al., 1978).

Thus, to explain coronal heating in term of Joule dissipation, the electric current density must somehow be enhanced. Parker (1972) proposed that electric currents may be concentrated into thin sheets as a result of «dynamical nonequilibrium» of the magnetic field. He suggests that a field whose footpoints are being shuffled by complex photospheric motions cannot, in general, remain in equilibrium, so that current sheets are inevitably formed. However, some doubt has been raised about Parker's analysis (Van Ballegooijen, 1985). We reconsider in this paper a simplified model of a closed coronal loop, an array of closely-packed flux tubes, which have been twisted by photospheric motions (Parker, 1983; Browning et al., 1986). It is shown that discontinuities in the magnetic field (current sheets) arise only if the velocity field at the photospheric boundary is itself a discontinuous function of position (at least, in linear approximation). This suggests that the coronal field can simply adjust to the slowly changing boundary conditions in the photosphere, and the so called «topological dissipation» of the winding patterns (Parker, 1972; 1983) does not take place. It is worth noting here that there is one other way to produce current sheets: the presence of an X-type neutral point within the magnetic configuration can lead to the development of a current sheet, even if the motions at the photosphere are continuous (Syrovatskii, 1971). But we do not consider such a case in this paper.

Other solution of the problem of coronal heating is magnetic reconnection, which provides a convincing mechanism for efficiently dissipating magnetic energy (Priest, 1982). It is well known that as non-potential field is generated by the footpoint motions, resistive instabilities are triggered, which can lead to the formation of thin current sheets (see, for instance, tearing-mode instability in coronal loops (Galeev et al., 1981). As a result permanent reconnection and release of the excess magnetic energy occur (e. g. Steinolson and Van Hoven, 1984). The consistent calculation of corresponding dissipation rate is rather complicated problem concerned to the nonlinear dynamics of a plasma turbulence. An important approach to the problem has been proposed by Heyvaerts and Priest (1984). The idea is to consider complex magnetic reconnection processes phenomenologically as the relaxation to the minimum magnetic energy state in highly conductive plasmas. The latter is determined by the extension (Heyvaerts and Priest, 1984) of the Taylor's hypothesis well known in fusion plasma physics (Taylor, 1974), which suggests that the global magnetic helicity of the configuration is conserved during the relaxation.

The aim of this paper is to analyse such a relaxation plasma heating in an array of closely-packed flux tubes (Parker, 1972, 1983). In the recent paper (Browning et al., 1986) this model has been also considered, but only for the very slow photospheric motions, in the sense that $\omega \sim v/l \ll \tau_r^{-1}$, where v is the photospheric pulsation velocity, l—the length scale of a convective cell and τ_r—the characteristic relaxation time. Our results are valid for an arbitrary sub-alfvenic fluid motions and express the coronal heating rate for any given photospheric velocity field.

2. MODEL AND BASIC EQUATIONS

Let us consider a simplified model of a closed coronal loop (Parker, 1972, 1983). Neglecting the curvature of the loop, the initial magnetic field was taken to be uniform:

$$\mathbf{B}_0 = B_0 \mathbf{e}_z \tag{1}$$

extending between two flat parallel plates $z = \pm h$, which represent the photosphere at the ends of the loop (Fig. 1). The field is then perturbed by prescribed motions of the footpoints in both of the boundary plates. We assume photospheric pulsations to be incompressible and represent them in the form:

$$\mathbf{v}|_{z=+h} = \mathbf{v}_+ = [\nabla \Phi_+ \times \mathbf{e}_z], \qquad \mathbf{v}|_{z=-h} = \mathbf{v}_- = [\nabla \Phi_- \times \mathbf{e}_z] \tag{2}$$

with the given stream functions $\Phi_+(x, y, t)$ and $\Phi_-(x, y, t)$ in the planes $z = \pm h$. Since the photospheric motions may be represented as a number of convective cells, we restrict our analysis only to one isolated cell, bounded by the closed curve S in the plane (x, y) (Fig. 2). It is possible to consider now that $\Phi_{+,-}(x, y, t)|_S = 0$. Furthermore, without loss of generality we choose at first only one harmonic of the photospheric motions, so that $\Phi_{+,-} \infty e^{-i\omega t}$.

