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G.V. Stupakov

FINITE LARMOR RADIUS PLASMA EQUILIBRIUM IN MIRRORS

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ABSTRACT

The problem of FLR effect on plasma equilibrium in mirror geometry is studied. The equation is developped which governs equilibrium pressure distribution in the trap. We find that qualitatively new equilibria occur not only for large Larmor radius plasma but even in the case of small FLR effect. For small Larmor radius, an analytic solution is found that describes MHD equilibrium perturbed by a small scale vortex-like structures.

1. INTRODUCTION

In many modern mirror machines plasmas with relatively large Larmor radius are confined in a magnetic field with small curvature of the lines of force. It has long been known that for such plasmas finite Larmor radius effect (FLR) significantly influences characteristics of the MHD oscillations and, in particular, stabilizes the interchange instability [1]. In the limit of strong FLR effect, when

$$\frac{\varrho_i}{a} \gtrsim \varkappa a, \tag{1}$$

where ϱ_i is the ion Larmor radius, a is the plasma radius, $\varkappa$ is the magnetic field line curvature, all the interchange modes are stable except for $m=1$ mode corresponding to the rigid displacement of the plasma.

Considerably less attention was paid to the FLR effect on plasma equilibrium, although, as it was noted in Ref. [2], this effect is also very important for the equilibrium problem when inequality (1) holds. Only in a few papers [3—5] the effect of FLR was studied in connection with distortion of the equilibrium flux surfaces caused by the parallel current in a long nonaxisymmetric tandem mirrors [6].

In this paper we consider finite Larmor radius plasma equilibrium in a paraxial (in general, nonaxisymmetric) mirror trap. The requirement of paraxiality means that the plasma radius a is small compared with the vacuum magnetic field scale length L, $a \ll L$.

An important small parameter in the problem is the ratio of the ion Larmor radius to the plasma radius:

$$\varepsilon = \frac{\varrho_i}{a} \ll 1.$$

Together with finite value of this ratio we allow for the azimuthal plasma rotation caused by the radial electric field E in the plasma. This field is assumed to be not very large, $E \lesssim W_i/ae$, where W_i is the ion energy. In what follows, in «FLR effects» we also include the effects of plasma rotation having the same order of magnitude as «true» ion finite Larmor radius effects.

The paper is organized as follows. In Sec. 2 the equations are derived governing equilibrium of a collisionless anisotropic FLR plasma in a paraxial trap. The FLR terms in these equations are derived in Sec. 3, in the case of the magnetic field with straight field lines. In Sec. 4, plasma equilibrium in a quadrupole mirror cell is considered and, in Sec. 5, the main results are discussed.

2. EQUILIBRIUM EQUATIONS

Our starting point is the force balance equation which for collisionless plasma reads

$$\nabla \hat{p} = \frac{1}{c}[\vec{j}, \vec{B}], \tag{2}$$

wher $\vec{j}$ is the current density, $\vec{B}$ is the magnetic field and $\hat{p}$ stands for the plasma pressure tensor $((\nabla \hat{p})_\alpha \equiv \partial p_{\alpha\beta}/\partial x_\beta)$. The eq. (2) is an exact consequence of the steady state kinetic equations for electrons and ions if the tensor $p_{\alpha\beta}$ is understood as the density of the momentum flux:

$$p_{\alpha\beta} = \sum_{e,i} m \int v_\alpha v_\beta f \, d^3v, \tag{3}$$

where $f=f(\vec{r}, \vec{v})$ is the distribution function of a plasma species.

Neglecting FLR effects, the tensor $\hat{p}$, as is well known, takes the form (see, for example, [2]):

$$p_{\alpha\beta} = p_{\alpha\beta}^{(0)} \equiv p_\parallel h_\alpha h_\beta + p_\perp (\delta_{\alpha\beta} - h_\alpha h_\beta), \tag{4}$$

where $p_\parallel$ and $p_\perp$ are pressures along and perpendicular to the magnetic field and $\vec{h}=\vec{B}/B$. Formally, one can derive the expression (4) from (3) neglecting the difference between a particle's position and that of its guiding centre and identifying particle distribution function f with the guiding centre distribution function $f_{g.c.}$. The latter does not depend on the gyrophase, $f_{g.c.}=f_{g.c.}(v_\parallel, v_\perp, \vec{r})$ and putting it to (3) yields (4). Allowing for the difference between f and $f_{g.c.}$ gives a correction to $p_{\alpha\beta}^{(0)}$ which we denote by $p'_{\alpha\beta}$:

$$p_{\alpha\beta} = p_{\alpha\beta}^{(0)} + p'_{\alpha\beta}. \tag{5}$$

In general, the tensor $p'_{\alpha\beta}$ has nonzero offdiagonal terms[1]. The magnitude of $p'_{\alpha\beta}$ is small compared with $p_{\alpha\beta}^{(0)}$; one can show that $p'_{\alpha\beta} \sim \varepsilon^2 p_{\alpha\beta}^{(0)}$ [7].

From eq. (2), one can find current density $\vec{j}_\perp$ which is perpendicular to the magnetic field

$$\vec{j}_\perp = \frac{c}{B^2}[\vec{B}, \nabla \hat{p}]. \tag{6}$$

The parallel component of $\vec{j}$ is determined by the charge balance equation

$$\operatorname{div} \vec{j}_\parallel = B\frac{\partial}{\partial s}\frac{j_\parallel}{B} = -\operatorname{div} \vec{j}_\perp, \tag{7}$$

where $\partial/\partial s$ is «along the field line» derivative, $\partial/\partial s = \vec{h}\nabla$. Putting (6) and (5) into (7) after considerable algebra one can cast (7) into the following form

$$\frac{\partial}{\partial s}\left(\frac{j_\parallel}{B}\left[1+4\pi\frac{p_\perp - p_\parallel}{B^2}\right]\right) = -\frac{c}{B^2}\vec{h}[\vec{\varkappa}, \nabla(p_\perp + p_\parallel)] - \\ -\frac{2c}{B^2}\vec{h}[\vec{\varkappa}, \nabla \hat{p}'] + \frac{c}{B^2}\vec{h} \operatorname{rot} \nabla \hat{p}', \tag{8}$$

where $\vec{\varkappa}$ is the curvature of the field line. In the limit $\hat{p}' \to 0$, eq. (8) reduces to the well known equation for $j_\parallel$ (see, e. g., [8]) which says that the parallel current in plasma is due to the nonzero curvature of the field lines. Now, allowing for finite $\hat{p}'$, an important fact is that the last term on the right hand side of eq. (8) does not

