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# SOLUTION OF THE ANOMALY PUZZLE IN SUSY GAUGE THEORIES AND THE WILSON OPERATOR EXPANSION

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Solution of the Anomaly Puzzle in SUSY Gauge Theories
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A b s t r a c t

The present paper completes a series of works on $\beta$ functions and the anomaly problem in supersymmetric theories. Exact expressions for the $\beta$ functions are obtained within the framework of the standard perturbation theory. The key observation is that the Wilson effective action $S_W(\mu)$ does not coincide with the sum of vacuum loops in the external field $\Gamma(\mu)$. The difference is due to infrared effects. The coefficient $1/g^2$ in front of the operator $W^2$ in $S_W$ is renormalized only at one-loop level (extension of the non-renormalization theorem for F terms). This fact results in one-loop form of the anomalous operator equation for the supercurrent (generalization of the Adler-Bardeen theorem). The full Gell-Mann-Low function emerges after passing to matrix elements of the operators. The quantity entering observable amplitudes differs from $1/g^2$ by $\sum_i \ln Z_i$ where the factors $Z_i$ describe renormalization of the fields. (In this sense the Z factors of the matter fields become observable). We discuss relation with the calculations of the instanton type.

## I. Introduction

It is well-known that the perturbative series in supersymmetric (SUSY) models [1] possess miraculous properties. Thus, for F terms the loop corrections are absent at all (the so called non-renormalization theorems [2]), while the Gell-Mann-Low functions in N=2 gauge theories are exhausted by the first loop [3]. In the present work we will discuss calculation of the effective action in N=1 SUSY gauge theories, i.e., in particular, the gauge coupling constant renormalization.

In the literature the term "effective action" is used, actually, in two distinct senses. According to one procedure we calculate vacuum loops in external (background) fields. The functional of the external fields obtained in this way $\Gamma(\mu)$ is often called "effective action", although more exact is another name — generator of one-particle-irreducible vertices. We will stick to the latter terminology. The second construction is calculation of the effective action $S_W(\mu)$ a lá Wilson [4]. The difference between $S_W(\mu)$ and $\Gamma(\mu)$ is due to the fact that in the vacuum loops for $S_W$ we keep only the contribution of virtual momenta $p > \mu$. The action $S_W(\mu)$ is the normal action with respect to the low-frequency fields. The subscript W introduced above emphasizes the distinction between the two notions. Thus, within the framework of the Wilson procedure we deal with the normal operator product expansion. In order to pass from $S_W$ to $\Gamma$ one must take matrix elements of $exp(i S_W(\mu))$

$$e^{i\Gamma(\mu)} = \langle e^{i S_W(\mu)} \rangle$$

Let us emphasize that the difference of two definitions is due to the contribution of the infrared domain $p \lesssim \mu$ (p is the momentum flowing in loops).

In particular, in SUSY gauge theories the coefficients in front of $W^{\alpha}W_{\alpha}$ ( $1/g^2$ where $g$ is the gauge constant) are different in $S_W$ and $\Gamma$ (starting from the second loop). We will show that the coefficient of $W^2$ in $S_W$ is renormalized only at one loop. The conventional gauge coupling is defined from $\Gamma$. Two-loop and higher order contributions to the latter coupling correspond in the Wilson language to calculation of some matrix elements.

Quite clearly, it is convenient to study the ultraviolet behaviour of the theory in terms of $S_W$. Moreover, in solving the anomaly problem, the use of $S_W$ becomes the necessity, not the question of convenience, if we are going to deal with the anomaly equations in the operator form. As a reflection of the fact that the coefficient of $W^2$ in $S_W$ is renormalized only at one loop the anomaly in the supertrace $D^{\dot\alpha} J_{\alpha\dot\alpha}$ contains the operator $W^2$ with the one-loop coefficient. Just this observation solves the problem of higher orders.

The issues touched upon above have a rather long history. Let us sketch the situation which will allow us to formulate the results in more concrete form.

We know,already for a few years , the exact expression for the Gell-Mann-Low function $\beta(\alpha)$ in SUSY non-abelian models with matter,

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\,\frac{3T(G) - \sum_i T(R_i)(1-\gamma_i)}{1-(T(G)\,\alpha/2\pi)} \qquad (1)$$



where the sum in the right-hand side runs over all matter multiplets, $\gamma_i$ is the anomalous dimension of the i-th matter superfield,

$$\gamma_i(\alpha) = -\frac{d\ln Z(\mu)}{d\ln\mu} = -C_2(R_i)\frac{\alpha}{\pi} + \dots ,$$

$C_2(R_i)$ is the quadratic Casimir operator,

$$T^a T^a = C_2(R_i)\,I ,$$

while the coefficients $T(R_i)$ determine the normalization of the generators:

$$Tr\,(T^a T^b) = T(R)\,\delta^{ab} .$$

We have introduced $T(G) = T(\text{Adjoint})$. Recall that

$$T(R_i) = C_2(R_i)\,\frac{dim\,(R_i)}{dim\,(Adjoint)} .$$

In supersymmetric gluodynamics (hereafter referred to as SSYM), in particular, the exact $\beta$ function is fixed unambiguously,

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\,\frac{3T(G)}{1-(T(G)\alpha/2\pi)} . \qquad (2)$$

Eqs.(1),(2) have been obtained in ref.[5] within the framework of instanton calculus. From the very beginning, however, it was clear that there should exist a direct derivation in the standard perturbation theory. The fact that the $\beta$ function can be written in a simple form, - for instance, it reduces to a geometrical progression (2) in SSYM- certainly, could not be accidental, and hence, one has to give answers to the obvious questions:

- How relations like (1) and (2) emerge in perturbation theory?
- What properties of the theory are responsible for the specific structure of the $\alpha$ series for $\beta(\alpha)$ ? Do these properties show up in other quantities?

A partial answer, primarily to the first question, has been given in refs. [6,7]. These works reproduce eqs.(1),(2) with no reference to instantons. Here we shift the emphasis from the computational aspect to the conceptual one.

Another line of research which also led us to the necessity of carrying out this work is the notorious anomaly problem in SUSY theories.

About 10 years ago Ferrara and Zumino have noticed that the classical supercurrent $S_{\mu\alpha}$ and the energy-momentum tensor $\theta_{\mu\nu}$ are connected with each other by a supersymmetry transformation [8]. In ref. [8] a prescription has been formulated accoridng to which one could construct the supermultiplet $\mathcal{J}_{\alpha\dot{\alpha}}$ including $S_{\mu\alpha}$, $\theta_{\mu\nu}$, and, apart that, the axial current $a_\mu$.

All three objects, $a_\mu$, $S_{\mu\alpha}$ and $\theta_{\mu\nu}$, are classically conserved and, as well-known, have anomalies at the quantum level. Grisaru has indicated [9] that if $a_\mu$, $S_{\mu\alpha}$ and $\theta_{\mu\nu}$ enter one and the same supermultiplet, just the same property should be inherent to the corresponding anomalies. In ref. [9] it has been demonstrated that this is indeed the case at one-loop level.

The problem arises at two loops. The Adler-Bardeen theorem [10], establishing the one-loop nature of $\partial_\mu a_\mu$, seems to contradict the multiloop expression for the trace of the energy-momentum tensor



$$\theta_{\mu\mu} = \frac{\beta(\alpha)}{4\alpha} G^a_{\mu\nu} G^a_{\mu\nu} \qquad (3)$$

which is usually quoted in the literature. Many papers are devoted to attempts of solving the anomaly puzzle in SUSY theories. A list of references which is far from being complete includes more than 10 papers [11-22]. One of the first detailed investigations has been undertaken by Piguet and Sibold [11]. Unfortunately, in spite of definite progress no real breakthrough has been achieved.

Let us elucidate our basic assertions concerning $\beta$ functions and anomalies by a simple example - supersymmetric electrodynamics (SQED). The action of the model can be written as:

$$S_W = \frac{1}{4e^2(\mu)} \int d^4x\, d^2\theta\, W^2 + \frac{Z(\mu)}{4} \int d^4x\, d^4\theta \left(\bar{T} e^V T + \bar{U} e^{-V} U\right) \qquad (4)$$

where W is the supergeneralization of the strength tensor

$$W_\alpha = \frac{1}{8} \bar{D}^2 D_\alpha V = i\lambda_\alpha(x_L) - \theta_\alpha D(x_L) - i\theta^\beta F_{\alpha\beta}(x_L) + \theta^2 \partial_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x_L), \qquad (5)$$

$$(x_L)_{\alpha\dot{\alpha}} = x_{\alpha\dot{\alpha}} - 2i\theta_\alpha \bar{\theta}_{\dot{\alpha}}.$$

$T(x_L, \theta)$ and $U(x_L, \theta)$ are chiral matter superfields with charges +1 and -1, respectively. The action (4) is to be understood in the sense of Wilson, i.e. all operators in the right-hand side are normalized at $\mu$ and $1/e^2(\mu)$ and $Z(\mu)$ are the corresponding coefficient functions. The mass term $mTU|_F$

is omitted in eq.(4), since $\mu$ is assumed to be much larger than m, $\mu >> m$. The maximal value of $\mu$ is equal to $M_0$, the ultraviolet cut off parameter. At this point the action (4) is just the original SQED action and the coefficients $1/e^2(M_0)$, $Z(M_0)$ are bare parameters. For arbitrary $\mu$ the coefficient functions are determined by normal graphs of perturbation theory constructed starting from the original action with the following constraint - and this is the most crucial point for us - the integration domain over momenta k <u>in all loops</u> is limited by the condition $\mu < k < M_0$.

Assume that we would like to find the amplitudes of physical processes with the external momenta $p \sim \mu$. The central statement is that they cannot be read off directly from the action (4). The adequate quantity determining the amplitudes is $\Gamma(\mu)$, the generator of 1 PI vertices. Although superficially $\Gamma$ contains just the same structures as the action (4), their meaning is different: in $\Gamma$ they are c-number functions while in $S_W$ they are operators. A reflection of this fact is the distinction in the coefficients. We will denote the coefficients in $\Gamma(\mu)$ by the same letters but in square brackets, $1/[e^2(\mu)]$, $[Z(\mu)]$. The normalization point $\mu$ in $\Gamma$ is to be understood as the momentum of the external field.