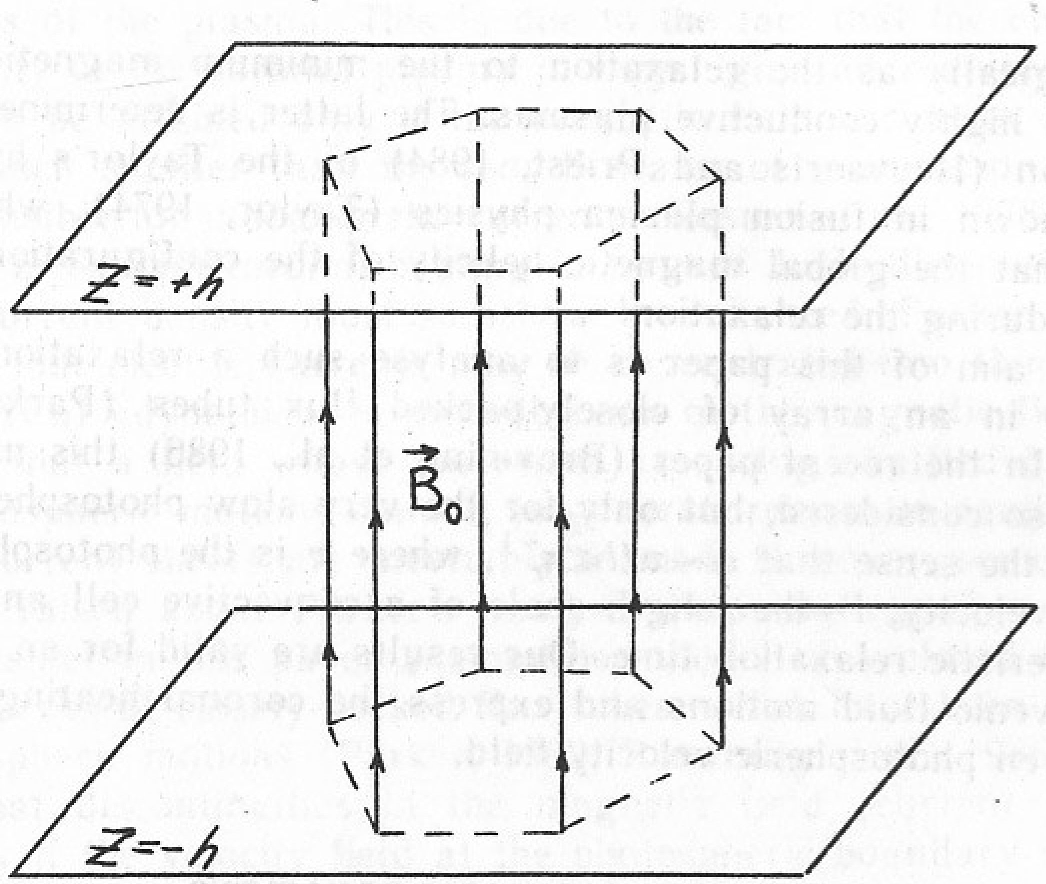

Fig. 1. The simple model of a coronal loop: uniform magnetic field $\mathbf{B}_0$ extending from $z=-h$ to $z=+h$.