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[1] Strictly speaking, the partition of the tensor $p_{\alpha\beta}$ on $p_{\alpha\beta}^{(0)}$ and $p'_{\alpha\beta}$ is not unique since a part of the tensor $p'_{\alpha\beta}$ having the structure (4) can be put into $p_{\alpha\beta}^{(0)}$. For what follows, this is not important.

contain curvature. For that reason, in spite of the smallness of the tensor $\hat{p}'$ compared with $p_\parallel$ and $p_\perp$, in a paraxial system with small curvature, parallel current generation caused by the FLR effects may compare with and even dominate the contribution to $j_\parallel$ coming from the zero Larmor radius magnetohydrodynamics.

We shall henceforth neglect the second term on the right hand side of eq. (8) since it is proportional to the product of two small quantities $\varkappa$ and $\hat{p}'$.

We assume that plasma confined in a mirror trap is isolated from the end plates by a vacuum region where j vanishes. Integrating eq. (8) along a field line over the plasma region and making use of the boundary condition $j_\parallel=0$ at the ends of the integration interval, we find

$$\int \frac{ds}{B^2}\, \vec{h}[\vec{\varkappa}, \nabla(p_\perp+p_\parallel)] - \int \frac{ds}{B^2}\, \vec{h} \text{ rot } \nabla\hat{p}'=0. \qquad (9)$$

This is the main equation governing the plasma equilibrium in the mirror trap. In the small β limit, one can neglect distortions of the magnetic field by the plasma pressure and use the vacuum magnetic field for evaluating integrals in (9). eq. (9) then determines possible pressure distributions in the given magnetic field. If $\beta\sim 1$, then the mirror magnetic field should be selfconsistently found taking into account the currents flowing in the plasma.

3. PLASMA EQUILIBRIUM IN A UNIDIRECTIONAL MAGNETIC FIELD

Since FLR effects, as it follows from eq. (8), survive even in the limit $\varkappa\to 0$ we consider first the simplest case of a plasma equilibrium in the magnetic field having straight field lines [9]. This problem is a zero approximation for studying more complex equilibria in curvilinear magnetic fields.

Let magnetic field be directed along the z axis, $\vec{h}=\vec{e}_z$, and all of the pertinant quantities depend on x and y only. eqs (2) and (4) then reduce to

$$\nabla\left(p_\perp+\frac{B^2}{8\pi}\right)=0, \qquad (10)$$

where we neglected small FLR corrections. As it follows from (10), the functions $p_\perp(x,y)$ and $B(x,y)$ have the same lines of constant

value in x, y plane and introducing a new function $\psi(x,y)$ which is constant along these lines we can put $p_\perp=p_\perp(\psi)$. eq. (10) reduces then to

$$p_{\perp\psi}+\frac{1}{4\pi}BB_\psi=0, \qquad (11)$$

where the subscript ψ denotes differentiation with respect to ψ.

In the case under consideration, eq. (8) takes the form

$$\vec{e}_z \text{ rot } \nabla\hat{p}'=0. \qquad (12)$$

The tensor $\hat{p}'$ in this equation is derived in Appendix 1 basing on the kinetic treatment of the problem similar to Newcomb's approach [10]. In this derivation we use the following statement [11]: not only $p_\perp$ but also the ion and electron distribution functions (and, as a consequence, all their moments) are constant on the lines $\psi=\text{const}$.[2] This is also valid for plasma potential φ, $\varphi=\varphi(\psi)$. Having calculated $\hat{p}'$, eq. (12) can be written in the following form [9, 11]

$$\vec{e}_z \text{ rot } \nabla\hat{p}'=A\frac{\partial(\psi,\Delta\psi)}{\partial(x,y)}+\frac{1}{2}A_\psi\frac{\partial(\psi,(\nabla\psi)^2)}{\partial(x,y)}=0 \qquad (13)$$

with $A=A(\psi)$ given by

$$A=\frac{mc^2}{e}\left[\frac{en}{B^2}\varphi_\psi^2+\frac{1}{B^3}\varphi_\psi(p_\perp B)_\psi+\frac{m}{4eB^3}B_\psi q_\psi\right], \qquad (14)$$

where m and e are the ion mass and charge and

$$\frac{\partial(a,b)}{\partial(x,y)}=\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}.$$

In (14) $n=n(\psi)$ is the plasma density and $q=q(\psi)$ is the forth moment of the ion distribution function:

$$q=\frac{1}{2}m\int v_\perp^4 f\, d^3v.$$

As it follows from eq.(14), there are two independent sources of the FLR effects which contribute to A. In a small β plasma, when one can put $B_\psi=0$ the parameter A is determined by the radial

[2] In Ref. [10] this property was referred to as isorrhopy.

electric field only. The last term in (14) which is proportional to the magnetic field gradient becomes essential at $\beta \simeq 1$.

We shall not discuss here the equilibria governed by eq. (13) because this equation was studied in Ref. [9] where the limit $\beta \ll 1$ has been considered. Finite β adds to A only the last term in (14) but does not alter eq. (13). Hence, the results of Ref. [9] are also valid in the general case of finite β.

4. PLASMA EQUILIBRIUM IN PARAXIAL MIRRORS

For a given mirror magnetic field we employ the flux coordinates μ, ν defined so that $\vec{B}=[\nabla\mu, \nabla\nu]$. The functions $\mu(\vec{r})$ and $\nu(\vec{r})$ are constant along every field line. As a third coordinate we use also the arclength s counted along a field line from the mirror central plane.