As will be shown below, the connection between $\alpha(\mu) = e^2(\mu)/4\pi$ and $[\alpha(\mu)]$ is as follows*)

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*) The Z factors for <u>the matter superfields</u> in $S_W$ and $\Gamma$, $Z(\mu)$ and $[Z(\mu)]$ respectively, seem to coincide. This is definitely the case if the Konishi anomaly is purely one loop, see below. There are various arguments in favour of the equality and in what follows we will often make no distinction between the two quantities. If necessary, one can easily trace which particular Z factor appears in this or that expression.



$$\frac{2\pi}{[\alpha(\mu)]} = \frac{2\pi}{\alpha(\mu)} - 2\ln[Z(\mu)] \tag{6}$$

where $\alpha = e^2/4\pi$. The term with $\ln Z$ emerges in calculating the photonic matrix element of the operator $\int d^4\theta d^4x (\bar{T}e^V T + \bar{U}e^{-V} U)$ in eq.(4). The matrix element is fixed by the so called Konishi anomaly [23-25]. The observable quantity is $[\alpha(\mu)]$. The fact that it explicitly depends on the Z factor is a new and surprising element. Notice that eq.(6) refers to the bare quantities $(\mu = M_0)$ as well. Therefore, the following two models will be physically equivalent: in the first one the coefficient of $W^2$ in $S_W$ is equal to $(4\pi\alpha_0)^{-1}$ while the coefficient of $\bar{T}T + \bar{U}U$ is equal to $Z_0$ and in the second model these coefficients are $(4\pi\alpha_0)^{-1} - (1/4\pi^2)\ln Z_0$ and 1, respectively. The quantity $\alpha(\mu)$ is renormalized only at one-loop level, as was mentioned above, i.e.

$$\frac{2\pi}{\alpha(\mu)} = \frac{2\pi}{\alpha_0} + 2\ln\frac{M_0}{\mu} \tag{7}$$

Combining eqs.(6) and (7), we get

$$\frac{2\pi}{[\alpha(\mu)]} = \frac{2\pi}{[\alpha_0]} + 2\ln\frac{M_0}{\mu} - 2\ln\frac{[Z(\mu)]}{[Z_0]} \tag{8}$$

Apart from the one-loop log the $\mu$ dependence enters only via the Z factor. Differentiation over $\ln\mu$ yields the $\beta$ function for the observable constant $[\alpha(\mu)]$:

$$\beta(\alpha) = \frac{\alpha^2}{\pi}\left(1 - \gamma(\alpha)\right) \qquad (9)$$

where $\gamma(\alpha)$ is the anomalous dimension of the matter superfield

$$\gamma = -\frac{d \ln Z(\mu)}{d \ln \mu} = -\frac{\alpha}{\pi}, \qquad (10)$$

Let us describe now the anomaly equations in SQED. The supercurrent $\mathcal{J}_{\alpha\dot\alpha}$ in this model has the form

$$\mathcal{J}_{\alpha\dot\alpha} = -\frac{1}{e^2} W_\alpha \bar W_{\dot\alpha} + Z\Big\{\frac{1}{6}\left(D_\alpha(e^V T)\right) e^{-V} \bar D_{\dot\alpha}(e^V \bar T) -$$

$$-\frac{1}{6} T e^V D_\alpha\left(e^{-V} \bar D_{\dot\alpha}(e^V \bar T)\right) - \frac{1}{6} T \bar D_{\dot\alpha}\left(e^V D_\alpha \bar T\right) + \qquad (11)$$

$$+ \left(T \to U, V \to -V\right)\Big\}$$

Its supertrace is

$$\bar D^{\dot\alpha} \mathcal{J}_{\alpha\dot\alpha} = \frac{1}{24} D_\alpha\left[\frac{1}{2\pi^2} W^2 - \gamma Z \bar D^2\left(\bar T e^V T + \bar U e^{-V} U\right)\right] \qquad (12)$$

The coefficients in the right-hand side result from differentiation of the action $S_W$ (see eq.(4)) over the cut-off parameter $M_0$. The coefficient in front of $W^2$ in the operator relation (12) is exclusively one-loop. Eq.(12) is actually a superextension of the Adler-Bardeen theorem for the axial current.

The one-loop result for the coefficient of $W^2$ is general for all SUSY gauge theories. It is worth noting that the second term in the right-hand side (12), formally equal to zero by



equations of motion, cannot be omitted. Indeed, the photonic matrix element of this operator, as was discussed above, is just the source of difference between $S_W$ and $\Gamma$. Calculation of the matric element of $\bar T T + \bar U U$ can be carried out by virtue of the Konishi anomaly[23-25]

$$Z \bar D^2\left(\bar T e^V T + \bar U e^{-V} U\right) = \frac{1}{2\pi^2} W^2 \qquad (13)$$

Due to this relation the complete $\beta$ function is restored in the photonic matrix element of the anomaly (12).

Let us pass now to non-abelian theories. Since the effects associated with the matter fields can be treated essentially in the same way as in SQED let us concentrate on purely gauge model (SSYM) with the action

$$S = \frac{1}{2g^2}\int d^2\theta\, d^4x\, \mathrm{Tr}\, W^2, \qquad W_\alpha = \frac{1}{8}\bar D^2\left(e^{-V} D_\alpha e^V\right) \qquad (14)$$

Non-abelian fields play the role of sources for each other; therefore even in the absence of matter multiplets rescaling of fields changes the magnitude of the gauge coupling in the Wilson action, just in the same way as it occured in SQED under rescaling of the matter fields. Specifically, if one passes from the field V to $\eta V$, i.e.

$$W_\alpha \to \frac{1}{8}\bar D^2\left(e^{-\eta V} D_\alpha e^{\eta V}\right) \qquad (15)$$

in order to get the action equivalent to (14) one must, simultaneously with (15), change the coupling, $g^2 \to g_\eta^2$, according to the law

$$\frac{8\pi^2}{g^2} \to \frac{8\pi^2}{g_\eta^2} = \frac{8\pi^2}{g^2} - 2T(G)\ln\eta \tag{16}$$

The coupling constants in $S_W$ and $\Gamma$ will coincide if the kinetic term of the V field is normalized to unity. Under such normalization the matrix element of the operator $W^2$ can be obtained by mere substituting $W_\alpha$ by the external (c-number) field. Normalization to unity means that $\eta=[g_\eta]$. Hence, the observable charge $[g^2]$ and the one in the Wilson action are related as follows

$$\frac{8\pi^2}{g^2} = \frac{8\pi^2}{[g^2]} + T(G)\ln[g^2] \tag{17}$$

Since the Wilson $g^2$ is renormalized only at one loop,

$$\frac{8\pi^2}{g^2} = \frac{8\pi^2}{g_0^2} - 3T(G)\ln\frac{M_0}{\mu}, \tag{18}$$

differentiating eq.(17) over $\ln\mu$ we get the $\beta$ function (2) for the observable coupling $[g^2]$.

Notice that an explicit computation of the two-loop contribution to the effective action in SUSY gauge theories has been undertaken recently in very interesting and stimulating works [26]. The covariant supergraph technique has been used in combination with supersymmetric regularization by dimensional reduction (SRDR). From explicit formulae given in ref. [26], it is seen that the two-loop part of the $\beta$ function emerges from an infrared-uncertain expression of the type $p^2/p^2$ ($p^2 \to 0$) where p is the external field momentum. Actually the integral corresponding to one of the loops is totally determined by



the domain of virtual momenta k of order p. In our terminology this loop is to be interpreted as computation of the matrix element of operator $W^2$.

A few words about the supertrace anomaly in the non-abelian theory (14). The anomaly equation has the form:

$$\bar{D}^{\dot\alpha} \mathcal{J}_{\alpha\dot\alpha} = -\frac{T(G)}{16\pi^2} D_\alpha \operatorname{Tr} W^2 \tag{19}$$

The original Lagrangian can be written in the form

$$\mathcal{J}_{\alpha\dot\alpha} = -\frac{2}{g^2}\operatorname{Tr}\left(W_\alpha e^{-V}\bar{W}_{\dot\alpha} e^{V}\right) \tag{20}$$

where the definition of $W_\alpha$ is given in eq.(14).

The absence of higher orders in $g^2$ in this relation is in one-to-one correspondence with the one-loop law of the coupling constant renormalization in $S_W$. As in SQED the complete $\beta$ function emerges at the stage of taking the matrix element of the operator $W^2$.

The paper is organized as follows. In Sec.2 we consider a simple (non-supersymmetric) example of electrodynamics of a scalar field. In this example we demonstrate the difference between $S_W$ and $\Gamma$ by analysing the charge renormalization in two loops. Sec.3 is devoted to SQED. Specific features of non-abelian models are discussed in Sec.4. Comparison with calculations of the instanton type is presented in Sec.5. Here we generalize the instanton-based approach of ref. [5] to the case of arbitrary self-dual background. Sec.6 presents comments on the literature, summary of the results and conclusions.

## 2. Electrodynamics of Scalar Field

In this Section we will discuss an instructive example - scalar electrodyhamics. This model will allow us to elucidate in a simplified situation, some aspects of the results referring to SUSY theories. A concrete computation of the two-loop $\beta$ function with the special emphasis on the points we will need below is described in detail in ref. [27]. Here we focus on interpretation of this calculation within the framework of OPE.

The original lagrangian can be written in the form

$$\mathcal{L} = -\frac{1}{4e_0^2} F_{\mu\nu} F_{\mu\nu} + (\mathcal{D}_\mu \varphi)^* \mathcal{D}_\mu \varphi , \tag{21}$$

where $\varphi$ is the complex scalar field and $\mathcal{D}_\mu = \partial_\mu - iA_\mu$.

Below we will construct, to two loops, both $S_W$ and $\Gamma$, the Wilson action and the generator of 1 PI vertices. The background field method will be consistently used throughout the paper.

First of all, let us explain the procedure of introducing the normalization point $\mu$ in $S_W$ and $\Gamma$. In both cases $\mu$ is the external field momentum. (The $\varphi$ field mass is assumed to be negligibly small in comparison with $\mu$ ). As we will see, in one of the loop integrals a definite contribution comes from the infrared domain of virtual momenta, $k \simeq \mu$. This contribution should be included into $\Gamma(\mu)$, but excluded from $S_W(\mu)$. As far as the ultraviolet cut off is concerned, within our approach we will deal only with single-log integrals which can be cut off at the upper limit $M_0$ in a step-like way. In principle, one may think that the theory is regularized by



the Pauli-Villars fields, partners to $\varphi$, plus higher derivatives for the vector field $A_\mu$.

