ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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PREPRINT 86-67

НОВОСИБИРСК
1986

ON THE GREEN FUNCTION CALCULATION FOR HEAVY QUARKONIUM POTENTIAL MODELS

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Abstract

A new method for numerical calculation of the Green functions of the Schrödinger equation is proposed in case of spherically symmetric potentials. This method can be applied to the potentials describing the heavy quarkonium as well as to other problems of quantum mechanics . The case of Coulomb potential with a running coupling constant is considered in detail.

The potential models are widely used now for the description of heavy quark-antiquark systems (see e.g. [1-3]). All these models successfully describe the $c\bar{c}$ and $b\bar{b}$ systems in spite of the essentially different behaviour of the corresponding potentials at short and long distances. It is suggested that the discovery of toponium will allow one to determine the correct potential.

The Green functions of the Schrödinger equation are a very convenient tool for the solution of different problems connected with heavy quarkonium (see e.g. [4,5]). However, the explicit forms of the Green functions are known only for a few potentials and numerical calculation of these Green functions directly from Schrödinger equation is a very hard problem. The use of Feynman path-integral representation for the Green function is one of the possible methods of calculation (see e.g. [6,7]).

In the present paper a new method for numerical calculation of the Green functions of the Schrödinger equation is proposed in case of spherically symmetric potentials. This method seems to be effective not only for the quarkonium problems but for other problems of quantum mechanics as well. Let us consider the Green function $G(\vec{R}',\vec{R}|\tau)$:

$$G(\vec{R}',\vec{R}|\tau) = \langle \vec{R}'| e^{-\tau H} |\vec{R}\rangle \ , \ H = \frac{\vec{p}^2}{m} + U(r) \ , \qquad (1)$$

which corresponds to the Schrödinger equation with imaginary time, where m is the quark mass, $\vec{p} = -i\vec{\nabla}$, $U(r)$ is a potential. The consideration for the real time is similar. We shall represent $G(\vec{R}',\vec{R}|\tau)$ in the form

$$G(\vec{R}',\vec{R}|\tau) = \int \frac{d\vec{K}}{(2\pi)^3} \langle \vec{R}'| e^{-\tau H} |\vec{K}\rangle e^{-i\vec{K}\vec{R}} \qquad (2)$$

where $|\vec{K}\rangle$ is a state with momentum $\vec{K}$. We introduce the function $G_1(\vec{R}',\vec{K}|\tau)$ by the relation

$$G_1(\vec{R}',\vec{K}|\tau) = e^{\tau K^2/m} \langle \vec{R}'| e^{-\tau H} |\vec{K}\rangle \ . \qquad (3)$$

Then,

$$G(\vec{R}',\vec{R}|\tau)=\int\frac{d\vec{K}}{(2\pi)^3}e^{-\tau\frac{K^2}{m}-i\vec{K}\vec{R}}G_1(\vec{R}',\vec{K}|\tau) \quad (4)$$

We define the function $F(\vec{R}',\vec{R}|S,\tau)$ as follows:

$$F(\vec{R}',\vec{R}|S,\tau)=\int\frac{d\vec{K}}{(2\pi)^3}e^{-S\frac{K^2}{m}-i\vec{K}\vec{R}}G_1(\vec{R}',\vec{K}|\tau) \quad (5)$$

The Green function $G(\vec{R}',\vec{R}|\tau)$ is connected with $F(\vec{R}',\vec{R}|S,\tau)$ by the relation

$$G(\vec{R}',\vec{R}|\tau)=F(\vec{R}',\vec{R}|\tau,\tau) \quad (6)$$

Our further purpose is to obtain the relatively simple equation for the function F suitable for numerical calculations. Differentiating both sides of expression (3) with respect to variable τ, we get

$$\frac{\partial}{\partial\tau}G_1(\vec{R}',\vec{K}|\tau)=-\langle\vec{R}'|e^{-\tau H}U(\tau)|\vec{K}\rangle e^{\tau\frac{K^2}{m}} \quad (7)$$