As it was noted in the previous section, in the unidirectional magnetic field all the pertinant quantities $(p_\perp, n, \varphi, B)$ depend on x and y through the function ψ. Looking for equilibria in a paraxial mirror we assume that such dependance on ψ persist but, on the other hand, all of the aformentioned quantities vary along field lines being functions of the magnetic field modulus B:

$$p_{\perp,\parallel}=p_{\perp,\parallel}(\psi, B),$$
$$\varphi=\varphi(\psi, B), \qquad (15)$$
$$n=n(\psi, B),$$

where $\psi=\psi(\mu, \nu)$. This assumption, as it was shown in Ref. [2], results from the fact that particle's drift trajectories in a quadrupole mirror lie on the surfaces $\psi=\text{const}$. We, however, do not give a derivation of eqs (15) from kinetic theory, but take them as a constraint on a class of equilibria under consideration. Note that this assumption is also used in Refs [3—5].

Since the last term in eq. (8) is small in parameter ε^2, we do not take into account the effect of finite curvature in this term and use for $\vec{h}\,\text{rot}\,\nabla\hat{p}'$ the expression (13) from the planar problem. The ψ derivative in (13) now is to be understood as partial and the gradient and Laplacian in (13) should be changed by $\nabla_\perp$ and $\Delta_\perp$, respectively, acting on x and y coordinates only (we assume that the axis of the trap points to z-direction).

In order to transform eq. (8) to coordinates μ, ν we employ the relation

$$\frac{\partial(\mu,\nu)}{\partial(x,y)}=B$$

which is valid in paraxial approximation [2]. This casts eq. (8) to

$$\frac{\partial}{\partial s}\left(\frac{j_\parallel}{B}\left[1+4\pi\frac{p_\perp-p_\parallel}{B^2}\right]\right)=-\frac{c}{B^2}\vec{h}\,[\vec{\varkappa}, \nabla_\perp\psi]\,(p_{\perp\psi}+p_{\parallel\psi})+$$
$$+\frac{c}{B}\left[A\frac{\partial(\psi,\Delta_\perp\psi)}{\partial(\mu,\nu)}+\frac{1}{2}A_\psi\frac{\partial(\psi,(\nabla_\perp\psi)^2)}{\partial(\mu,\nu)}\right] \qquad (16)$$

and eq. (9) takes the form

$$\int\frac{ds}{B^2}\vec{h}\,[\vec{\varkappa}, \nabla\psi]\,(p_{\perp\psi}+p_{\parallel\psi})-$$
$$-\int\frac{ds}{B}\left[A\frac{\partial(\psi,\Delta_\perp\psi)}{\partial(\mu,\nu)}+\frac{1}{2}A_\psi\frac{\partial(\psi,(\nabla_\perp\psi)^2)}{\partial(\mu,\nu)}\right]=0 \qquad (17)$$

To proceed further analytically we constraint ourselves by the limiting case $\beta\ll 1$ and consider a quadrupole mirror cell having Ing-Yang symmetry. Using the conventional representation of the magnetic field of such a mirror (see, e. g., [8]) one can perform integration along field lines in (17) and obtain an equation for the function ψ (see the derivation in Appendix 2)

$$D(\psi)\frac{\partial\psi}{\partial\theta}-\bar{A}\frac{\partial(\psi,\Delta_0\psi)}{\partial(\xi,\eta)}-\frac{1}{2}\bar{A}_\psi\frac{\partial(\psi,(\nabla_0\psi)^2)}{\partial(\xi,\eta)}=0. \qquad (18)$$

Here ψ is supposed to be a function of field line coordinates in the central plane of the mirror ξ, η (rectangular coordinates) or r, θ (polar coordinates in this plane); Δ_0 and ∇_0 are two dimensional Laplacian and gradient, respectivly. The function $D(\psi)$ is given by (2.10) and the bar over A means an averaging determined by eq. (2.8). Note that in our limiting case $\beta\ll 1$ one can neglect the last term in eq. (14):

$$A=\frac{mc^2}{e}\left(\frac{en}{B^2}\varphi_\psi^2+\frac{1}{B^2}\varphi_\psi\,p_{\perp\psi}\right).$$

The first term in (18) describes MHD equilibria in pure MHD approximation without FLR. It vanishes when $\psi=\psi(r)$, i. e. when the cross section of the surfaces $\psi=\text{const}$ constitute a set of concentric circles. We shall refer to the case $\psi=\psi(r)$ as the MHD solution. Note that this is also a solution of eq. (18) since both Jacobians in (18) vanish when $\psi=\psi(r)$. However, as we shall show later eq.(18) has also other solutions different from MHD ones.

Let us examine the relative order of different terms in eq. (18). Taking as a rough estimate $\partial\psi/\partial\eta\sim\partial\psi/\partial\xi\sim\psi/a$ and assuming $e\varphi\sim W_i$ one can find that the ratio of any of the last two terms in (18) to the first one in this equation is equal to[3] the parameter $\varrho_i^2/\varkappa a^3$ which determines the role of FLR effects in the interchange oscillations (see (1)). In the case $\varrho_i^2\gg\varkappa a^3$ one can neglect, in zero approximation, the first term in eq.(18). It then reduces to eq. (13) in which A should be changed by the averaged quantity $\bar A$. In this limit the curvature effects completly come out from the equations.

In the other limiting case, $\varrho_i^2\ll\varkappa a^3$, when the last two terms are small compared with the first one, at a first glance, one can neglect these terms and retain only MHD contribution. However, the situation is more subtle, since there is highest (forth order) derivative in the second term of the equation, so neglecting this term decreases the order of the equation which, in general, results in loosing solutions.