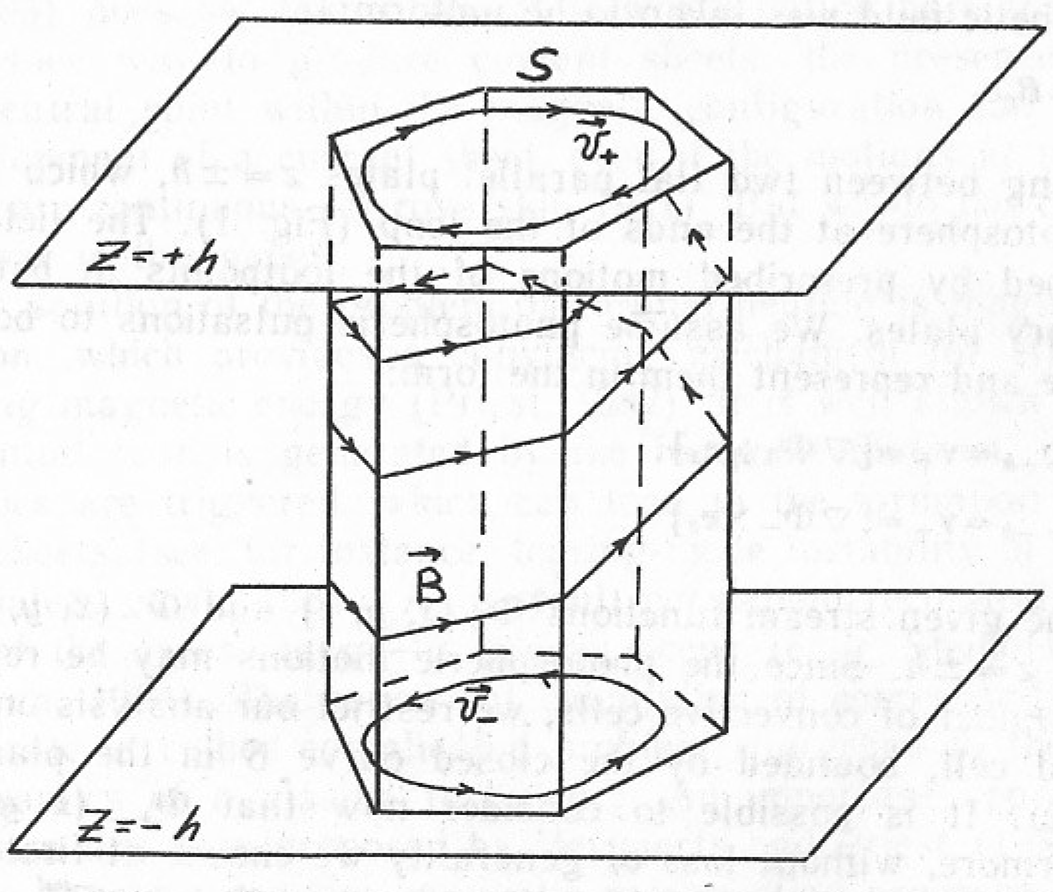

Fig. 2. A sketch of the perturbed magnetic configuration.

In what follows we consider photospheric perturbations to be rather small, so that coronal magnetic field $\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1$ with $B_1 \ll B_0$, and find $\mathbf{B}_1$ in linear approximation. As it was mentioned above the processes of interest occur in time scale which is small in comparison with the diffusion time for a magnetic field τ_d, determined by finite resistivity of a plasma σ ($\tau_d \approx 4\pi\sigma l^2/c^2$). Therefore the resistive term can be omitted from the equations and the processes of magnetic reconnection in narrow current sheets may be described as relaxation of the magnetic field to the minimum energy state with some characteristic relaxation time τ_r (Vekstein, 1986):

$$\partial(\mathbf{B}-\mathbf{B}^{(M)})/\partial t = -(\mathbf{B}-\mathbf{B}^{(r)})/\tau_r \,. \tag{3}$$

Here $\mathbf{B}^{(M)}$ is the force-free magnetic field obtained from the perfect MHD equation for the given velocity fields in the planes $z=\pm h$, and $\mathbf{B}^{(r)}$ — the relaxed field which provides the magnetic energy minimum for the same photospheric motions and with the constraints resulting from high conductivity of a plasma (see later). Obviously the relaxation time in (3) $\tau_r \ll \tau_d$. The value of τ_r may be roughly estimated as the tearing-mode instability growth time (Furth et al., 1963). For a given frequency of the photospheric pulsation it follows from (3) that perturbation of the field

$$\mathbf{B}_1 = \frac{i\omega\tau_r(\mathbf{B}_1^{(r)}-\mathbf{B}_1^{(M)})+\mathbf{B}_1^{(r)}+\omega^2\tau_r^2\mathbf{B}_1^{(M)}}{1+\omega^2\tau_r^2} \tag{4}$$

So we need to know the fields $\mathbf{B}_1^{(M)}$ and $\mathbf{B}_1^{(r)}$.