Let us multiply both sides of eq. (7) by $\exp(-i\vec{K}\vec{R}-SK^2/m)$ and take the integral over $\vec{K}$. Using (5), one obtains:

$$\frac{\partial}{\partial\tau}F(\vec{R}',\vec{R}|S,\tau)=-\int\frac{d\vec{K}}{(2\pi)^3}\langle\vec{R}'|e^{-\tau H}U(\tau)|\vec{K}\rangle e^{-(S-\tau)\frac{K^2}{m}-i\vec{K}\vec{R}}= \quad (8)$$

$$=-\int\frac{d\vec{q}}{(2\pi)^3}\langle\vec{R}'|e^{-\tau H}|\vec{q}\rangle\int\frac{d\vec{K}}{(2\pi)^3}\langle\vec{q}|U(\tau)|\vec{K}\rangle e^{-(S-\tau)\frac{K^2}{m}-i\vec{K}\vec{R}}$$

Then, let us consider a very wide class of potentials, which can be represented in the following form:

$$U(\tau)=\int_0^\infty d\sigma f(\sigma,\frac{\partial}{\partial\sigma})e^{-\sigma\tau^2} \quad (9)$$

All potentials used in the quarkonium potential models satisfy this condition. In particular, all potentials for which $U(\sqrt{x})$ is a Laplace transform of some function $f(\sigma)$ satisfy the relation (9). For example, if $U(\tau)=A/\tau^\beta$, then $f(\sigma)=$ $=A\sigma^{\beta/2-1}/\Gamma(\beta/2)$; for the Coulomb potential $f(\sigma)=\alpha/\sqrt{\pi\sigma}$. In case of the power law $U(\tau)=A\tau^\beta$, as for the Martin potential [3], one has $f=-A\sigma^{-\beta/2}\frac{\partial}{\partial\sigma}/\Gamma(1-\beta/2)$ and so on. Note that in case of real time $\tau\to\pm i\tau$, $S\to\mp iS$ we have to use the relation (9) with $\sigma\to\mp i\sigma$

Substituting (9) into (8), taking the integral over $\vec{K}$ and using the relations (3) and (5), we get finally for F:

$$\frac{\partial}{\partial\tau}F(\vec{R}',\vec{R}|S,\tau)=-\int_0^\infty d\sigma f(\sigma,\frac{\partial}{\partial\sigma})W^{3/2}e^{-\sigma WR^2}F(\vec{R}',W\vec{R}|\tau+W(S-\tau),\tau) \quad (10)$$

where $W=(1+4\sigma(S-\tau)/m)^{-1}$. Here, for simplicity, we restrict ourselves to the function f independent of $\partial/\partial\sigma$ (see eq. (9)). The general case can be considered quite similarly. Introducing the variable $y=1-W$ one has

$$\frac{\partial}{\partial\tau}F(\vec{R}',\vec{R}|S,\tau)=-\frac{m}{4(S-\tau)}\int_0^1\frac{dy}{\sqrt{1-y}}f\left(\frac{my}{4(1-y)(S-\tau)}\right)e^{-\frac{ymR^2}{4(S-\tau)}}F(\vec{R}',\vec{R}(1-y)|S-$$
$$-y(S-\tau),\tau). \quad (11)$$

The boundary condition can be easily obtained from the definition (5) and (3):

$$F(\vec{R}',\vec{R}|S,0)=\left(\frac{m}{4\pi S}\right)^{3/2}e^{-\frac{m(\vec{R}-\vec{R}')^2}{4S}} \quad (12)$$

Emphasize that equation (11) contains only the variables $|\vec{R}|$, S and τ as independent ones. The variables $\vec{R}'$ and the angle between vectors $\vec{R}'$ and $\vec{R}$ are the parameters and present only in the boundary condition (12). It is very convenient to solve eq. (11) numerically. Equation (11) becomes essentially simpler when $R=0$:

$$\frac{\partial F(s,\tau)}{\partial \tau} = -\frac{m}{4(s-\tau)}\int_0^1 \frac{dy}{\sqrt{1-y}} f\left(\frac{my}{4(1-y)(s-\tau)}\right) F\left(s-y(s-\tau),\tau\right) \quad (13)$$

The boundary condition in this case is

$$F(s,0) = \left(\frac{m}{4\pi s}\right)^{3/2} e^{-\frac{mR'^2}{4s}} \quad (14)$$

So, eqs. (10),(12) and relation (6) allow one to solve the problem of the Green function calculation.