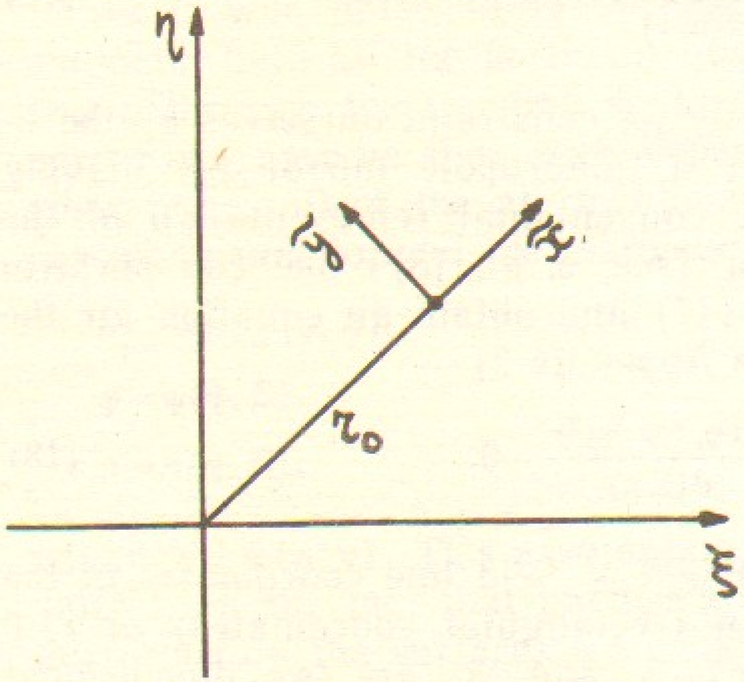

Fig. 1. Local coordinate system $\tilde x$, $\tilde y$.

We present now an example which shows that eq. (18) does have solutions different from $\psi=\psi(r)$ even in the limit $\varrho_i^2/\varkappa a^3\to 0$. These solutions describe small scale structures localized on the MHD solution background with a scale length l small compared with the plasma radius a (but large compared with the ion Larmor radius ϱ_i).

Given a MHD-solution $\psi=\Psi(r)$ with corresponding functions $\bar A(\psi)$, $D(\psi)$, we choose a local coordinate system $\tilde x$, $\tilde y$ so that the axis $\tilde y$ points in θ direction (see Fig. 1).

[3] If one assumes $e\varphi\ll W_i$, then this ratio is equal to $\varrho_i^2 e\varphi/W_i\varkappa a^3$.

In this local system, $\partial/\partial\theta=r_0\partial/\partial\tilde y$ where r_0 is the radius of the origin of the coordinate system. Since the radial length scale of the functions $\bar A$ and D is equal to the plasma radius a, these functions can be considered to be constant in the vicinity of the origin of the local system, the constants being the values of $\bar A$ and D at the point $r=r_0$. For constant $\bar A$ and D, in $\tilde x$, $\tilde y$ coordinates eq. (18) comes to

$$\gamma\frac{\partial\psi}{\partial\tilde y}+\frac{\partial(\psi,\tilde\Delta\psi)}{\partial(\tilde x,\tilde y)}=0, \tag{19}$$

where

$$\gamma=-r_0\frac{D(\Psi(r_0))}{\bar A(\Psi(r_0))},\qquad \tilde\Delta=\frac{\partial^2}{\partial\tilde x^2}+\frac{\partial^2}{\partial\tilde y^2}.$$

The boundary condition for eq.(19) reads: $\psi\to\tilde x\,\partial\Psi/\partial r|_{r=r_0}$ at $\tilde x^2+\tilde y^2\to\infty$. It means that at large distances the function ψ fits the MHD-solution which locally looks like a linear profile.

Using a method proposed in Ref. [12] one can find a class of localized solutions of eq. (19). These solutions exist only if $\gamma/\Psi'<0$ and have a form

$$\psi=\left[\varrho-\frac{\lambda}{K_1(\lambda)}K_1(\varrho)\right]\cos\alpha,\qquad \varrho>\lambda;$$

$$\psi=\frac{1}{k^2}\left[-\varrho+\frac{\lambda}{J_1(k\lambda)}J_1(k\varrho)\right]\cos\alpha,\qquad \varrho<\lambda; \tag{20}$$

where

$$\varrho=\sqrt{\tilde x^2+\tilde y^2},\qquad \alpha=\operatorname{arc\,tg}\frac{\tilde y}{\tilde x},$$

and parameter k is defined by

$$\frac{1}{k}\,\frac{J_2(k\lambda)}{J_1(k\lambda)}=-\frac{K_2(\lambda)}{K_1(\lambda)},$$

where K_1 and K_2 are McDonald functions, J_1 and J_2 are Bessel functions and λ is an arbitrary parameter. The characteristic length scale of the solution (20) is: $l\sim\sqrt{\varrho_i^2/\varkappa a}$. Figure 2 shows the equipotential lines of the solution (20) and ψ vs. $\tilde x$ at $\tilde y=0$.

The structure of the equilibrium corresponding to MHD-solution with a perturbation given by eq. (20) is illustrated by Fig. 3 which shows that the perturbation consists of two vortices immersed in the background axisymmetric equilibrium configuration. Note that, in