In perfect MHD limit

$$\partial\mathbf{B}/\partial t = \text{rot}\ [\mathbf{v}\times\mathbf{B}] \,, \tag{5}$$

so in a linear approximation we have $\mathbf{B}^{(M)}=\mathbf{B}_0+\mathbf{B}_1^{(M)}$ with

$$\mathbf{B}_1^{(M)} = \frac{i}{\omega}\,\text{rot}\,[\mathbf{v}\times\mathbf{B}_0] \,. \tag{6}$$

Here $\mathbf{v}$ is the velocity of a plasma flow between the planes $z=\pm h$, which is fitted in such a way that the field $\mathbf{B}^{(M)}$ retaines to be force free and $\mathbf{v}$ is equal to $\mathbf{v}_{+,-}$ at $z=\pm h$. For incompressible photospheric motions (2) we can put $\mathbf{v}=[\nabla\Phi(x, y, z)\times\mathbf{e}_z]$, so that $[\mathbf{v}\times\mathbf{B}_0] = -B_0\nabla\Phi+\mathbf{e}_z B_0\,\partial\Phi/\partial z$. It follows now that the vector potential perturbation $\mathbf{A}_1^{(M)}$ may be taken in the form:

$$\mathbf{A}_1^{(M)}=\mathbf{e}_z B_0\frac{i}{\omega}\frac{\partial\Phi}{\partial z};\quad \mathbf{B}_1^{(M)}=\text{rot}\,\mathbf{A}_1^{(M)}=B_0\frac{i}{\omega}\left[\nabla\frac{\partial\Phi}{\partial z}\times\mathbf{e}_z\right]. \tag{7}$$

Since in this case

$$\text{rot}\,\mathbf{B}_1^{(M)}=B_0\frac{i}{\omega}\nabla\frac{\partial^2\Phi}{\partial z^2}-\mathbf{e}_z\cdot\frac{i}{\omega}B_0\nabla\Phi \tag{8}$$

such a field would be force free, i. e. rot $\mathbf{B}_1^{(M)}\parallel\mathbf{B}_0$, if $\partial^2\Phi/\partial z^2=0$. Taking into account the boundary conditions $\Phi|_{z=\pm h}=\Phi_{+-}$, it is easy to find that

$$\Phi(x,y,z)=\frac{\Phi_+(x,y)+\Phi_-(x,y)}{2}+\frac{z}{2h}(\Phi_+-\Phi_-). \tag{9}$$

Putting this expression for Φ in (7) we obtain, that perturbation of a magnetic field is determined by the difference of footpoints velocities in the planes $z=\pm h$. Denoting $\Psi(x,\ y)=(\Phi_+-\Phi_-)/2h$, we have now:

$$\mathbf{A}_1^{(M)}=\mathbf{e}_z B_0\frac{i}{\omega}\Psi;\quad \mathbf{B}_1^{(M)}=B_0\frac{i}{\omega}[\nabla\Psi\times\mathbf{e}_z]. \tag{10}$$

A sketch of the perturbed magnetic configuration is shown in Fig. 2.

The important conclusion following from (10) is that for an arbitrary continuous velocity field in the photosphere plane the corresponding equilibrium solution of perfect MHD equation exist which has not the magnetic discontinuities like current sheets. Hence, the coronal field can simply adjust to the slowly changing footpoints displacements. Although this result has been obtained here only in linear approximation, it gives rise to doubt about «topological dissipation» of magnetic energy proposed by Parker (e. g. Van Ballegooijen, 1985).

Let us consider now the relaxed field $\mathbf{B}^{(r)}$, which corresponds to the minimum energy state. It is well known that potential magnetic field possesses the minimum of magnetic energy. However the release of all the magnetic energy above potential take place over the diffusion time τ_d, which is unacceptably large in a highly conductive coronal plasma. So we are interested in some intermediate minimum energy state that takes into account constraints resulting from high conductivity. The answer is well known in fusion plasma physics (Taylor, 1974, 1985) and tells that during the local magnetic reconnections in narrow sheets the global magnetic helicity of a configu-

ration $K=\int\mathbf{A}\cdot\mathbf{B}\,dV$ is conserved, where $\mathbf{A}$—vector potential, and integration extends over the whole volume of a system. The physical grounds for this rule, known as Taylor's hypothesis, can be explained as follows. Let us write the equations for the evolution of magnetic field $\mathbf{B}$ and vector potential $\mathbf{A}$, taking into account the finite resistivity of a plasma σ:

$$\partial\mathbf{B}/\partial t=\text{rot}\,\{[\mathbf{v}\times\mathbf{B}]-c\mathbf{j}/\sigma\}, \tag{11}$$