The Green function $G(\vec{\sigma},\vec{\sigma}'|\tau) \equiv g(\tau)$ is of great interest because it presents in QCD sum rules for heavy quarks [4,5,8]. This function has been obtained exactly for the Coulomb potential with a fixed coupling constant in the first paper of ref. [4]. The analytic calculation of $g(\tau)$ for the case of Coulomb potential with a running coupling constant has been performed for the first time in [5] in the one-loop approximation. In the recent paper [8] the last case of potential corresponding to the quark-antiquark interaction in the colour singlet state has been considered numerically, with the use of the path-integral representation for the Green function (1). In view of great importance of this problem we apply our method to the case of Coulomb potential with a running coupling constant too. The case of pure Coulomb potential has been used to control the calculations. The potential under consideration is [9]:

$$V_c(\tau) = \frac{4}{3}\cdot\frac{2\pi}{\beta_0 \ln \Lambda_V \tau}\cdot\frac{1}{\tau} \quad (15)$$

where $\beta_0 = 11 - 2n_f/3$ (n_f is the number of light quarks). The scale parameter Λ_V is connected with the parameter $\Lambda_{\overline{MS}}$ used in QCD by the relation: $\Lambda_V = \Lambda_{\overline{MS}} e^{\gamma+c}$, where $\gamma = 0.577...$ is Euler constant, $c = (93 - 10\, n_f)/18\beta_0$ (if $n_f = 3$ then $\beta_0 = 9$, $\Lambda_V \approx 2.628\,\Lambda_{\overline{MS}}$). Since the bend of potential $V_c(\tau)$ has no physical sense, we use the regularized potentials $V_{1,2}(\tau)$, which practically

coincide with $V_c(\tau)$ up to the maximum of $V_c(\tau)$:

$$V_1(\tau) = B\left(\frac{1}{x\ln x^2} + \frac{x^2}{1-x^2}\right), \quad (16)$$

$$V_2(\tau) = B\left(\frac{1}{x\ln x^2} + \frac{2x^2}{1-x^2} + \frac{x^2\ln x^2}{(1-x^2)^2}\right),$$

where $x = \Lambda_V \tau$, $B = 16\pi\Lambda_V/27$. The potentials $V_c(\tau)$, $V_1(\tau)$ and $V_2(\tau)$ are shown in Fig. 2. Making the inverse Laplace transform one can represent $f_{1,2}(\sigma)$ (see (9)) in the form:

$$f_1(\sigma) = B\left[\frac{1}{\pi\Lambda_V^2}\int_0^\infty \frac{dx}{\sqrt{x}}\frac{\ln x}{\ln^2 x+\pi^2} e^{-\sigma x/\Lambda_V^2} - \delta(\sigma)\right], \quad (17)$$

$$f_2(\sigma) = f_1(\sigma) + B\left[\frac{1}{\Lambda_V^2}\int_0^\infty \frac{x\,dx}{(1+x)^2} e^{-\sigma x/\Lambda_V^2} - \delta(\sigma)\right].$$