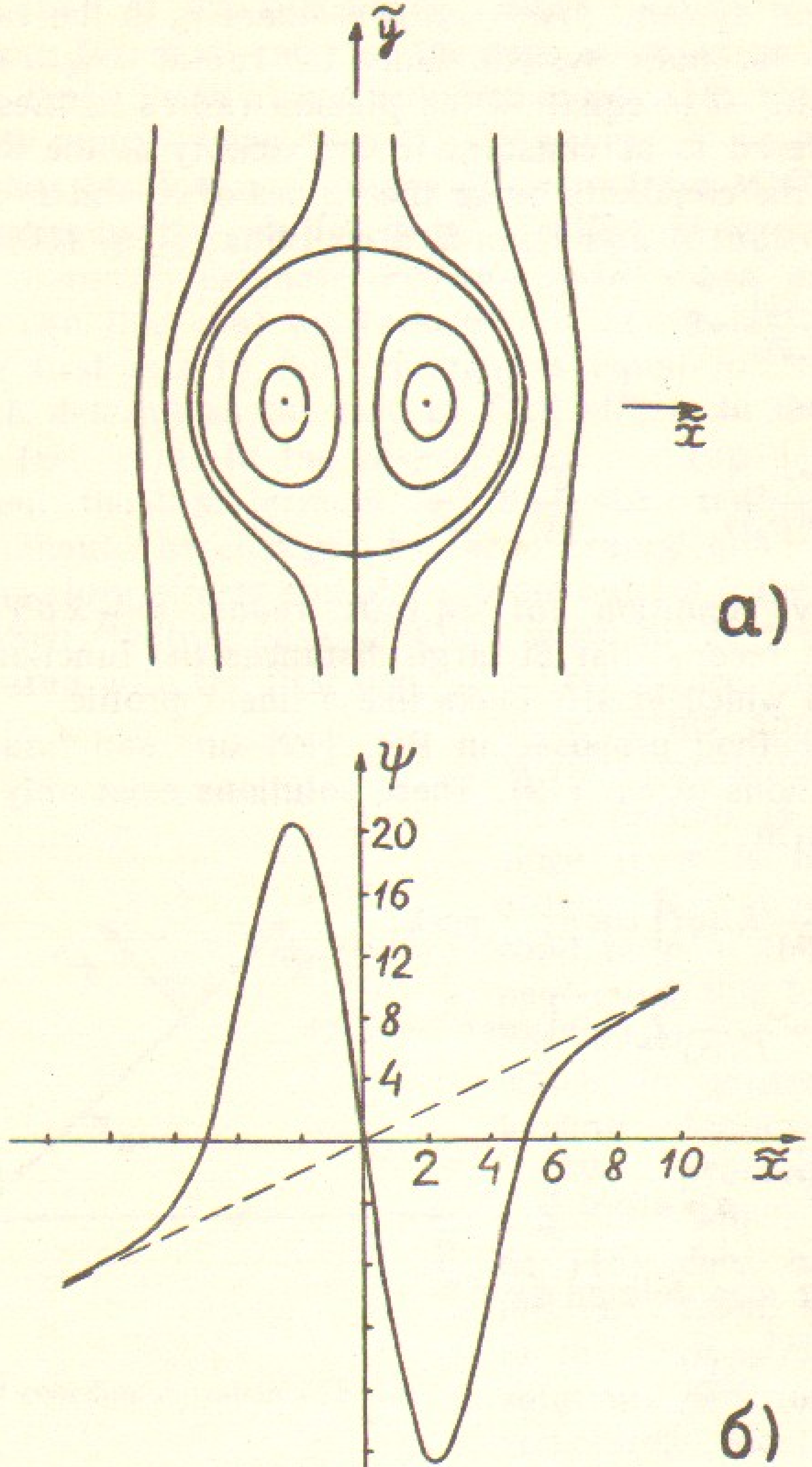

Fig. 2. The lines of constant ψ (a) and ψ vs. $\tilde{x}$ at $\tilde{y}=0$ (b) given by eqs (20) for $\lambda=5$ ($k=0{,}88$).

addition to arbitrary parameter λ, the position of the vortices in the ξ, η plane is also arbitrary. Due to small scale of the vortices (compared with the plasma radius) it is evident that the equilibrium equations exhibit also solutions corresponding to many nonoverlapping pairs of vortices.

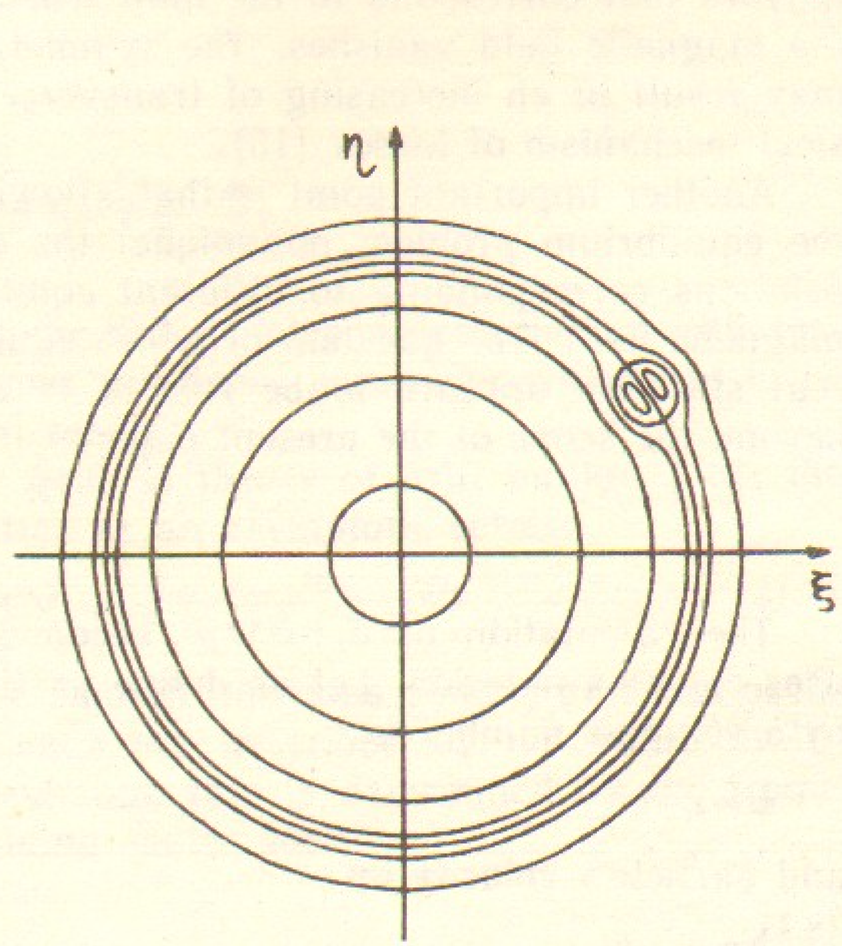

Fig. 3. The global picture of the lines of constant ψ in the mirror midplane for MHD equilibrium with FLR local perturbation (20).

Concluding this section, we note a close relation between our solution (20) and flute solitons studied in the Ref. [13] (see also [14]).[4] This relation becomes clear if one transform to a frame moving with the soliton velocity; in this frame the soliton is at rest and it can be considered as an equilibrium configuration.