$$\partial\mathbf{A}/\partial t=[\mathbf{v}\times\mathbf{B}]-c\mathbf{j}/\sigma+\nabla\partial g/\partial t, \tag{12}$$

where g is a gauge function. Using (11) and (12), we get for the magnetic energy of a system $W_M=\int dV\cdot B^2/8\pi$ and global helicity $K=\int\mathbf{A}\cdot\mathbf{B}\,dV$:

$$\frac{dW_M}{dt}=\int\frac{\mathbf{B}}{4\pi}\frac{\partial\mathbf{B}}{\partial t}dV=-\int\frac{(\mathbf{j})^2}{\sigma}dV+\frac{1}{c}\int\mathbf{j}[\mathbf{v}\times\mathbf{B}]dV+$$

$$+\frac{1}{4\pi}\int\mathbf{B}\,[d\mathbf{S}\times[\mathbf{v}\times\mathbf{B}]], \tag{13}$$

$$\frac{dK}{dt}=\int\frac{\partial\mathbf{A}}{\partial t}\mathbf{B}\,dV+\int\mathbf{A}\frac{\partial\mathbf{B}}{\partial t}dV=-2c\int\frac{\mathbf{j}\cdot\mathbf{B}}{\sigma}dV+c\int d\mathbf{S}\frac{[\mathbf{A}\times\mathbf{j}]}{\sigma}+$$

$$+\int(\mathbf{A}\cdot\mathbf{v})(\mathbf{B}\cdot d\mathbf{S})-\int(\mathbf{A}\cdot\mathbf{B})(\mathbf{v}\cdot d\mathbf{S})+\int(\mathbf{B}\cdot d\mathbf{S})\,\partial g/\partial t. \tag{14}$$

It is seen now from (13) that the enhanced dissipation rate of the magnetic energy is possible if narrow current sheets are formed with the thickness $\delta\ll l$. In this case the current density in a sheet $\mathbf{j}=\frac{c}{4\pi}\text{rot}\,\mathbf{B}\sim cB/4\pi\delta$, so that dissipation power

$$Q_d=-\int\frac{(\mathbf{j})^2}{\sigma}dV\sim c^2B^2l^2/16\pi^2\sigma\delta$$

and the magnetic energy dissipate in a time scale

$$\tau_W\sim\frac{B^2}{8\pi}l^3/Q_d\sim\tau_d\cdot\delta/l\ll\tau_d, \tag{15}$$

where $\tau_d\simeq4\pi\sigma l^2/c^2$ is the diffusion time. On the other hand, as it follows from (14), the current sheet formation has not drastic influence on the global helicity evolution rate since

$$\frac{dK}{dt}\approx-2c\int\frac{\mathbf{j}\cdot\mathbf{B}}{\sigma}dV\sim\frac{c^2B^2l^2}{4\pi\sigma};\quad \tau_K\sim K\left(\frac{dK}{dt}\right)^{-1}\sim\tau_d. \tag{16}$$

So in a highly conductive plasma, when magnetic reconnetions occur in narrow current sheets, the magnetic energy and global helicity change in rather different time scales, so that $\tau_W \ll \tau_K$. That is why we have to consider the minimum of a magnetic energy with the constraint of global helicity conservation.

Solving the corresponding variation problem we have:

$$\delta(W_M-\mu K)=\int dV\left\{\frac{\operatorname{rot}\mathbf{B}}{4\pi}-2\mu\mathbf{B}\right\}\delta\mathbf{A}+\int\left(\frac{\mathbf{B}}{4\pi}-\mu\mathbf{A}\right)[d\mathbf{S}\times\delta\mathbf{A}] \qquad (17)$$

(here μ is a Lagrange multiplier). Let us assume now that the relaxation of a magnetic configuration occurs independently in every cell. It means that the normal component of a magnetic field does not change at the boundary of a cell: $\delta B_n|_S=0$. So the surface integral in (17) is equal to zero and the relaxed magnetic field satisfies the equation (Taylor, 1974):

$$\operatorname{rot}\mathbf{B}^{(r)}=\alpha\mathbf{B}^{(r)}; \quad \alpha=\text{const}. \qquad (18)$$