The terms with δ -function lead to the multiplication of $g_1(\tau)$ and $g_2(\tau)$ by factors $e^{B\tau}$ and $e^{2B\tau}$ correspondingly. The consideration of two potentials $V_1(\tau)$ and $V_2(\tau)$ enable one to investigate the dependence of the Green function on the regularization. Let us define the function $C(\tau)$ as follows:

$$C(\tau) = \left(\frac{4\pi\tau}{m}\right)^{3/2} g(\tau) . \quad (18)$$

If $V \equiv 0$, then $C(\tau) = 1$. Fig. 2 shows the dependence of $\ln C_{1,2}(\tau)$ on $\sqrt{m\tau}$ for some values of Λ_V/m (see figure caption). Fig. 3 shows the dependence of $\frac{1}{m}\frac{d}{d\tau}\ln C_{1,2}(\tau)$ on $\sqrt{m\tau}$ at the same values of Λ_V/m . For comparison, the results obtained by formulas (16-18) of ref. [5] are shown in Fig. 3 too. One can see from Fig. 3 that the one-loop approximation results can be come to an agreement with our results in some domain of

τ by recalculation of Λ_V. For example, $\Lambda_V = 0.25$ GeV corresponds to $\Lambda_V \approx 0.35$ GeV from [5]. Taking into account the terms $(\ln z_{\tau_0}/\ln \Lambda_V \tau_0)^2$ and higher (in the notation of [5]) results in a change of $R(\tau)$ obtained in [5]. Since the argument of the coupling constant α_s is $\Lambda_V R(\tau)$, the change of $R(\tau)$ for certain values of τ is equivalent to the recalculation of Λ_V in [5]. Notice that $m\tau \gg 1$ in the nonrelativistic approximation. It is well known that $\frac{d}{d\tau} \ln g(\tau) \to -E_0$ for $\tau \to \infty$, where E_0 is the ground-state energy. This is valid also for $\frac{d}{d\tau} \ln C(\tau)$ (see (18)). For the values of parameter Λ_V/m under consideration the bound states are absent. Therefore, E_0 coincides with the continuum threshold (-B for $V_1(r)$ and -2B for $V_2(r)$, see (16)). That is confirmed by numerical calculations. Then, it is seen from Figs.2 and 3 that the dependence of $C(\tau)$ on regularization may be very essential. This fact has been pointed out in ref.[8]. The corresponding results of [8] concerning the calculation of $C(\tau)$ are in agreement with our results. The example considered above shows that application of our method gives an essential decrease of the computation time in comparison with the method of [8].

I would like to thank V.N.Baier and V.S.Fadin for useful discussions.

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Figure captions

Fig. 1. $V(z)/B$ ($B = 16\pi\Lambda_V/27$) vs. $z\Lambda_V$ for the potentials $V_c(z)$ (curve 1), $V_1(z)$ (curve 2) and $V_2(z)$ (curve 3).

Fig. 2. $\ln C(\tau)$ vs. $\sqrt{m\tau}$ for different values of Λ_V/m. For m = 4.9 GeV the scale parameter Λ_V is taken to be 0.25 GeV, 0.35 GeV and 0.45 GeV (curves 1,2,3 for the potential $V_1(z)$ and curves 1',2',3' for the potential $V_2(z)$ correspondingly).

Fig. 3. The dependence of $\frac{1}{m}\frac{d}{d\tau}\ln C(\tau)$ on $\sqrt{m\tau}$. The notation is the same as in Fig. 2. The dashed curves correspond to the results of ref. [5]; for Λ_V=0.25 GeV (curve 1") and Λ_V = 0.35 (curve 2").