5. CONCLUSION

The most remarcable feature of the equilibria governed by the FLR effects is a breaking of the initial symmetry of the MHD-solutions. These solutions, as it follows from the results of the previous sections, are axisymmetric in a sence that the lines of constant pressure in the mirror midplane are concentric circles. Vortices arising

[4] This relation was pointed out to the author by V.P. Pastukhov.

in the limit $\varrho_i^2 \ll a^3 \varkappa$ clearly have not this symmetry. One should emphasize that although we have concentrated on the quadrupole mirror geometry above our results are also applicable to axisymmetric mirrors that correspond to the limit when quadrupole component of the magnetic field vanishes. The symmetry breaking in equilibrium may result in an increasing of transverce transport due to neoclassical mechanism of losses [15].

Another important point is that allowing for FLR effects makes the equilibrium problem nonunique: the equations now have many solutions corresponding to different equilibria in the same vacuum magnetic field. The question of which equilibrium is established in a real situation appears to be related to stability analysis that lies beyond the scope of the present paper.

APPENDIX 1

The calculation of $\vec{e}_z \operatorname{rot} \nabla \hat{p}'$ becomes much more easy if one uses complex notation. Let us define particle's position in x, y plane by a complex number w:

$$w = x - iy, \tag{1.1}$$

and particle's velocity by

$$\dot{w} = v_x - iv_y, \tag{1.2}$$

We introduce also «complex pressure» P instead of the tensor $\hat{p}$ given by eq. (3)

$$\mathrm{P} = m \int \dot{w}^2 f\, d^3v = m \int (v_x^2 - v_y^2 - 2iv_xv_y)\, f\, d^3v = p_{xx} - p_{yy} - 2ip_{xy}, \tag{1.3}$$

where f is the ion distribution function. In eq. (3) the electron contribution is neglected since FLR effects are due to the ion species only. Defining complex differentiation

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial w^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{1.4}$$

one finds that

$$\vec{e}_z \operatorname{rot} \nabla \hat{p} = -2 \operatorname{Im} \frac{\partial^2 \mathrm{P}}{\partial w^2}. \tag{1.5}$$

Before proceeding to the calculation of the complex pressure P we should derive the ion drift equations within the order ε^2 (to the

first order in ε these equations are given by the standard drift theory).

The ion equation of motion in the complex form can be written as

$$\frac{\varepsilon}{\Omega}\ddot{w} - i\dot{w} = \varepsilon \mathrm{E}, \tag{1.6}$$

where

$$\Omega = \frac{eB}{mc}, \quad \mathrm{E} = \frac{c}{B}(E_x - iE_y) = -2\frac{c}{B}\frac{\partial \varphi}{\partial w^*},$$

φ is electrostatic potential, the dot denotes differentiation with respect to time and ε is a formal smallness parameter which has to be equated to unity in the final result.

In accordance with the general theory of drift motion [16], the solution of eq. (1.6) is written as an asymptotic series:

$$w = w_0 + \varepsilon w_1 e^{i\tau} + \varepsilon^2 w_2 e^{2i\tau} + \varepsilon^3 w_{-1} e^{-i\tau} + \varepsilon^3 w_3 e^{3i\tau} + o(\varepsilon^3), \tag{1.7}$$

where w_k $(k = 0, \pm 1, \ldots)$ are smooth functions of time and fast oscillations with Larmor frequency are described by the phase multipliers $\exp(ik\tau)$. The phase change rate is determined by the frequency Ω at the particle's guiding center position:

$$\dot{\tau} = \frac{1}{\varepsilon}\Omega_0 \equiv \frac{1}{\varepsilon}\Omega(x_0, y_0), \tag{1.8}$$

where x_0 and y_0 are connected with the first term in the series (1.7):

$$x_0 = \frac{1}{2}(w_0 + w_0^*), \quad y_0 = -\frac{1}{2i}(w_0 - w_0^*). \tag{1.9}$$

The amplitudes w_k with $k > 1$ and $k < 0$ should be expressed through w_0, w_1 and their derivatives at the guiding centre position as asymptotic series in nonnegative powers of ε. The time evolution of w_0 and w_1 is given by the following equation:

$$\dot{w}_0 = \varepsilon A_0 + o(\varepsilon^2),$$
$$\dot{w}_1 = \varepsilon A_1 + o(\varepsilon^2), \tag{1.10}$$

where A_0 and A_1 depend on x_0, y_0, $\operatorname{Re} w_1$, $\operatorname{Im} w_1$. The problem now consists of calculation of A_0, A_1 and also w_1, w_2, w_3 in (1.7).

Putting the series (1.7) in eq. (1.6) and making Taylor expansion of the functions Ω and E with respect to the difference $w - w_0$ to the second order in parameter ε, one finds

$$A_0 = i\,\mathrm{E} - i\,|w_1|^2 \frac{\partial \Omega}{\partial w^*},$$

$$w_2 = \frac{w_1^2}{2\Omega}\frac{\partial \Omega}{\partial w}, \tag{1.11}$$

$$w_{-1} = \frac{1}{4} w_1 w_1^{*2} \frac{1}{\Omega^2}\left(\frac{\partial \Omega}{\partial w^*}\right)^2 - \frac{1}{4} w_1 w_1^{*2}\,\Omega \frac{\partial^2 \Omega^{-1}}{\partial w^{*2}} - \frac{1}{2}\frac{w_1^*}{\Omega}\frac{\partial \mathrm{E}}{\partial w^*}.$$

We do not give here expressions for A_1 and w_3 since we shall not need them in what follows. The functions E and Ω on the right hand side of eqs (1.11) are computed at the guiding centre position.

Let us now introduce real variables u and γ so that

$$w_1 = \frac{u}{\Omega_0} e^{i\gamma}. \tag{1.12}$$

We shall characterize the guiding centre position by parameters x_0, y_0, u and $\theta \equiv \tau + \gamma$.