In one loop the issue of finding $S_W(\mu)$ is certainly trivial. The result reduces to

$$S_W = \int d^4x \left\{ -\frac{1}{32\pi^2} \left( \frac{2\pi}{\alpha_0} + \frac{1}{3} \ln \frac{M_0}{\mu} \right) F_{\mu\nu}^2 + Z\left(\frac{M_0}{\mu}\right) \mathcal{D}_\mu \varphi^* \mathcal{D}_\mu \varphi \right\} \tag{22}$$

where for the Z factor we have

$$Z\left(\frac{M_0}{\mu}\right) = 1 - \left(1 + \frac{\xi}{2}\right) \frac{\alpha_0}{\pi} \ln \frac{M_0}{\mu} \tag{23}$$

Notice that Z depends on the gauge of the photon field whose propagator is chosen in the form

$$\mathcal{D}_{\mu\nu} = e_0^2 \left( -g_{\mu\nu} + \xi \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2}$$

In this approximation $\Gamma(\mu)$ superficially coincides with $S_W(\mu)$ since the photonic matrix element of $(Z-1)(\mathcal{D}_\mu \varphi)^* \mathcal{D}_\mu \varphi$ is to be taken into account only in the two-loop order.

Let us proceed now to two-loop analysis. In the two-loop approximation the coefficient in front of $F^2$ in $S_W$ is determined by the diagram of Fig.1. Let us single out integration over the virtual photon and perform it at the very end. Then, before this last integration, calculation of $S_W(\mu)$ is equivalent to a calculation of the photon polarization operator in one loop (Fig.2). More strictly, one needs to find only one term in the operator expansion for $\Pi_{\mu\nu}$, namely $C F_{\alpha\beta} F_{\alpha\beta}$. The coefficient of this term is finite and well-defined. Then the last integration over the photon momentum k will yield a logarithmic integral of the type $\int d^4k / k^4$ which can be simply cut off from above at $M_0$ and from below at $\mu$. (For further details see ref. [27]).

Specifically, in the $x$ representation

$$\mathcal{L}^{(2)}_{eff} = \frac{1}{2}\int_{M_0^{-1}<x<\mu^{-1}} d^4x \; i D_{\mu\nu}(x)\, \Pi^{(F^2)}_{\mu\nu}(x) \tag{24}$$

where *)

$$\Pi_{\mu\nu} = i\langle T\{ J_\mu(x) J_\nu(0)\}\rangle ,$$

$$J_\mu = i\varphi^* \overleftrightarrow{\partial}_\mu \varphi = i\left[\varphi^* D_\mu\varphi - (D_\mu\varphi)^*\varphi\right] \tag{25}$$

while the superscript $(F^2)$ indicates that only the operator $F^2$ should be kept in $\Pi_{\mu\nu}$.

In eq.(24) $D_{\mu\nu}$ stands for the free photon propagator

$$D_{\mu\nu} = e_0^2\left(-g_{\mu\nu} + \zeta\frac{\partial_\mu\partial_\nu}{\partial^2}\right)\frac{i}{4\pi^2 x^2} =$$

$$= e_0^2\left[-g_{\mu\nu} + 2\zeta\left(g_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}\right)\right]\frac{-i}{4\pi^2 x^2}\,. \tag{26}$$

The operator expansion for $\Pi_{\mu\nu}$ has been constructed in ref. [27] where an explicit expression for the scalar field Green function $G(x,0)$ in the photonic background has been found:

$$G(x,0) = \langle x|\frac{1}{\mathcal{P}^2}|0\rangle = \frac{i}{4\pi^2 x^2} + \frac{i}{512\pi^2}x^2 F_{\alpha\beta}F_{\alpha\beta} \tag{27}$$

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*) We omit in $\Pi_{\mu\nu}$ the piece of the form $-2g_{\mu\nu}\langle\varphi^*\varphi\rangle$, i.e. the tadpole type graphs which are irrelevant for the present analysis.



(in the Fock-Schwinger gauge for $A_\mu^{ext}$, $x_\mu A_\mu^{ext} = 0$). The crucial point is that in constructing $S_W(\mu)$ we must keep only the first, singular, term in eq.(27). Just this piece corresponds to large ($\sim k$, see Fig.2) virtual momenta in the loop of Fig.2. The non-singular term (in the momentum space it is proportional to $(\partial^2/\partial p_\mu \partial p_\nu)\,\delta^4(p)$ represents an infrared effect. In the diagram of Fig.2 it corresponds to virtual momenta of order $\mu$, the external field momentum. (In eq.(27) $\mu \to 0$).

In other words, this latter domain is irrelevant to the OPE coefficient and will be accounted for in taking the matrix element, see below.

Let us quote here eq.(25) from ref. [27] (the quantity we need in that work was denoted $\Pi^{(4)}_{\mu\nu}$):

$$\Pi^{(F^2)}_{\mu\nu} = -\frac{i}{192\pi^4 x^4}\left(x^2 g_{\mu\nu} - x_\mu x_\nu\right) F^2_{\alpha\beta}(0) \tag{28}$$

Let us draw the reader's attention to the fact that the expression (28) is not transversal; as a reflection of this fact in $S_W(\mu)$ there emerges a dependence on the gauge parameter $\zeta$.

Combining eqs.(24),(26) and (28) we get

$$S_W(\mu) = \int d^4x \left\{ -\frac{1}{32\pi^2}\left[\frac{2\pi}{\alpha_0} + \frac{1}{3}\ln\frac{M_0}{\mu} + \frac{\alpha_0}{2\pi}\left(1-\frac{\zeta}{2}\right)\ln\frac{M_0}{\mu}\right]F^2_{\alpha\beta} + \right.$$

$$\left. + Z\left(\frac{M_0}{\mu}\right)(D_\mu\varphi)^* D_\mu\varphi \right. \tag{29}$$

At first sight the situation is paradoxical. Indeed, the two-loop coefficient in front of $F^2_{\alpha\beta}$ in $S_W(\mu)$ is gauge-dependent. The corresponding constant, clearly, cannot be observable. How can one reconcile eq.(29) with the well-known expression for the charge renormalization in scalar electrodynamics, which, of course, contains no gauge parameter?

The answer must be clear to the reader from Sec.1. The coefficient in front of $F^2$ in $S_W(\mu)$ actually <u>does not coincide</u> with the observable charge. In determining $1/[\alpha]$ one should take into account that a non-vanishing contribution to the amplitudes (in external photonic field in the case at hand) comes from the matrix element of the operator $(\mathcal{D}_\mu\varphi)^*\mathcal{D}_\mu\varphi$ over the external field.

Formally, the operator $\int d^4x(\mathcal{D}_\mu\varphi)^*\mathcal{D}_\mu\varphi$ is zero by equations of motion. However, one can convince oneself that in the external gauge field there exists an anomalous relation

$$\langle(\mathcal{D}_\mu\varphi)^*\mathcal{D}_\mu\varphi\rangle=\frac{1}{64\pi^2}F_{\alpha\beta}F_{\alpha\beta} \tag{30}$$

stemming from eq.(27) for the Green function $G(x,0)$.

The complete propagator $G(x,0)$ certainly satisfies the equations of motion, $-\mathcal{D}^2G(x,0)=\delta^4(x)$. In our computations, however, we decompose it in two pieces: the first part, singular in $x$, is included in the OPE coefficient while the non-singular part is referred to the matrix element. Indeed,

$$\langle(\mathcal{D}_\mu\varphi)^*\mathcal{D}_\mu\varphi\rangle=\lim_{x\to0}\left[-i\mathcal{D}^2G^{Reg}(x,0)\right]=\frac{1}{64\pi^2}F_{\alpha\beta}F_{\alpha\beta},$$

where $G^{Reg}(x,0)$ is the second term in the right-hand side of eq.(27). (Recall that the analysis is carried out in the Fock-Schwinger gauge for the background field).



Returning now to the action (27) we can find the observable coupling constant by passing from $S_W$ to $\Gamma$. Taking the matrix element over the external photon field we get for the structure $F_{\alpha\beta}F_{\alpha\beta}$ :

$$\Gamma=\int d^4x\left\{-\frac{1}{32\pi^2}\left[\frac{2\pi}{\alpha_0}+\frac{1}{3}\ln\frac{M_0}{\mu}+\frac{\alpha_0}{2\pi}\left(1-\frac{\xi}{2}\right)\ln\frac{M_0}{\mu}\right]+\right.$$
$$\left.+\frac{1}{64\pi^2}\left[Z\left(\frac{M_0}{\mu}\right)-1\right]\right\}F_{\alpha\beta}F_{\alpha\beta}+\dots \tag{31}$$

Invoking eq.(23) we convince ourselves that in the sum the gauge dependence cancels, as expected on general grounds, and

$$\frac{2\pi}{[\alpha]}=\frac{2\pi}{\alpha_0}+\frac{1}{3}\ln\frac{M_0}{\mu}+\frac{\alpha_0}{\pi}\ln\frac{M_0}{\mu}+O(\alpha_0^2) \tag{32}$$

We pause here to make a few explanatory comments. In passing from $S_W$ to $\Gamma$, as usually in perturbation theory, we have calculated the matrix element only of that part of $S_W$ that should be treated as perturbation with respect to the operator basis chosen. Specifically, we have calculated $\langle(Z-1)\int d^4x(\mathcal{D}_\mu\varphi)^*\mathcal{D}_\mu\varphi\rangle$.

Notice also that quite definite procedure for the ultraviolet and infrared regularization has been used, which is by no means unique. In the literature Feynman diagrams in gauge theories are often calculated within the dimensional regularization, both, in the ultraviolet and infrared. Actually, all these calculations, irrespectively of the procedure of infrared regularization, refer to $\Gamma$ and yield a correct answer for the observable charge. The answer automatically includes the infrared domain.

In the example considered above - scalar electrodynamics - the decomposition in two pieces, ultraviolet and infrared (i.e. OPE coefficients and matrix elements) is not unique. In particular, the anomalous relation (30) is not related to the well-known conformal anomaly; the coefficient in the right-hand side of eq.(30) depends on the procedure adopted. Naturally, within each particular computational scheme both, the coefficient in front of $F^2$ in OPE and $\langle (\mathcal{D}_\mu\varphi)^+ \mathcal{D}_\mu \varphi \rangle$, are fixed unambiguously and have quite definite values. However, in other schemes some redistribution can take place.

Confirmation of the latter point can be easily found in SQED (see Sec.3), which admits two types of analyses: the component one (with no mention of superfields) and the analysis based on the superfield formalism. Scalar particles are a part of the matter sector of SQED. The spinor fields, also figuring in SQED, have no anomaly analogous to (30) under the component treatment, $\langle \bar\psi \not{\mathcal{D}} \psi \rangle = 0$. In other words, spinor electrodynamics has no infrared piece in the charge renormalization. On the other hand, within the superfield treatment eq.(30) is substituted by the Konishin anomaly (13).