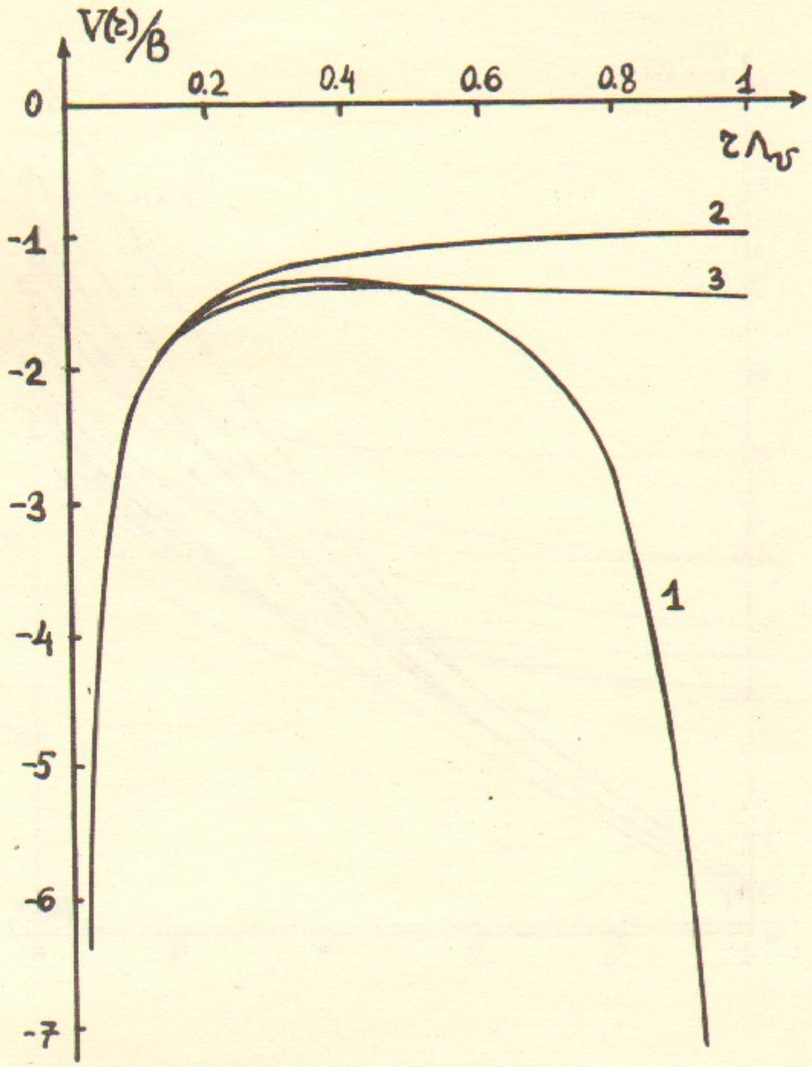

Fig.1

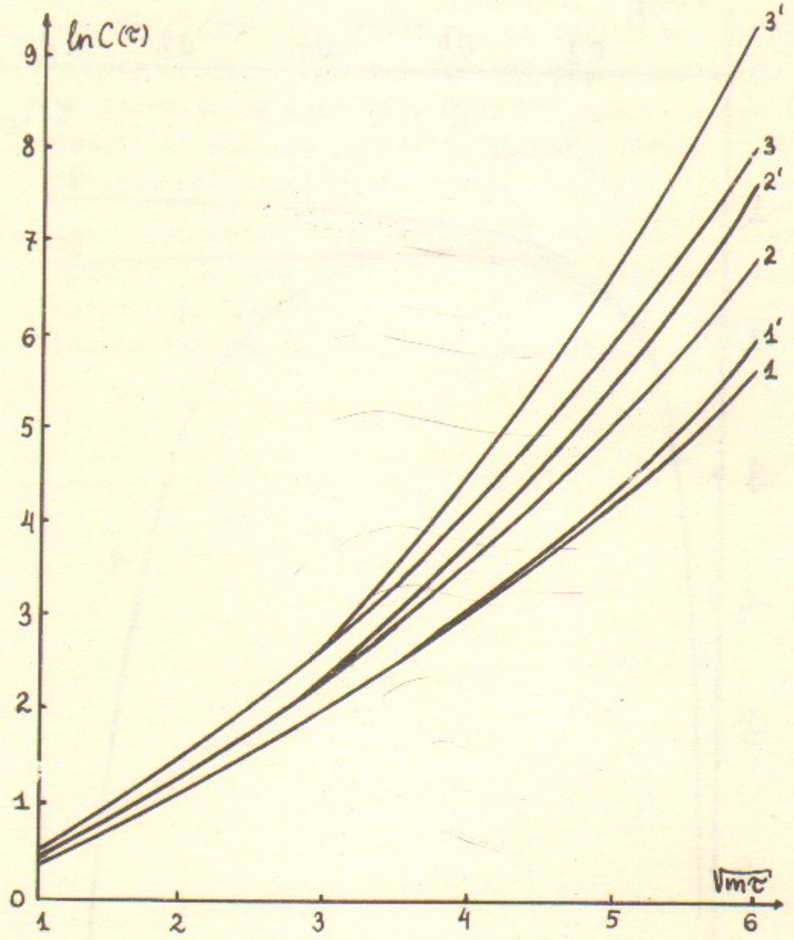
ln C(τ)
9
8
7
6
5
4
3
2
1
0
1
2
3
4
5
6
√mτ
3'
3
2'
2
1'
1