In order to calculate P, we utilize guiding centre distribution function $f_{\text{g.c.}}$ which, in steady state, depends on x_0, y_0 and u [9]. With this function, the number dN of the guiding centres in the infinitesimel area $dx_0\,dy_0$ having values u, θ in the intervals du, $d\theta$ is equal to

$$dN = f_{\text{g.c.}}(x_0, y_0, u)\, dx_0\, dy_0\; u\, du\, d\theta. \tag{1.13}$$

It is easy to see that P is given by the following formula

$$\mathrm{P}(x', y') = m \int dx_0\, dy_0\; u\, du\, d\theta\; f_{\text{g.c.}}\, \dot{w}^2 \delta(x - x')\, \delta(y - y'), \tag{1.14}$$

where the δ-function accounts for the difference between a particle's position in x, y plane and that of its guiding centre. We recall that x and y coordinates in (1.14) are functions of x_0, y_0, u, θ given by eq. (1.7) in the complex form. We find the complex velocity $\dot{w}$ to the order of ε^2 by differentiating (1.7) with respect to time and using (1.8) and (1.10):

$$\dot{w} = -i\, w_1 \Omega_0 e^{i\tau} + \varepsilon A_0 + 2 i \varepsilon\, \Omega_0 w_2 e^{2i\tau} + \varepsilon^2 A_1 e^{i\tau} -$$

$$- i \varepsilon^2 \Omega_0 w_{-1} e^{-i\tau} + 3\varepsilon^2 i\, w_2 \Omega_0 e^{3i\tau} + o(\varepsilon^2). \tag{1.15}$$

Now, it is useful to come from the real quantities x, y, x', y', x_0,

y_0 in eq. (1.14) to the complex variables w, w', w_0 using an identity

$$dx_0\, dy_0\, \delta(x - x')\, \delta(y - y') = dw_0\, dw_0^*\, \delta(w - w')\, \delta(w^* - w'^*), \tag{1.16}$$

so that

$$\mathrm{P}(w', w'^*) = m \int dw_0\, dw_0^*\, u\, du\, d\theta\; f_{\text{g.c.}}\, \dot{w}^2 \delta(w' - w)\, \delta(w'^* - w^*). \tag{1.17}$$

Finally, putting (1.15) and (1.7) into (1.17), expanding δ-function in the Taylor series according to

$$\delta(w + \varepsilon z) = \delta(w) + \varepsilon z \frac{\partial}{\partial w}\delta(w) + \frac{1}{2}\varepsilon^2 w^2 \frac{\partial^2}{\partial w^2}\delta(w) + \dots \tag{1.18}$$

and integrating over w_0, w_0^* and θ, we find

$$\mathrm{P}(w, w^*) = \varepsilon^2 \Big[\frac{\partial}{\partial w^*}\langle w_1^2 w_2^* \Omega^2 \rangle - \frac{1}{2}\frac{\partial^2}{\partial w^{*2}}\langle |w_1|^4 \Omega^2 \rangle -$$

$$- 2i \frac{\partial}{\partial w^*}\langle A_0 \Omega\, |w_1|^2 \rangle + \langle A_0^2 \rangle + 2\langle w_1 w_{-1} \Omega^2 \rangle \Big], \tag{1.19}$$

where $\langle \dots \rangle$ denotes an averaging

$$\langle \dots \rangle = 2\pi m \int u\, du\; f_{\text{g.c.}}(w, w^*, u) \dots \tag{1.20}$$

Defining the moments of the function $f_{\text{g.c.}}$

$$n = 2\pi \int u\, du\; f_{\text{g.c.}},$$

$$p_\perp = \pi m \int u^3\, du\; f_{\text{g.c.}}, \tag{1.21}$$

$$q = \pi m \int u^5\, du\; f_{\text{g.c.}},$$

we rewrite (1.19) as follows

$$\mathrm{P}(w, w^*) = \varepsilon^2 \Big[-mn\,\mathrm{E}^2 - \frac{2p_\perp}{\Omega}\frac{\partial \mathrm{E}}{\partial w^*} + 4\frac{\partial}{\partial w^*}\frac{\mathrm{E}\,p_\perp}{\Omega} + 4p_\perp \frac{\mathrm{E}}{\Omega^2}\frac{\partial \Omega}{\partial w^*} -$$

$$- \frac{\partial^2}{\partial w^{*2}}\frac{q}{\Omega^2} - 3\frac{\partial}{\partial w^*}\left(\frac{q}{\Omega^3}\frac{\partial \Omega}{\partial w^*}\right) - \frac{q}{\Omega^4}\left(\frac{\partial \Omega}{\partial w^*}\right)^2 - \frac{q}{\Omega}\frac{\partial^2 \Omega^{-1}}{\partial w^{*2}} \Big]. \tag{1.22}$$

Now, let us use the isorrhopy condition supposing that

$$n = n(\psi),\quad p_\perp = p_\perp(\psi),\quad q = q(\psi),\quad \varphi = \varphi(\psi), \tag{1.23}$$

and the function ψ depends on w and w^*. Taking into account (1.23) one easily finds that eq. (1.22) takes the form:

$$P=\lambda(\psi)\left(\frac{\partial\psi}{\partial w^*}\right)^2+\frac{\partial}{\partial w^*}\chi(\psi)\frac{\partial\psi}{\partial w^*}, \tag{1.24}$$

where λ and χ are real functions and

$$\lambda=-\frac{4mc^2}{e}\left(\frac{en\varphi_\psi'^2}{B^2}+\frac{\varphi_\psi}{B^3}(p_\perp B)_\psi+\frac{1}{4}\frac{m}{e}\frac{q_\psi B_\psi}{B^3}\right). \tag{1.25}$$

The prime in (1.25) denotes differentiation with respect to ψ. Acting by the operator $\operatorname{Im}\partial^2/\partial w^2$ on (1.24) we vanish the second term and the first term gives

$$\operatorname{Im}\frac{\partial^2 P}{\partial w^2}=\operatorname{Im}\left(2\lambda\frac{\partial\psi}{\partial w}\frac{\partial^3\psi}{\partial w^2\partial w^*}+\lambda_\psi\frac{\partial\psi}{\partial w^*}\frac{\partial}{\partial w}\left(\frac{\partial\psi}{\partial w}\frac{\partial\psi}{\partial w^*}\right)\right)=$$

$$=\frac{1}{8}\left(\lambda\frac{\partial(\psi,\Delta\psi)}{\partial(x,y)}+\frac{1}{2}\lambda_\psi\frac{\partial(\psi,(\nabla\psi)^2)}{\partial(x,y)}\right). \tag{1.26}$$

This result, together with (1.25) and (1.5) is equivalent to (13), (14).