## 3. Supersymmetric Quantum Electrodynamics

The Wilson action in SQED has the form

$$S_W = \frac{1}{4e^2}\int d^4x\, d^2\theta\, W^2 + \frac{Z}{4}\int d^4x\, d^4\theta \left(\bar T e^V T + \bar U e^{-V} U\right) \tag{33}$$

Using the same general approach sketched in the previous section we will show here that transition from $S_W$ to $\Gamma$ results in eq.(6) for the observable charge. Besides that, a general



theorem will be proven establishing the fact that the renormalization of the coefficient of $W^2$ is exhausted by the first loop. The assertion of one-loop nature of renormalizations, close in spirit to ours, has been made in ref.[5]. Since the parameters $e^2$ and Z in eq.(33) are not observable and depend on the quantization procedure the validity of both points formulated above essentially rely on the superfield formalism. In other words, it is important that the off-shell continuation is performed explicitly supersymmetrically.

To analyse the relation between $S_W$ and $\Gamma$ we might follow the same program as in Sec.2. It is necessary to find the propagator of the matter superfield in the background gauge field, decompose it in two pieces - singular and non-singular - refer the first piece to calculation of the coefficient $1/e^2$ and then the non-singular part will fix the matrix element of the operator

$$\int d^4x\, d^4\theta \left(\bar T e^V T + \bar U e^{-V} U\right) = -\frac{1}{2}\int d^4x\, d^2\theta\, \bar D^2 \left(\bar T e^V T + \bar U e^{-V} U\right) \tag{34}$$

In the problem at hand realization of the program simplifies because there are no second and higher order loops in $1/e^2$, and therefore, no "redistribution" between different terms in $S_W$. In other words, within the superfield formalism the separation of OPE coefficients and matrix elements is performed in a unique way. As a manifestation of this situation one can formulate calculation of the matrix element of (34) as the anomalous Konishi relation (13).

In this point there is a direct analogy with the Adler anomaly in the axial current [10]

$$\partial_\mu a_\mu = \partial_\mu \bar\psi \gamma_\mu \gamma_5 \psi = -\frac{1}{8\pi^2} F_{\mu\nu}\tilde F_{\mu\nu} \qquad (35)$$

As well known, this anomaly has two faces. On one hand it can be revealed as an infrared effect in the transition of the axial current into two photons:

$$a_\mu \sim \frac{q_\mu}{q^2} F_{\alpha\beta}\tilde F_{\alpha\beta},$$

where q is the axial current momentum (the photons are assumed to be on mass shell). From this expression it is immediately seen that the fermion loop is saturated by small virtual momenta, of order q. Multiplying by $q_\mu$ we arrive at eq.(35). This derivation [29] emphasizing the infrared nature of the effect corresponds to computation of the matrix element of $(\mathcal{D}_\mu\varphi)^* \mathcal{D}_\mu\varphi$ in Sec.2 by virtue of separation of the non-singular part of the Green function in the external field.

On the other hand, since we deal with the divergence, the anomalous relation (35) can be obtained as a result of the ultraviolet regularization, say, by the Pauli-Villars method.

In the language of the spectral flow in the external field the double-face nature of the anomaly means that the number of levels coming to zero is equal to the number of levels crossing the ultraviolet cut off.

The operator $\bar D^2(\bar T e^V T + \bar U e^{-V} U)$ of interest is a direct generalization of $\partial_\mu a_\mu$ in SUSY theories. Its matrix element is unambiguously fixed by the Konishi anomaly (13). From this we deduce that to the first order in Z-1 eq.(33) implies the following expression for $\Gamma$:



$$\Gamma = \left[\frac{1}{4e^2} - \frac{1}{16\pi^2}(Z-1)\right]\int d^4x\, d^2\theta\, W^2 + (\text{terms with matter}) \qquad (36)$$

As was already noted, the one-loop law is valid for $1/e^2$:

$$\frac{8\pi^2}{e^2} = \frac{8\pi^2}{e_0^2} + 2\ln\frac{M_0}{\mu} \qquad (37)$$

If we consider for a moment only the first order in Z-1, the result (36) can be represented as a substitution of the ultraviolet cut off $M_0$ by $M_0/Z$. This fact is not a simple coincidence, of course; it is in one-to-one correspondence with the ultraviolet derivation of the Konishi anomaly.

Indeed, let us introduce explicitly the Pauli-Villars regulator fields $T_R$ and $U_R$. In other words, the action (33) is supplemented by the regulator part

$$S_{reg} = \frac{Z}{4}\int d^4x\, d^4\theta \left(\bar T_R e^V T_R + \bar U_R e^{-V} U_R\right) + \left(\frac{M_0}{2}\int d^4x\, d^2\theta\, T_R U_R + h.c.\right) \qquad (38)$$

When the regulators are included explicitly the naive equations of motions are valid. Therefore, we can use these equations in the perturbation, proportional to (Z-1). Then the perturbation reduces to

$$\Delta S = (Z-1)\left(\frac{M_0}{2}\int d^4x\, d^2\theta\, T_R U_R + h.c.\right) \qquad (39)$$

After taking the matrix element the additional term (39) reproduces the term with (Z-1) in eq.(36). Thus, we have used the term O(Z-1) to establish the following: the mass term of the regulator fields in eq.(38) does not contain Z. After this observation is made it is quite evident that summation of all orders in (Z-1) in the matrix element $\langle$ gauge field $|\exp(i\Delta S)|$ gauge field $\rangle$ is equivalent to the substitution $M_0 \to M_0/Z$. Hence, the observable charge figuring in $\Gamma$ is equal to

$$\frac{8\pi^2}{[e^2]} = \frac{8\pi^2}{e_0^2} + 2\ln\frac{M_0}{\mu Z} \tag{40}$$

Expression (40) is our final result for SQED. Differentiating over $\ln\mu$ we arrive at the $\beta$ function quoted in eq.(9).

As far as the supertrace anomaly (eq.(12)) is concerned it can be obtained by differentiating the action (33) over $\ln M_0$. In doing so one should keep in mind that the one-loop law (37) is valid for $1/e^2$ and the factor Z depends on the ratio $M_0/\mu$.

The last thing to be done in this Section is to prove the theorem of the absence of higher orders in $1/e^2$ declared above. Actually, we will merely reformulate arguments of ref.[5]. Thus, let us assume for definiteness that we are calculating the two-loop coefficient in front of the operator $W^2$ in the effective action. Within the background field method the latter is determined by the graph of Fig.3.

Notice that in the abelian case the superfield V does not interact with the external field. Regularization can be performed just in the same way as in Sec.2. Namely, we cut the V line and get the sub-block which is finite both in the ultraviolet and in infrared. The last integration over the virtual momentum of the V propagator is cut off at $M_0$ in the ultraviolet domain, and at $\mu$ in the infrared one. The ultraviolet cut



off can be introduced via higher derivatives for the field V. Keeping in mind any n-loop graph one can formulate the general regularization procedure - the Pauli-Villars fields $T_R$, $U_R$ with the mass $M_0$ combined with higher derivatives for V. Only the very fact of existence of a superfield regularization in 4 dimensions is important for us.
Within the background field technique the expression for the diagram 3 has the form

$$\Delta S(\text{fig. }3) \sim \int d^8z_1\, d^8z_2\, \mathcal{D}(z_1,z_2)\, G(z_1,z_2)\, G(z_2,z_1), \tag{41}$$

where $z=(x,\theta,\bar\theta)$ and $\mathcal{D}(z_1,z_2)$ and $G(z_1,z_2)$ are the Green functions of the vector and covariantly chiral superfields in the external field. The operator representation for these propagators can be found, for instance, in refs.[28, 26]:

$$\mathcal{D}(z_1,z_2) = \langle z_1 | \left(\nabla_\mu^2 - iW^\alpha\nabla_\alpha + i\bar W^{\dot\alpha}\bar\nabla_{\dot\alpha}\right)^{-1} | z_2\rangle$$

$$G(z_1,z_2) = \langle z_1 | -\nabla^\alpha\nabla_\alpha \left(\nabla_\mu^2 - iW^\alpha\nabla_\alpha - \frac{i}{2}[\nabla^\alpha, W_\alpha]\right)^{-1} \bar\nabla_{\dot\alpha}\bar\nabla^{\dot\alpha} | z_2\rangle \tag{42}$$

Of principle importance is the fact that eqs.(42) contain no $V_{ext}$, the external field enters only via $W_\alpha^{ext}$ and covariant derivatives. Therefore, the method is explicitly gauge invariant with respect to the external field. In particular, under gauge transformations of the external field the propagator D does not change at all while the propagator G goes into

$$G(z_1,z_2) \to e^{iK(z_1)}\, G(z_1,z_2)\, e^{-iK(z_2)} \tag{43}$$

where $K(z)$ is a real superfield of the general form. The integrand in eq.(41) is obviously invariant with respect to the transformation (43). In the non-abelian case K is a matrix in the colour space and the analogue of eq.(41) contains the trace over colour. The gauge invariance of the integrand in eq.(41) means that the latter is expressible in terms of $W_\alpha^{ext}$ and covariant derivatives. After integration we arrive at the expression of the type

$$\Delta S(fig.3) = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, f(x,\theta,\bar{\theta})$$

where $f$ is a function of $W_\alpha^{ext}$ (it does not depend on $x, \theta, \bar{\theta}$ explicitly).

If $f(x,\theta,\bar{\theta})$ can be locally expressed in terms of $W_\alpha^{ext}(x,\theta)$ and the covariant derivatives, then, quite obviously, the structure of the type $\int d^4x\, d^2\theta\, W^2(x,\theta)$ cannot emerge. On the other hand, in concrete two-loop calculations [26] this structure has been obtained. A paradox? The explanation is a non-local function of W, for instance

$$f = W^\alpha \frac{D_\beta D^\beta}{\Box} W_\alpha$$

An infrared singularity here is undeniable. If the momentum of the external field W is p then the $1/p^2$ pole could appear only from the domain of virtual momenta $k \sim p$. As explained above this domain should not be included in the OPE coefficients in $S_W$, it should be accounted for in matrix elements. This remark exhausts the proof for the diagram of Fig.3.

The arguments presented above are not applicable to the one-loop graph with the chiral superfield inside. Indeed,



to prove the vanishing it was crucial to have a quantum interaction vertex accompanied by integration over $d^2\theta\, d^2\bar{\theta}$. The "extra" $d^2\bar{\theta}$ cannot be eliminated then. The one-loop supergraph in the external field technique does not reduce to expression like (41) at all.

The proof of vanishing of the two-loop graph in $S_W$ is readily generalizable to include higher loops and non-abelian theories. Our assertion is actually an extension of the well-known non-renormalization theorem for F-terms [2].