Since initial magnetic field $\mathbf{B}_0$ is a potential one, the constant α is, in general case, proportional to the photospheric velocity amplitude. Thus in the linear approximation we can write that $\mathbf{B}^{(r)}=\mathbf{B}_0+\mathbf{B}_1^{(r)}$ with

$$\operatorname{rot}\mathbf{B}_1^{(r)}\approx\alpha\mathbf{B}_0=\alpha B_0\mathbf{e}_z. \qquad (19)$$

Introducing the vector potential perturbation $\mathbf{A}_1^{(r)}$ by $\mathbf{B}_1^{(r)}=\operatorname{rot}\mathbf{A}_1^{(r)}$ we have from (19):

$$\Delta\mathbf{A}_1^{(r)}=-\alpha B_0\mathbf{e}_z. \qquad (20)$$

It means that $\mathbf{A}_1^{(r)}$ may be chosen in the form: $\mathbf{A}_1^{(r)}=\mathbf{e}_z\cdot\alpha B_0\varphi(x,y)$ with

$$\Delta\varphi=-1 \qquad (21)$$

and the perturbation of a magnetic field

$$\mathbf{B}_1^{(r)}=\alpha B_0[\nabla\varphi\times\mathbf{e}_z]. \qquad (22)$$

The boundary condition for φ has to be so that the normal component of a magnetic field retaines to be equal to zero on S: $(\mathbf{B}_1^{(r)}\cdot\mathbf{n})|_S=0$. As it follows from (22) it needs that $\varphi|_S=\text{const}$. We assume here $\varphi|_S=0$. It is useful to note that the solution of eq. (21) with such the boundary condition represents at the same time the electrostatic potential of a uniformly and positively charged cylindrical cavity with a cross-section S (Fig. 2) surrounded by conductor. It makes it clear at once that $\varphi>0$ inside the cell (this fact will be used later).

The constant α that appears in eqs (18—22) has to be obtained from the global helicity evolution resulting from photospheric footpoints displacements (Heyvaerts and Priest, 1984; Browning et al., 1986; Vekstein, 1986). Omitting in eq. (14) all the terms connected with a finite conductivity of a plasma (they are not essential here, as it was mentioned above) and taking into account that the normal velocity of a plasma vanishes on the boundary of a cell, $(\mathbf{v}\cdot d\mathbf{S})=0$, we have:

$$\frac{dK}{dt}=\int(\mathbf{A}\cdot\mathbf{v})(\mathbf{B}\cdot d\mathbf{S})+\int(\mathbf{B}\cdot d\mathbf{S})\,\partial g/\partial t. \qquad (23)$$

So the effect of the motions of the photospheric footpoints is to inject helicity into the system. In a linear approximation

$$\delta K=\frac{iB_0}{\omega}\int dS\{\mathbf{A}_0(\mathbf{v}_+-\mathbf{v}_-)\}+B_0\int dS\,(g_+-g_-), \qquad (24)$$

where $g_{+,-}$ are the gauge function values in the planes $z=\pm h$. Taking a vector potential $\mathbf{A}_0$ of the initial field $\mathbf{B}_0$ in the form $\mathbf{A}_0=\frac{1}{2}[\mathbf{B}_0\times\mathbf{r}]$, we obtain for the first integral in (24):

$$\frac{iB_0}{\omega}\int dS-\{\mathbf{A}_0(\mathbf{v}_+-\mathbf{v}_-)\}=\frac{iB_0}{2\omega}\int dS\,[\mathbf{B}_0\times\mathbf{r}]\cdot 2h[\nabla\Psi\times\mathbf{e}_z]=$$

$$=\frac{-ihB_0^2}{\omega}\int dS\,(\nabla\Psi\cdot\mathbf{r})=\frac{2hiB_0^2}{\omega}\int\Psi\,dS. \qquad (25)$$

The gauge function values $g_{+,-}$ can be found from eq. (12) for the vector potential evolution (resistive term is irrelevant as before). After integration of (12) over time in the planes $z=\pm h$ we obtain:

$$\delta\mathbf{A}_{+,-}=\frac{i}{\omega}[\mathbf{v}_{+,-}\times\mathbf{B}_0]+\nabla g_{+,-} \qquad (26)$$