APPENDIX 2

In paraxial approximation, the vacuum magnetic field of a quadrupole mirror is given by the magnetic potential χ:

$$\chi=\int^z\tilde{B}(\zeta)\,d\zeta-\frac{1}{4}\tilde{B}'(x^2+y^2)-\tilde{b}(x^2-y^2), \tag{2.1}$$

where $\vec{B}=\nabla\chi$, x, y and z are cartezian coordinates with z-axis directed along the mirror axis. In eq. (2.1) $\tilde{B}=\tilde{B}(z)$ is the magnetic field strength on the mirror axis, $\tilde{b}=\tilde{b}(z)$ determines the quadrupole component of the field and the prime denotes differentiation with respect to z. So called Ing-Yang symmetry means that the functions $\tilde{B}$ and $\tilde{b}$ are even. Magnetic field line coming from the point with coordinates $x=\xi$, $y=\eta$ in the mirror midplane is given by the following equations [8]:

$$x=\xi\sqrt{\frac{\tilde{B}_0}{\tilde{B}}}e^{-\Phi/2},$$

$$y=\eta\sqrt{\frac{\tilde{B}_0}{\tilde{B}}}e^{\Phi/2}, \tag{2.2}$$

where $\tilde{B}_0=\tilde{B}(0)$ and

$$\Phi(z)=4\int_0^z\frac{\tilde{b}(\zeta)}{\tilde{B}(\zeta)}\,d\zeta.$$

From (2.2) one finds how ξ and η depend on x, y, z:

$$\xi=\xi(x,z)=x\sqrt{\frac{\tilde{B}}{\tilde{B}_0}}e^{\Phi/2},$$

$$\eta=\eta(y,z)=y\sqrt{\frac{\tilde{B}}{\tilde{B}_0}}e^{-\Phi/2}. \tag{2.3}$$

One can identify the flux coordinates μ and ν with $\sqrt{\tilde{B}_0}\,\xi$ and $\sqrt{\tilde{B}_0}\,\eta$, respectively, so that

$$\frac{\partial(f,g)}{\partial(\mu,\nu)}=\frac{1}{\tilde{B}_0}\frac{\partial(f,g)}{\partial(\xi,\eta)}.$$

Let us now calculate $\Delta_\perp\psi$ and $(\nabla_\perp\psi)^2$, considering ψ as a function of ξ and η and taking into account linear dependence ξ on x and y on η:

$$\Delta_\perp\psi=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\frac{\partial^2\psi}{\partial\xi^2}\left(\frac{\partial\xi}{\partial x}\right)^2+\frac{\partial^2\psi}{\partial\eta^2}\left(\frac{\partial\eta}{\partial y}\right)^2,$$

$$(\nabla_\perp\psi)^2=\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2=\left(\frac{\partial\psi}{\partial\xi}\right)^2\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial\eta}\right)^2\left(\frac{\partial\eta}{\partial y}\right)^2. \tag{2.4}$$

Using (2.4) we find Jacobian in (17)

$$\frac{\partial(\psi,\Delta_\perp\psi)}{\partial(\mu,\nu)}=\frac{1}{\tilde{B}_0}\left[\frac{\partial(\psi,\partial^2\psi/\partial\xi^2)}{\partial(\xi,\eta)}\left(\frac{\partial\xi}{\partial x}\right)^2+\frac{\partial(\psi,\partial^2\psi/\partial\eta^2)}{\partial(\xi,\eta)}\left(\frac{\partial\eta}{\partial y}\right)^2\right],$$

$$\frac{\partial(\psi,(\nabla_\perp\psi)^2)}{\partial(\mu,\nu)}=$$

$$=\frac{1}{\tilde{B}_0}\left[\frac{\partial(\psi,(\partial\psi/\partial\xi)^2)}{\partial(\xi,\eta)}\left(\frac{\partial\xi}{\partial x}\right)^2+\frac{\partial(\psi,(\partial\psi/\partial\eta)^2)}{\partial(\xi,\eta)}\left(\frac{\partial\eta}{\partial y}\right)^2\right]. \tag{2.5}$$

Note, that according to (15) A is a function of ψ and B. Since, with required accuracy, $B=\tilde{B}(z)$ and $\tilde{B}(z)$ is an even function, A is an odd function of z. Taking $ds\simeq dz$ and using (2.5) we find

$$\int \frac{ds}{B}\left[A\frac{\partial(\psi, \Delta_\perp \psi)}{\partial(\mu, \nu)} + \frac{1}{2}A_\psi \frac{\partial(\psi, (\nabla_\perp \psi)^2)}{\partial(\mu,\nu)}\right] =$$

$$= \bar{A}\frac{\partial(\psi, \Delta_0\psi)}{\partial(\xi,\eta)} + \frac{1}{2}\bar{A}_\psi \frac{\partial(\psi, (\nabla_0\psi)^2)}{\partial(\xi,\eta)} \tag{2.6}$$

where

$$\Delta_0\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}, \qquad (\nabla_0\psi)^2 = \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2. \tag{2.7}$$

and the bar denotes averaging

$$\bar{A} = \frac{1}{\tilde{B}_0^2}\int dz\, A e^{\Phi}. \tag{2.8}$$

In deriving (2.6) we took into account that Φ is an odd function of z.

As far as the first integral in (17) is concerned, it is, in fact, calculated in Appendix 1 of [8]. Here we give only the final result

$$\int \frac{ds}{B^2} h[\varkappa, \nabla\psi]\,(p_{\perp\psi} + p_{\parallel\psi}) = D\frac{\partial\psi}{\partial\theta}, \tag{2.9}$$

where

$$D = \int \frac{dz}{\tilde{B}^4}(p_{\perp\psi} + p_{\parallel\psi})\left[\left(4\tilde{b}^2 - \frac{1}{2}\tilde{B}\tilde{B}'' + \frac{3}{4}\tilde{B}'^2\right)\operatorname{ch}\Phi - \right.$$

$$\left. - (4\tilde{B}'\tilde{b} - 2\tilde{B}\tilde{b}')\operatorname{sh}\Phi\right]. \tag{2.10}$$

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G.V. Stupakov

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Г.В. Ступаков

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