Let us mention one more interesting point. Renorm group eqs. have usual form in terms of $[\alpha(\mu)]$ and not $\alpha(\mu)$. In particular $d\ln Z/d\ln\mu = -\gamma([\alpha(\mu)])$ So it is convinient to choose $[\alpha_0]$ and $Z_0$ as initial data for these eqs. Then coefficient Z in $S_W$ is proportional to $Z_0$ and coefficient $1/\alpha$ at operator $W^2$ depends on $Z_0$ logarithmically, $2\pi/\alpha = (2\pi/[\alpha]) + 2\ln Z$ The variation of $S_W$ induced by rescaling of $Z_0$ gives the renorm invariant operator of the form

$$Z\bar{D}^2(\bar{T}e^V T + \bar{U}e^{-V}U) - \frac{1}{2\pi^2}W^2 \qquad (44)$$

The physical independence on $Z_0$ means that operator (44) is zero and it is our derivation of the Konishi anomaly.

Moreover we have two renorm invariant combinations of $W^2$ and $\bar{D}^2(\bar{T}e^V T + \bar{U}e^{-V}U)$ One is given by eq.(44) and another enters eq.(12) It implies that operator

$$(1-\gamma)Z(\bar{T}e^V T + \bar{U}e^{-V}U) \qquad (45)$$

also is renorm invariant. This relation defines the anomalous dimension of $Z(\bar{T}e^V T + \bar{U}e^{-V}U)$.

## 4. Non-abelian Gauge Theories

The basic assertions referring to this case have been formulated in Sec.1. Here we will elucidate derivation of the results by the simplest example - supersymmetric gluodynamics (SSYM). Introduction of matter fields require no special consideration because this aspect does not differ from the procedure discussed in detail in SQED.

The crucial distinction of non-abelian gauge theory from SQED is evident: now the gauge fields are sources for each other and, therefore, the matrix element $\langle W^2 \rangle$ does not reduce to the c-number $W^2_{ext}$. To make the point more transparent let us turn to SU(2) model and assume that the external field is oriented along the third axis in the colour space, $W_\alpha^{ext} = (W_\alpha^3)^{ext}$. Then the third field can be treated as "neutral" while $W^\pm = \frac{1}{\sqrt{2}}(W^1 \mp i W^2)$ play the role of charged fields with respect to the U(1) subgroup singled out by the orientation of the external field. The operator $W^2$ is representable as a sum

$$W^2 = 2W^+W^- + W^3W^3$$

The matrix element of $W^3W^3$ over the "neutral" external field is trivial, while to fix $\langle 2W^+W^- \rangle$ we actually must consider electrodynamics of the charged vector superfield. Now we see that conceptually the situation is just the same as in SQED. Due to an anomaly the matrix element of $W^+W^-$ over the "neutral external field does not vanish.

More specifically, the following relation takes place



$$\langle W^2 \rangle = \frac{\beta(\alpha)}{\beta_{1loop}(\alpha)} W^2_{ext} = \left(1 + \frac{T(G)\alpha}{2\pi} + \dots\right) W^2_{ext} \quad (46)$$

where $\beta(\alpha)$ and $\beta_{1loop}(\alpha)$ denote the complete and one-loop Gell-Mann-Low functions, respectively. The fact that $\langle \beta_{1loop} W^2 \rangle$ reduces to $\beta(\alpha) W^2_{ext}$ is obvious from the renormalization - group-invariance of the both quantities. Thus, the right-hand side of eq.(46) can be considered as a definition of the Gell-Mann-Low function. Our aim is constructive calculation of $\beta(\alpha)$. By virtue of eq.(46) the $(\ell+1)$-loop coefficient in $\beta(\alpha)$ is fixed in terms of the $\ell$-th coefficient in the matrix element $\langle W^2 \rangle$.

First of all, let us recall that the Wilson action in SSYM is exhausted by one loop

$$S_W(\mu) = \frac{1}{2}\left(\frac{1}{g_0^2} - \frac{3T(G)}{8\pi^2} \ln\frac{M_0}{\mu}\right) \int d^4x\, d^2\theta\, \mathrm{Tr}\, W^\alpha W_\alpha \quad (47)$$

This fact has been proven in Sec.3. At one-loop level the expression for $\Gamma$ has, evidently, just the same form. Higher loops in $\Gamma$ appear from calculation of matrix elements of $S_W$ (more exactly, $\exp i\Gamma = \langle \exp iS_W \rangle$). Let us establish now the relation between $\Gamma(\mu)$ and $S_W(\mu)$ at two-loop level. In this approximation

$$\Gamma(\mu) = \frac{1}{2g_0^2} \int d^4x\, d^2\theta\, \mathrm{Tr}\, W^2_{ext} - \frac{1}{2} \cdot \frac{3T(G)}{8\pi^2} \ln\frac{M_0}{\mu} \int d^4x\, d^2\theta \langle \mathrm{Tr}\, W^2 \rangle \quad (48)$$

The subscript "ext" marks a c-number external field in which we calculate the matrix element $\langle \mathrm{Tr}\, W^2 \rangle$ of the operator $W^2$.

In passing from eq.(47) to eq.(48) we have taken into account that the matrix element should be calculated only for additional piece, absent in bare action. Therefore, it is sufficient to find $\langle Tr\, W^2 \rangle$ to order $O(\alpha)$.

The one-loop part of the result (46) can be extracted from ref.[22] (eqs.(A.21) and (A.22)). The main complication in finding the matrix element $\langle Tr\, W^2 \rangle$ is the necessity of the <u>infrared</u> regularization. In particular, the authors of ref.[22] have used dimensional reduction for this purpose. We would like, however, to present here another derivation, referring directly to four-dimensional space. A transparent physical meaning of this derivation will help us to establish a few useful facts and to discuss the relation with previous analyses.*)

We concentrate on the G-component of the superfield $W^2$ which has the form

$$Tr\, W^2 \big|_G = \frac{1}{4}\left(G_{\mu\nu}\tilde{G}_{\mu\nu} - 2\partial_\mu a_\mu\right), \qquad (49)$$

where $a_\mu = -\bar{\lambda}^a \sigma_\mu \lambda^a$ is the gluino axial current, $\lambda^a_\alpha$ ($\alpha = 1,2;\ a = 1,2,3$) is the Weyl spinor describing gluino. In the Majorana notation $\lambda^a_\alpha$ ( $\alpha$ = 1,2,3,4) the same current can be written as $a_\mu = \frac{1}{2}\bar{\lambda}\gamma_\mu\gamma_5\lambda$ .

It is extremely essential that not only fermionic, but also the bosonic part of $W^2\big|_G$ is representable as a full derivative

$$G_{\mu\nu}\tilde{G}_{\mu\nu} = \partial_\mu K_\mu ; \quad K_\mu = 2\epsilon_{\mu\nu\gamma\delta}\left(A^a_\nu \partial_\gamma A^a_\delta + \frac{1}{3} f^{abc} A^a_\nu A^b_\gamma A^c_\delta\right) \qquad (50)$$

*) This derivation was done in collaboration with V.A.Novikov



Therefore, as will be seen below, we will manage to formulate the calculation of $\langle G\tilde{G} \rangle$ in terms of a certain anomaly just in the same way as it happens with $\partial_\mu a_\mu$ . More exactly, both $a_\mu$ and $K_\mu$ have infrared poles of the type $(q_\mu/q^2)G\tilde{G}$ with coefficients which can be fixed unambiguously. The assertion refers to the following kinematics: the gluons in the final state have momenta $k_1$ and $k_2$ with $k_1^2 = k_2^2 = 0$ and $q^2 = (k_1+k_2)^2 \neq 0$. The presence of the infrared pole in $a_\mu$ is a well-known fact [29,30] reflecting the existence of the axial anomaly. The analogous pole in seems to escape attention and was not discussed in the literature.

In calculation of $\langle K_\mu \rangle$ we will use the background field formalism,

$$A^a_\mu = (A^a_\mu)_{ext} + a^a_\mu \qquad (51)$$

where $(A^a_\mu)_{ext}$ and $a^a_\mu$ are the external and quantum fields. In one-loop approximation we need only the part of $K_\mu$ quadratic in $a^a_\mu$ ,

$$K^{(2)}_\mu = 2\epsilon_{\mu\nu\gamma\delta}\, a^a_\nu \mathcal{D}_\gamma a^a_\delta \qquad (52)$$

The matrix element of interest can be obtained from eq.(52) by substituting the Green function for the quantum field,

$$\langle K_\mu \rangle = -2g^2 \epsilon_{\mu\nu\gamma\delta}\, Tr_{colour} \langle x | \mathcal{P}_\gamma \left[\frac{1}{\mathcal{P}^2 - 2G}\right]_{\delta\nu} | x \rangle \qquad (53)$$

Here all quantities are matrices in the colour space,

$$(\mathcal{P}_\gamma)^{ab} = i\left(\delta^{ab}\partial_\gamma + f^{acb}A^c_\gamma\right)$$

$$[\mathcal{P}_\mu, \mathcal{P}_\gamma]^{ab} = -G^{ab}_{\mu\gamma} = f^{acb}G^c_{\mu\gamma}$$

To simplify the expressions we have scipped the subscript "ext" for the external fields. The formalism we exploit is explicitly gauge invariant with respect to the external field. As far as the quantum field gauge is concerned, eq.(53) implies the Feynman gauge, $\mathcal{L}_{g.f.} = -\frac{1}{2}(D_\mu a^a_\mu)^2$. Needless to say that the final answer should be independent of the quantum field gauge. We will return to the issue later.