Fig.2

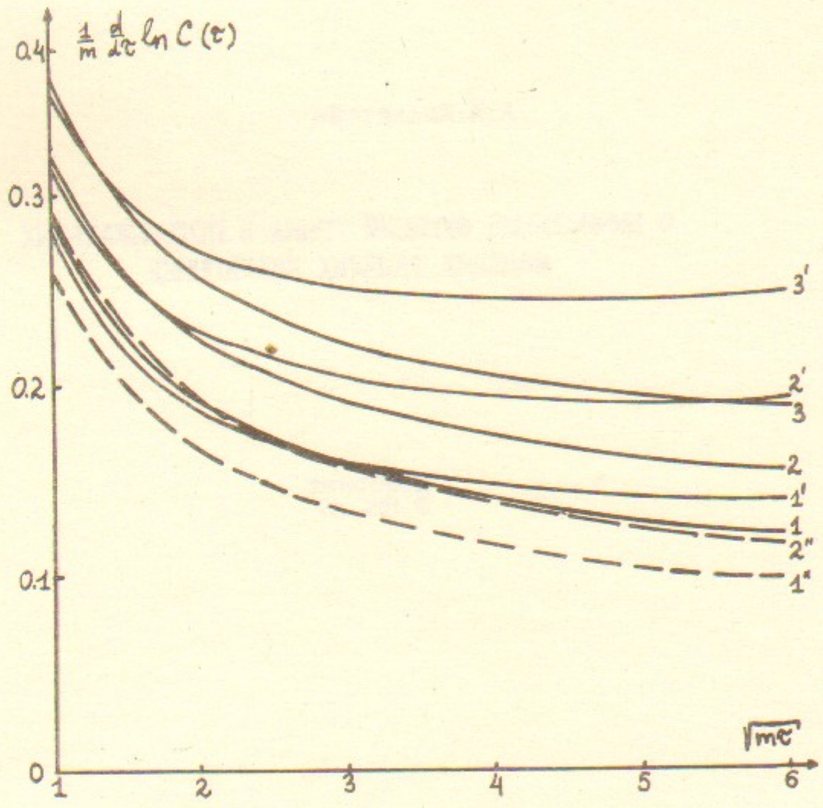
1/m d/dτ ln C (τ)
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
√mτ
3'
2'
3
2
1'
1
2"
1"

Fig.3

А.И.Мильштейн

О ВЫЧИСЛЕНИИ ФУНКЦИЙ ГРИНА В ПОТЕНЦИАЛЬНЫХ МОДЕЛЯХ ТЯЖЕЛЫХ КВАРКОНИЕВ

Препринт
№ 86-67

Работа поступила - 31 марта 1986 г.

Ответственный за выпуск - С.Г. Попов
Подписано к печати 23.04.86г. МН 11712
Формат бумаги 60x90 1/16 Усл.0,9 печ.л., 0,8 учетно-изд.л.
Тираж 220 экз. Бесплатно. Заказ № 67

Ротапринт ИЯФ СО АН СССР, г.Новосибирск, 90