In our gauge only the z-component of a vector potential is changed (see eq. (10)), thus it follows from (26) that

$$\nabla g_{+,-} = -\frac{i}{\omega}[\mathbf{v}_{+,-} \times \mathbf{B}_0] = \frac{i}{\omega}[\mathbf{B}_0 \times [\nabla \Phi_{+,-} \times \mathbf{e}_z]] = \frac{i}{\omega} B_0 \nabla \Phi_{+,-}, \quad (27)$$

i. e. $g_{+,-} = \frac{i}{\omega} B_0 \Phi_{+,-}$ (an additive constant cancels in the final results). Putting these quantities into eq. (24) we get

$$\delta K = \frac{4ihB_0^2}{\omega} \int \Psi \, dS. \quad (28)$$

From other hand, in our gauge the global helicity of the initial field $K_0 = 0$, since $\mathbf{A}_0 \cdot \mathbf{B}_0 = 0$, and the helicity of the relaxed state $\mathbf{B}^{(r)}$ in linear approximation is:

$$K^{(r)} = \int dV \{\mathbf{A}_0 \cdot \mathbf{B}_1^{(r)} + \mathbf{A}_1^{(r)} \cdot \mathbf{B}_0\} =$$

$$= 2h \int dS \{\frac{1}{2}[\mathbf{B}_0 \times \mathbf{r}] \cdot \alpha B_0 [\nabla \varphi \times \mathbf{e}_z] + \alpha B_0^2 \varphi\} = 4\alpha B_0^2 h \int \varphi \, dS. \quad (29)$$

Since $\delta K = K^{(r)} - K_0 = K^{(r)}$, we obtain finally that

$$\alpha = \frac{i}{\omega} \int \Psi \, dS / \int \varphi \, dS. \quad (30)$$

Now when we know the fields $\mathbf{B}_1^{(M)}$ and $\mathbf{B}_1^{(r)}$, it is possible to find from eq. (4) the magnetic field established in a plasma and thus to obtain the dissipation power in such a system.

3. CALCULATION OF THE HEATING

For the configuration under discussion here the power of a plasma heating is determined by the time averaged Poynting flux from the photospheric planes $z = \pm h$ (Fig. 2):

$$Q = \int dS \{\langle P_z \rangle|_{z=-h} - \langle P_z \rangle|_{z=+h}\}; \quad \mathbf{P} = \frac{c}{4\pi}[\mathbf{E} \times \mathbf{B}]. \quad (31)$$

The electric field $\mathbf{E}$ at the photosphere planes that arise due to the fluid motion is:

$$\mathbf{E}_{+,-} = -\frac{1}{c}[\mathbf{v}_{+,-} \times \mathbf{B}_0] = \frac{1}{c} B_0 \nabla \Phi_{+,-}. \quad (32)$$

So taking into account the expressions obtained above for the magnetic field perturbation we have:

$$Q = \frac{B_0^2 h \tau_r}{4\pi(1+\omega^2\tau_r^2)} \{\int (\nabla \Psi)^2 dS - (\int \Psi \, dS)^2 / \int \varphi \, dS\}. \quad (33)$$

It is seen now that Q represents as a difference of two positive terms (we would remind that $\varphi > 0$ inside the cell). At the same time it is obvious from the physical sense that dissipating power Q cannot be negative. Thus we have to check at first that $Q \geqslant 0$ for an arbitrary stream function $\Psi(x, y)$ (the only restiction is that $\Psi|_S = 0$). Let us consider the following functional:

$$J(\Psi) = \int (\nabla \Psi)^2 dS / (\int \Psi \, dS)^2. \quad (34)$$

From its definition it is clear that the upper limit for $J(\Psi)$ is infinity, so we would be interested in its minimum. Since $J(\Psi)$ is not changed after Ψ being multiplied by a constant, we may find the minimum of J with the restriction that the integral in the denominator of (34) is fixed: $\int \Psi \, dS = \text{const}$. Solving the corresponding variation problem, we obtain:

$$\delta\{\int (\nabla \Psi)^2 dS - \mu \int \Psi \, dS\} = \int dS \, (2 \nabla \Psi \nabla \delta \Psi - \mu \delta \Psi) =$$