Now, let us expand the propagator (25) in powers of $G/\mathcal{P}^2$. The zeroth order term in G drops out because of contraction with $\epsilon_{\mu\gamma\rho\sigma}$. The second order term

$$\mathcal{P}_\gamma \frac{1}{\mathcal{P}^2} G_{\delta\rho} \frac{1}{\mathcal{P}^2} G_{\rho\gamma} \frac{1}{\mathcal{P}^2}$$

contains two G's contracted over one index; hence there is no way of getting the only structure, $\mathcal{P}_\mu G_{\alpha\beta}\tilde{G}_{\alpha\beta}$, determining the longitudinal part of $K_\mu$. Thus, we are left with the linear in G term:

$$\langle K_\mu\rangle = -8g^2 Tr_{colour} \langle x| \mathcal{P}_\gamma \frac{1}{\mathcal{P}^2} \tilde{G}_{\mu\gamma} \frac{1}{\mathcal{P}^2} |x\rangle \qquad (54)$$

Instead of direct computation of (54) (which presents no difficulties, though) one can compare this expression with the matrix element of the spinor axial current $a_\mu$, whose anomaly is well-known. For the spinor current

$$\langle a_\mu\rangle = -\frac{i}{2} g^2 Tr_{colour} Tr_{spin} \langle x|\gamma_\mu\gamma_5 \frac{1}{\not{\mathcal{P}}}|x\rangle =$$

$$= -\frac{i}{2} g^2 Tr_{colour} Tr_{spin} \langle x|\gamma_\mu\gamma_5 \not{\mathcal{P}} \frac{1}{\mathcal{P}^2 - \frac{1}{2}G_{\alpha\beta}\sigma_{\alpha\beta}}|x\rangle .$$

As in the previous case one can convince oneself that in the longitudinal part of $\langle a_\mu\rangle$ only the linear in $\sigma G$ term survives,

$$\langle a_\mu\rangle = -2g^2 Tr_{colour} \langle x| \mathcal{P}_\gamma \frac{1}{\mathcal{P}^2} \tilde{G}_{\mu\gamma} \frac{1}{\mathcal{P}^2} |x\rangle \qquad (55)$$

A simple inspection of eqs.(54) and (55) shows that in there is just the same pole as in $\langle a_\mu\rangle$, $(\mathcal{P}_\mu/\mathcal{P}^2)G\tilde{G}$, but with an additional factor 4. Since

$$\langle \partial_\mu a_\mu\rangle = \frac{T(G)\alpha}{4\pi}\left(G^a_{\mu\gamma}\tilde{G}^a_{\mu\gamma}\right)_{ext} \qquad (56)$$

we get for $\langle G\tilde{G}\rangle$

$$\langle G\tilde{G}\rangle = \langle \partial_\mu K_\mu\rangle = \left(G^a_{\mu\gamma}\tilde{G}^a_{\mu\gamma}\right)_{ext}\left(1 + \frac{T(G)\alpha}{4\pi} + \dots\right), \qquad (57)$$

where we have added the unit term from the classical part.

Eqs.(56) and (57) imply for the G component of $W^2$ (see eq.(49)):

$$\langle Tr\, W^2|_G\rangle = W^2|^{ext}_G \left\{1 + \frac{T(G)\alpha}{\pi} - \frac{T(G)\alpha}{2\pi}\right\} = \qquad (58)$$

$$= W^2|^{ext}_G \left\{1 + \frac{T(G)\alpha}{2\pi}\right\}$$

Now, keeping in mind supersymmetry, we see that the one-loop piece of relation (46) is reproduced.

What remains to be done is to demonstrate independence of the quantum field gauge. In arbitrary gauge the propagator of the field $q_\mu^a$ has the form

$$\mathcal{D}_{\mu\nu}(x,y)=\langle x|\left(\mathcal{P}^2 g_{\mu\nu}-2G_{\mu\nu}-\xi\mathcal{P}_\mu\mathcal{P}_\nu\right)^{-1}|x\rangle \tag{59}$$

One can readily check the following operator equality

$$\mathcal{P}_\mu\left(\mathcal{P}^2 g_{\mu\nu}-2G_{\mu\nu}\right)=\mathcal{P}^2\mathcal{P}_\nu+i\mathcal{D}_\gamma G_{\gamma\nu} \tag{60}$$

If the external field is assumed to satisfy the equations of motion, $\mathcal{D}_\gamma G_{\gamma\nu}=0$, as required in the external field method, the second term in the right-hand side drops out, and the Green function is representable in the closed form:

$$\mathcal{D}_{\mu\nu}(x,y)=\langle x|\left[\frac{1}{\mathcal{P}^2-2G}\right]_{\mu\nu}+\frac{\xi}{1-\xi}\mathcal{P}_\mu\frac{1}{\mathcal{P}^4}\mathcal{P}_\nu|y\rangle$$

Returning now to the matrix element of $K_\mu$ (see eq.(53)) we write the $\xi$ dependent part as follows

$$\Delta_\xi\langle K_\mu\rangle=2g^2\frac{\xi}{1-\xi}\tilde{G}_{\mu\nu}(x)\langle x|\frac{1}{\mathcal{P}^4}\mathcal{P}_\nu|x\rangle. \tag{61}$$

Formally, due to gauge invariance with respect to the external field, $\langle x|\mathcal{P}^{-4}\mathcal{P}_\nu|x\rangle$ must be proportional to $\mathcal{D}_\gamma G_{\gamma\nu}$ which is zero. However, in the kinematics considered ($k_1^2=k_2^2=0$) the expression $\langle x|\mathcal{P}^{-4}\mathcal{P}_\nu|x\rangle$ is ill-defined in the infrared (it contains 1/k2 ). For regularization one can introduce an infrared mass m to the quantum field $q_\mu^a$. In other words, in all propagators $\mathcal{P}^{-2}\to(\mathcal{P}^2-m^2)^{-1}$. It is assumed that $m^2\ll q^2=(k_1+k_2)^2$. Just the same device is used in the fermion triangle (cf. ref.[29]). After the regularizing mass parameter m is introduced the vanishing of eq.(61) becomes valid not only formally.



A few remarks are in order here to comment the results obtained above. The infrared pole in $K_\mu$ is fixed absolutely unambiguously, and does not depend on the procedure of off-shell continuation. In other words, the matrix element of $W^2|_G$ must be one and the same both in the component and superfield formalisms. Extension to the other components of $W^2$ implies the superfield formalism. We have checked that eq.(46) results from ref.[22] which consistently exploits the superfield formalism and SRDR.

Notice that the bosonic anomaly in $W^2|_G$ has the coefficient twice larger than the fermionic one and of the opposite sign. Effectively this changes the sign of $\langle W^2|_G\rangle$ in comparison with the standard practice (refs.[13,16,19]) which totally neglects the bosonic anomaly. We will return to discussion of the point in Sec.6.

We have not computed explicitly the two-loop and higher-order terms in eq.(46). Let us sketch an indirect line of reasoning (the analogue of the analysis carried out in SQED) which will allow us to generalize the one-loop answer to all orders.

In SQED the second loop in $\Gamma$ emerged after taking the matrix element of the operator $(Z-1)\int d^4x\,d^4\theta\left(\bar{T}e^V T+\bar{U}e^{-V}U\right)$

This matrix element is saturated in the infrared domain; however, thanks to the universal nature of the anomaly the result for the second loop in $1/[e^2]$ can be formulated as the substitution of the regulator mass $M_o \to M_o/Z$, in the one-loop logarithm. In such a form the result is valid to all loops (see eqs.(37)-(40)).

In non-abelian theories no explicit procedure for deriving eq.(46) from the ultraviolet regularization is constructed at the moment, but beyond any doubt this can be done. Then it seems natural that the situation analogous to SQED should take place. Namely, the effect in $1/[g^2]$ due to higher loops must reduce to a substitution

$$\frac{M_o}{\mu} \to \frac{M_o}{\mu}\left[\frac{Z_o}{Z(\mu)}\right]^{\Delta}$$

in the one-loop logarithm. Here $\Delta$ is some number. Since the ultraviolet regularization procedure is not specified we have reserved the possibility of $\Delta \neq 1$. In the model considered (SSYM) the factor $[Z]$ obviously reduces to

$$[Z(\mu)] = \frac{1}{[g^2(\mu)]}$$

The two-loop result stemming from eqs.(48),(46) is representable in the form

$$\frac{1}{[g^2(\mu)]} = \frac{1}{[g_o^2]} - \frac{3T(G)}{8\pi^2} \ln\left[\frac{M_o}{\mu}\left(\frac{[Z_o]}{[Z(\mu)]}\right)^{\frac{1}{3}}\right] \qquad (62)$$



$$\frac{[Z_o]}{[Z(\mu)]} = \frac{[g^2(\mu)]}{[g_o^2]} = 1 + \frac{3T(G)}{8\pi^2} g_o^2 \ln\frac{M_o}{\mu} + O(g_o^4) \qquad (63)$$

Parallelizing the analysis in SQED we conclude that if we <u>do not expand</u> the ratio $[Z_o]/[Z(\mu)]$ in the gauge coupling constant, the expression (62) will be exact. Notice, that it immediately implies the Gell-Mann-Low function (2).

The fact that the exponent $\Delta$ turned out to be 1/3 in the case at hand finds a natural explanation within the instanton-type approach (See Sec.5).

## 5.<u>Comparison with Calculations of the Instanton Type</u>

The present investigation is essentially based on the observation that the coefficient of $\int d^2\theta\, W^2$ in $S_W$ is renormalized only in one loop. As has been already mentioned, this fact generalizes the well-known non-renormalization theorem [2] for F terms. Below a somewhat non-standard proof of the theorem will be given which, among other things, will show in what cases the theorem can be violated. Our arguments will simultaneously demonstrate why the one-loop renormalization of the structure $\int d^2\theta\, W^2$ is possible.

The basic idea of the approach we would like to propose is as follows. In any SUSY field theory there are several - at least four- supercharge generators, and one can pick up such an external field that will be invariant under the action of some part of the generators. For this specific external field some terms in the action S can vanish. The non-renormalization theorem will refer to those structures that do not vanish in the background field chosen.

For instance, in the Wess-Zumino model,

$$S^{WZ}=\frac{1}{4}\int d^4x\, d^4\theta\, \bar{\varphi}\varphi+\frac{1}{2}\int d^4x\left(d^2\theta\, W(\varphi)+h.c.\right) \qquad (64)$$

the appropriate external field is

$$\bar{\varphi}_{ext}=0\,,\qquad \varphi_{ext}=c_1+c_2^{\alpha}\theta_{\alpha}+c_3\theta^2 \qquad (65)$$

where $C_{1,2,3}$ are some constants. The relation (65) assumes that $\varphi$ and $\bar{\varphi}$ are treated as independent variables, not connected by the complex conjugation (a kind of analytic continuation). The $X$ independent chiral field (65) does not change under the action of $\bar{Q}_{\dot\alpha}$, the dotted supercharges, i.e. under the transformations

$$\delta\theta_{\alpha}=0\,,\quad \delta\bar{\theta}_{\dot\alpha}=\bar{\varepsilon}_{\dot\alpha}\,,\quad \delta x_{\alpha\dot\alpha}=2i\,\theta_{\alpha}\bar{\varepsilon}_{\dot\alpha}$$

Hence, in the quantum problem for the deviations $\varphi-\varphi_{ext}$ there exists the exact symmetry under the transformations generated by $\bar{Q}_{\dot\alpha}$.

In terms of quantum states we face here a boson-fermion degeneracy just as in the "empty" vacuum. This degeneracy is sufficient for ensuring cancellation of all quantum corrections to $\Gamma(\varphi_{ext})$, i.e.

$$\Gamma^{WZ}(\varphi_{ext})=S^{WZ}(\varphi_{ext}) \qquad (66)$$

The situation is absolutely analogous to that with quantum corrections to the vacuum energy in the empty space, i.e. for



$\bar{\varphi}_{ext}=0$.

Eq.(66) implies the absence of renormalization of the second term in (64). The first term vanishes in the external field (65), and nothing can be said about its renormalization.

What changes if one switches to gauges theories, for instance, SSYM (see eq.(14))?

The general line of reasoning stays the same. As in the Wess-Zumino model one can choose a purely chiral and $X$ independent external field. (More exactly, it is sufficient to impose these constraints on $W_{\alpha}$, $\bar{W}_{\dot\alpha}$ and not on the prepotential V because $\Gamma$ depends only on gauge invariant quantities). Another, more complex variant, also suitable for our purposes, is the instanton solution[31] (see ref.[5]). Although in the latter case the field does depend on x the invariance of the external field under the $\bar{Q}_{\dot\alpha}$-generated transformations still survives. (Strictly speaking, the statement refers to colourless combinations like $Tr\, W^2$). Starting from this point we might arrive at a non-renormalization "theorem" for the structure $\int d^2\theta\, W^2$, the conclusion analogous to the one made above for F-terms in the Wess-Zumino model. Such a conclusion is perfectly correct in the Wess-Zumino model but, as well-known, is incorrect in gauge models.

Where is the loophole? The point is that sometimes the fermion-boson symmetry can be broken, namely, in the cases when $\bar{Q}_{\dot\alpha}$ annihilate some states. In somewhat different language, more usual for the external field method, the effect reduces to appearance of zero modes.

Recall the instanton example in supersymmetric gluodynamics

[5]. For the vector field the number of modes with the eigenvalue $\lambda_n^2 \neq 0$ is equal to 4-2=2 *). Simultaneously, there are two fermion modes with the eigenvalue $\lambda_n$ and two more fermion modes with $(-\lambda_n)$ [32].

Such a relation between the bosonic and fermionic modes is a consequence of the invariance under $\bar{Q}_{\dot\alpha}$ and takes place in any field with this invariance. It is easy to see that just this balance - 1:2 - guarantees the cancellation of the quantum corrections. (For instantons the phenomenon has been first discovered in ref. [33]). In particular, the one-loop correction is proportional to $\sum_{bos}(1/2)\ln\lambda_n^2 - \sum_{ferm}(1/2)\ln|\lambda_n|$ and is vanishing because each bosonic level is accompanied by two fermionic. The cancellation in higher orders is ensured by the exact symmetry with respect to $\bar{Q}_{\dot\alpha}$.

Let us turn now to zero modes. The same symmetry with respect to $\bar{Q}_{\dot\alpha}$ results here in a "wrong" relation between the numbers of bosonic and fermionic zero modes, namely 2:1. Indeed, the zero modes of the vector field $A_{\alpha\dot\alpha}$ and the spinor one, $\lambda_\alpha$, are essentially the same and satisfy the relations

$$\mathcal{D}^{\alpha\dot\alpha}\lambda_\alpha = 0 \qquad \mathcal{D}^{\alpha\dot\alpha}A_{\alpha\dot\beta} = 0 \tag{67}$$

The second relation leaves the dotted index free, also a consequence of invariance with respect to $\bar{Q}_{\dot\alpha}$. Keeping in mind the "spectator" role of the dotted index it becomes evident

*) Literally speaking, the vector field in a fixed (covariant) gauge has four modes. However, taking account of the ghost determinant is equivalent to eliminating of two modes.



that the bosonic zero modes repeat fermionic and that $n_B^{zero\ modes} = 2n_F^{zero\ modes}$. The disbalance in the number of modes leads to the fact that quantum effects do not cancel completely, and $\Gamma(W_{ext}) \neq S(W_{ext})$. More specifically, the zero modes (in combination with their regulator counter-partners, of course) yield the following contribution to $\Gamma$:

$$(\Delta\Gamma)_{zero\ modes} = -\sum_{bos}\frac{1}{2}\ln\frac{M_o^2[Z_o]}{\mu^2[Z(\mu)]} + \frac{1}{2}\sum_{ferm}\ln\frac{M_o[Z_o]}{\mu[Z(\mu)]} \tag{68}$$

where $M_o$ is the regulator mass, and the sum runs over all bosonic and fermionic zero modes, respectively. The factors $Z_o/Z$ account for the fact that higher loops affect the normalization of the zero modes (and this is the only manifestation of higher loops). The corresponding renormalization coincides with that for the external field since the coefficients of the expansion in the zero modes have the meaning of collective coordinates of the external field. On the other hand, by definition the external field renormalization is just the charge renormalization,

$$\frac{[Z_o]}{[Z(\mu)]} = \frac{[g^2(\mu)]}{[g_o^2]} \tag{69}$$

where $[g_o] = [g(\mu = M_o)]$.

Let us rewrite now eq.(68) for $\Delta\Gamma$ in terms of $n_F$, the number of fermionic zero modes,

$$(\Delta\Gamma)_{zero\ modes} = -\frac{3}{2}n_F\left(\ln\frac{M_o}{\mu} + \frac{1}{3}\ln\frac{[Z_o]}{[Z]}\right) \tag{70}$$

Furthermore, the coefficient $n_F$ is fixed by the index theorem,

$$n_F = \frac{T(G)}{16\pi^2}\int d^4x\, G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} = \frac{T(G)}{16\pi^2}\int d^4x\, d^2\theta\, W^2 \tag{71}$$

The second relation in eq.(71) is due to the self-duality of the external field. Substituting eq.(71) into eq.(70) and comparing the result with the original action

$$S_0 = \frac{1}{4g_0^2}\left\{\int d^4x\, d^2\theta\, Tr W^2 + \int d^4x\, d^2\bar\theta\, Tr \bar W^2\right\} \tag{72}$$

we find the law of the charge renormalization (cf.eqs.(17,(18))

$$\frac{1}{[g^2]} = \frac{1}{[g_0^2]} - \frac{3T(G)}{8\pi^2}\left(\ln\frac{M_0}{\mu} + \frac{1}{3}\ln\frac{[g^2]}{[g_0^2]}\right) \tag{73}$$

It will be in order here to compare this derivation with the analysis of Sec.4. First of all, let us draw the reader's attention to the exponent 1/3 in $(M_0/\mu)(Z_0/Z)^{1/3}$ (cf. eq.(62)). This 1/3 emerged in a natural way; indeed, the coefficient of $\ln M_0/\mu$ in eq.(68) is $-(n_B - \frac{1}{2}n_F)$, while $\ln Z_0/Z$ is multiplied by $-\frac{1}{2}(n_B - n_F)$.

Moreover, the latter factor, $-\frac{1}{2}(n_B - n_F)$ is in one-to-one correspondence with the calculation of the matrix element of $Tr W^2$ presented in the previous Section. The residues of the poles in $K_\mu$ and $a_\mu$ count $n_B$ and $n_F$, respectively. Actually, we have discovered an index theorem for the zero modes of the non-abelian vector field. The fact that $n_B=2n_F$ manifests itself in perturbation theory in the following way: the contribution of the bosonic anomaly to $\langle Tr W^2/_G\rangle$ is twice larger (and of the opposite sign) in comparison with that



of the fermionic anomaly.

The arguments above do not rely on an explicit choice of the background field. All we need to know is $n_F$, the number fixed by the index theorem. Therefore, along with the instanton example an x-independent self-dual external field suits equally well for our purposes. However, for such a field the integral $\int d^4x\, G\tilde G$, figuring in the index theorem, is strictly speaking, ill-defined and calls for an accurate treatment. One of possible regularizations is introduction of a finite volume [34,35], thorus $L^4$. The simplest self-dual field in this case - toron - has been discovered by 't Hooft [34]. The $G_{\mu\nu}$ tensor for the toron field is constant (x independent); moreover, the action and topological charge constitute (1/2) of these quantities in the instanton (SU(2) colour is assumed). Our general argument in this case is realized as follows: there are two zero fermionic modes (they are generated by applying to the toron field) and four bosonic modes (conventional translations).

It is instructive to find the matrix element of the operator $Tr W^2$ in the toron field. Actually, this has been done in ref. [36] where the condensate $\langle\lambda^\alpha\lambda_\alpha\rangle$ has been determined. The result for the expectation value in the toron field reduces to

$$\langle Tr W^2\rangle = -\langle Tr \lambda^2\rangle = C L^{-3}[g(L)]^{-2} e^{-\frac{4\pi^2}{[g^2(L)]}}, \tag{74}$$

where L is the box size. We reproduce eq.(74) here in order to emphasize that this expression is exact - no corrections in the gauge coupling constant. Eq.(74) demonstrates that the operator $\mathrm{Tr}\,W^2$ , not $(\beta(\alpha)/\alpha^2)\,\mathrm{Tr}\,W^2$ , is renormalization-group invariant; just the matrix element of the former is expressible in terms of observable quantities and does not depend on $M_o$ .

## 6. Conclusions and Comments on the Literature

The present work- we hope - complete continuous efforts in investigation of two related problems in SUSY gauge theories: ultraviolet renormlizations and the structure of the anomaly supermultiplet. We have ascertain how the exact relations for $\beta$ functions emerge in ordinary perturbation theory. The key finding is non-coincidence of the Wilson action $S_W(\mu)$ and the sum of the vacuum loops $\Gamma(\mu)$ in the external field due to the presence of infrared effects. The famous non-renormalization theorem[2] for F-terms is generalized to include the operator $\int d^2\theta\,\mathrm{Tr}\,W^2$ in $S_W$ . The coefficient of this operator, $1/g^2$ , is not renormalized at two, three, etc. loops *) . The first coefficient in $\beta$ function for the observable charge reflects the renormalization of $1/g^2$ , the second and all other coefficients represent infrared effects coming from some matrix elements.

The observable gauge constant $1/[g^2]$ enters $\Gamma$ and differs from $1/g^2$ by $\sum_i C_i \ln Z_i$ where $Z_i$ is the renormalization factor for the i-th field and $C_i$ is a number emerging

*) Another example which falls under the generalized theorem is the so called Fayet- Iliopoulos D term $\int d^4\theta\, V$ in the abelian theory. The absence of higher loops for this term has been established in ref.[37]. In order to demonstrate the applicability of our proof let us rewrite this term as follows $\int d^4\theta\, V \sim \sim \int d\theta^\alpha \bar{D}^2 D_\alpha V \sim \int d\theta^\alpha W_\alpha$ . Now it is clear that it can be called F-term just in the same sense as $\int d^2\theta\,\mathrm{Tr}\,W^2$ .



in calculating the corresponding matrix element (all $C_i$'s are fixed by a one-loop computation). One can say that the Z factors of the matter fields become observable.

It is seen that the standard perturbation theory is extremely ineffective as far as calculations of the gauge constant renormalization are concerned. Working with the normal supergraphs one has to take into account a lot of superfluons things (ghosts, etc) - such things which all the same cancel from the final answer. In this sense the approaches of the instanton type are much more economic, they reduce the problem to essentially a classical one, with a finite number of degrees of freedom (a few zero modes). The Z factors encountered along this path evidently refer to external, not quantum fields.