$$= -\int dS \, (2\Delta \Psi + \mu) \delta \Psi \quad (35)$$

(we use here the condition that $\delta \Psi|_S = 0$). Thus J gets its extremum (it is easy to see that it is a minimum) under $\Psi = \Psi_m$, if $\Delta \Psi_m = \text{const}$. It means that such a stream function $\Psi_m(x, y)$ is proportional to the function $\varphi(x, y)$ obtained above from eq. (21), i. e. $\Psi_m = \gamma \varphi$. Therefore $\int (\nabla \Psi_m)^2 dS = -\int \Psi_m \Delta \Psi_m \, dS = \gamma^2 \int \varphi dS$ and, as it is seen now from (33), the dissipation power $Q = 0$ when $\Psi = \Psi_m$. So we prove that $Q > 0$ for every stream function $\Psi \neq \Psi_m$ and $Q = 0$ for $\Psi = \Psi_m$. The physical sense of this distinguished stream function is rather simple. In this case ($\Psi = \Psi_m$), as it follows from eqs (10) and (22), the perturbation of a magnetic field $\mathbf{B}_1^{(M)}$, obtained from the perfect MHD equations, becomes equal to the field $\mathbf{B}_1^{(r)}$, that corresponds to the minimum energy state. So there is no relaxation and resulting heating of a plasma in this case.

The general formula (33) for dissipation power shows that a negative contribution to Q is connected with the currents that produce the relaxed field $\mathbf{B}_1^{(r)}$. From the physical point of view it means that under the relaxation process considered here not the whole free energy of a nonpotential magnetic field may be released. The dissipated portion in general case is, roughly speaking, one half of the excess energy, but some times it may be almost the whole one. To see it let us rewrite the expression (33) for Q in terms of velocity field $\mathbf{v}(x, y) = \mathbf{v}_+ - \mathbf{v}_- = 2h[\nabla\Psi \times \mathbf{e}_z]$. Since $\int \Psi \, dS = -\frac{1}{2}\int dS\,(\nabla\Psi \cdot \mathbf{r})$, eq. (33) may be transformed as following:

$$Q = \frac{B_0^2 \tau_r}{16\pi h (1+\omega^2\tau_r^2)} \{\int (\mathbf{v})^2 dS - (\int dS\,[\mathbf{r}\times\mathbf{v}])^2 / 4\int \varphi\, dS\}. \qquad (36)$$

Introducing now the angular velocity $\Omega(x, y)$ by

$$\mathbf{v}(x, y) = [\Omega \mathbf{e}_z \times \mathbf{r}]$$

the integral in the second term in (36) becomes

$$\int dS\,[\mathbf{r}\times\mathbf{v}] = \mathbf{e}_z \int \Omega r_\perp^2 \, dS.$$

So the relaxed magnetic field $\mathbf{B}_1^{(r)}$ is proportional to a weighted average of $\Omega(x, y)$ inside the cell. Therefore the negative contribution to dissipation power would be relatively small if the photospheric motion represents a number of vortices with opposite twists. In the latter case the magnetic relaxation leads to damping of the electric currents in a plasma.

The formula (36), that is obtained above for the one harmonic of the photospheric motions, can be simply generalised for a rather arbitrary case of pulsations. For instance, if we have a periodic motion inside the cell, the contributions to the heating from the different harmonics are independent, so the dissipation power is equal to the sum of expressions like (36) over all the harmonics. Another case of practical interest is the stationary turbulent motion. Writing now the stream function in the form

$$\Psi(x, y, t) = \frac{1}{\sqrt{2\pi}} \int \Psi_\omega(x, y)\, e^{-i\omega t} d\omega; \quad \Psi_{-\omega} = \Psi_\omega^* \qquad (37)$$

the heating power Q may be expressed over the correlation functions

$$\langle \nabla\Psi_\omega \cdot \nabla\Psi_{\omega'} \rangle = \frac{2\pi}{4h^2} (v^2)_\omega\, \delta(\omega+\omega'),$$

$$\frac{\langle \int \Psi_\omega dS \cdot \int \Psi_{\omega'} dS \rangle}{\int \varphi\, dS} = \frac{2\pi}{4h^2} (\Gamma^2)_\omega\, \delta(\omega+\omega'). \qquad (38)$$

The final result becomes as follows:

$$Q = \frac{B_0^2 \tau_r}{8\pi h} \int \frac{d\omega}{1+\omega^2\tau_r^2} \{\int (v^2)_\omega\, dS - (\Gamma^2)_\omega\}. \qquad (39)$$

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