The geometrical meaning of the first coefficient in the $\beta$ function is absolutely transparent within the instanton type approach ($\beta_1 \sim (n_B - \frac{1}{2} n_F)$). In essence, this means that the first loop by itself is determined by infrared effects. Unfortunately, in the standard perturbation theory we failed to find the line of reasoning which would adequately reflect the phenomenon.

It seems - and we strongly hope - that the ordinary perturbation theory can be improved in such a way that the process of renormalization of $W^2$ will not require invoking quantum fields, ghosts, etc.

As regards the anomaly supermultiplet problem, within our approach its solution is straightforward. Since the renormalization of $\mathrm{Tr}\,W^2$ in $S_W$ is exhausted by one loop, the operator anomaly equation for $J_{\alpha\dot\alpha}$ (see eq.(19)) contains the operator $W^2$ with purely one-loop coefficient. Just this assertion is SUSY extension of the Adler-Bardeen theorem. Many readers may find rather unusual that the conformal anomaly in $\theta_{\mu\mu}$ is determined by the first loop *). The conventional expression for $\theta_{\mu\mu}$ proportional to the complete $\beta$ function is restored after averaging the operator equality over the external (gauge) field. Simultaneously the same $\beta$ function appears in the matrix element of $\partial_\mu q_\mu$. At this point we encounter another unusual aspect - the expectation value of the operator $G\tilde{G}$ by no means reduces simply to $(G\tilde{G})_{ext}$. The effect can be formulated as anomaly in the bosonic axial current $K_\mu$ (see eq.(57)).

Now, let us discuss in brief the relation of our results to those known in the literature.

Clark et al. [11] have undertaken a thorough investigation of $\Gamma$ and matrix elements of the supercurrent. The analysis has been carried out in terms of the Ward identities. The authors of ref. [11] came to the conclusion that supersymmetric construction of $\mathcal{J}_{\alpha\dot{\alpha}}$ and an anomaly supermultiplet is possible. However, their results - and the authors fully realized the fact - did not admit a direct operator formulation. Although in principle the program [11] is quite correct (and actually combining some expressions from ref. [11] one could extract, with some effort, eq.(9) for SQED), practically the construction is too overcomplicated. Our progress is due to the following additional elements:

(i) the language of the Wilson operator expansion (the

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*) Here there is a remote analogy with anomaly supermultiplet in the gravitational background [38-40]. As has been shown in ref. [40] the accurate treatment of scalar field contributions require a modification just in $\theta_{\mu\nu}$, not in $q_\mu$.



distinction between $S_W$ and $\Gamma$);

(ii) supersymmetric off-shell continuation essential for the assertion of one-loop renormalization of $1/g^2$ in $S_W$.

It is worth emphasizing that the supersymmetric off-shell continuation is important for analysing the coefficients in $S_W$; in $\Gamma$ details of how we treat the theory off shell are far less essential.

The next series of investigations [13-16] has been initiated by the paper of Jones [12]. In this cycle of works efforts were focused around the question: how to reconcile the Adler-Bardeen theorem for $\partial_\mu q_\mu$ with the existence of higher orders in the trace $\theta_{\mu\mu}$? The summary of the program most clearly formulated in ref. [13] is as follows: one introduces two different axial currents, $q_\mu^{AB}$ figuring in the Adler-Bardeen relation, and $q_\mu^{SS}$ entering the supermultiplet $\mathcal{J}_{\alpha\dot{\alpha}}$. Then one assumes that the difference between $q_\mu^{AB}$ and $q_\mu^{SS}$ is due to an ultraviolet subtraction constant, to be found order by order by comparison of supposedly two anomaly equations for $\partial_\mu q_\mu$.

As we understand now the very formulation of the problem was inconsistent and to a large extent associated with a wrong treatment of the anomaly status. Indeed, the starting presumption that in the operator form $\theta_{\mu\mu}=(\beta(\alpha)/4\alpha)G_{\mu\nu}^a G_{\mu\nu}^a$ does not take place and, as a consequence, the main stumbling block is eliminated. The Gell-Mann-Low function appears in the right-hand side only if the latter is understood as a matrix element. On the other hand, the original proof of the theorem [10] for $\partial_\mu q_\mu$ actually is the operator statement. Ref. [10] is based on a certain two-limit technique (for a recent discussion see [20]) with two regulator masses, $M_V$ and $M_F$ ($M_F >> M_V$). Within this technique there are no corrections of second and higher orders in $\partial_\mu a_\mu$. The assertion, as it stands, refers to the amplitudes with external momenta p in the interval $M_F >> p >> M_V$, i.e. it bears the operator character in our language *). In ref. [20] we have extended the two-limit-technique to supersymmetry and found that the situation with $\theta_{\mu\mu}$ was just the same as with $\partial_\mu a_\mu$. In other words, $\theta_{\mu\mu}$ is exhausted by one loop in the two-limit sense. The answer, however, did not satisfy us since of most interest is the one-limit regularization (evolution to $p << M_V$) and constructive computation of the $\beta$ - functions. In passing to the domain $p << M_V$ confrontation with SUSY seemed to be inevitable since calculation of matrix elements of $G\tilde{G}$ and $G^2$ seemed to produce different coefficients. It was tacitly assumed that $\langle G\tilde{G}\rangle$ coincides with c-number $G\tilde{G}$ while $\langle G^2\rangle$ does not.

The latter postulate $\langle G\tilde{G}\rangle = (G\tilde{G})_{ext}$ borrowed from ref. [10] served as a basis for a no-go theorem [14] ruling out the existence of the anomaly supermultiplet. Needless to say that the theorem [14] is invalid since $\langle G\tilde{G}\rangle \neq (G\tilde{G})_{ext}$. An attempt to circumvent the no-go theorem [14] has been undertaken by Kazakov [18] and Jones et al. [19]. The basic hypothesis of refs. [18,19] is that the operator $G\tilde{G}$ changes in supersymmetric calculation in comparison with the non-supersymmet-

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*) Ref. [10] also gives some arguments that further evolution to $p << M_V$, i.e. calculation of the matrix element does not change the coefficient. Unlike the first part of the theorem the arguments are not generally valid.



ric one, and the change is due to an ultraviolet subtraction constant (the full parallel with the alleged $a_\mu^{AB}$ and $a_\mu^{SS}$).

A constructive approach has been presented in works [21,22] which, in the technical sense, have produced a strong impression on us and partially stimulated the present investigation. The authors have performed a direct two-loop calculation for all relevant operators invoking dimensional reduction for regularization. Explicit formulae for $a_\mu^{AB}$ and $a_\mu^{SS}$ have been found in d=4-$\epsilon$ ($\epsilon > 0$). In d=4-$\epsilon$, apart from $W^2$, there exists another operator gauge invariant with respect to the external field, $\hat{\hat{\Gamma}}\hat{\hat{\Gamma}}$, where $\hat{\Gamma}$ is a connection, and the double caret according to [21,22] denotes projection on the "additional" $\epsilon$ dimensions. The answer for the two-loop diagram obtained in [21,22] reduces to the operator $\frac{c}{\epsilon}\int d^4x d^4\theta \hat{\hat{\Gamma}}^2$, and not to the operator $\frac{c}{\epsilon}\int d^4x d^2\theta W^2$ appearing in the one-loop graph. Then the authors have used the fact that in SRDR $\bar{\nabla}^2 \hat{\hat{A}}^2 = -\epsilon W^2$.

According to the picture developed here the solution of the anomaly problem does not require introduction of two axial currents, two operators $G\tilde{G}$, etc. Moreover, the two currents considered in ref. [22] actually differ not by an ultraviolet constant, but by an infrared singular non-local expression. As a manifestation, the difference $a_\mu^{AB} - a_\mu^{SS}$ from ref. [22] could not be written in the limit $\epsilon \to 0$.

In our language the situation is easily explainable: in essence, the two-loop computation of ref. [22] is a computation of the matrix element of $\int d^2\theta W^2$ within the SRDR procedure. The matrix element is completely saturated in the infrared domain. As regards the issue of different schemes for the operator $G\tilde{G}$, the main point is not the distinction between $G\tilde{G}$ in the different schemes but the distinction between the operator and its matrix element. The latter is fixed unambiguously.

Here it will be in order to explain to which renormalization scheme the $\beta$ functions quoted in eqs.(1),(2),(9) refer. Our definition is close to the MOM scheme. Specifically, we fix the gauge coupling $[g^2(\mu)]$ for some external field momentum $p\sim\mu$ and express $g^2(\mu)$ in terms of the bare charge and the ultraviolet cut off. Thus, we get a relation between $g_0$ and $M_0$. No subtractions are made at intermediate stages. The latter point seems to explain the disagreement between our three-loop coefficient in eq.(2) and that found in [41].

In conclusion, let us mention the paper [7] which presents two perturbative derivations of eq.(1). One derivation was based on an infrared regularization in a box of a finite volume. Although eq.(7) from ref. [7] for the relation between $1/\alpha$ and $1/\alpha_0$ is correct, the motivation used in its derivation (see eq.(6) in [7]) is literally speaking unjustified.

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## Figure Captions

Fig.1. The two-loop contribution to $S_W(\mu)$ in scalar electrodynamics. The solid line - the scalar particle propagator in the external field, wavy line - the photon propagator.

Fig.2. By cutting off the photon line in Fig.1 we arrive at the photon polarization operator $\Pi_{\mu\nu}$. We are interested in the coefficient in front of $F_{\alpha\beta}F_{\alpha\beta}$ in the operator expansion for $\Pi_{\mu\nu}$.

Fig.3. The two-loop contribution to $\Gamma(\mu)$ in SQED. The solid line - the matter superfield propagator in the external gauge field, the wavy line - the gauge superfield propagator.

Fig.1

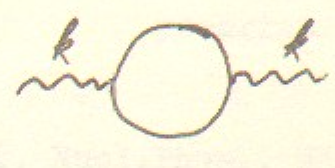


Fig.2

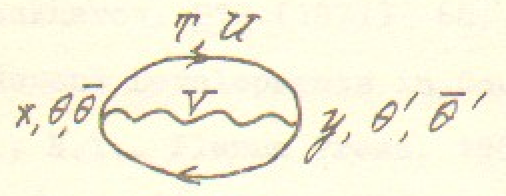


Fig.3



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# РЕШЕНИЕ ПРОБЛЕМЫ АНОМАЛИЙ В СУПЕРСИММЕТРИЧНЫХ КАЛИБРОВОЧНЫХ МОДЕЛЯХ И ОПЕРАТОРНОЕ РАЗЛОЖЕНИЕ

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