ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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# COLLECTIVE QUADRUPOLE EXCITATIONS OF SPHERICAL NUCLEI IN THE FRAMEWORK OF NONLINEAR VIBRATIONS MODEL I: THE THEORY

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ABSTRACT

Low-lying collective excitations of the soft spherical nuclei are considered in the framework of nonlinear quadrupole vibration model. The main nonlinear terms being originated from the microscopical estimations are included in Hamiltonian as well as in quadrupole operator to calculate the energy spectra and collective E2-transition probabilities and the static momenta. Analytical expressions are obtained for all those quantities.



## 1. INTRODUCTION

The structure of low-lying quadrupole collective excitations in spherical nuclei attracts attention of experimentalists as well as theorists. Besides the urgent necessity to understand the rapidly expanding amount of exsperimental spectroscopic information, the physical interest to collective motion in soft nuclei is connected with unsolved problems of description of finite fermi-systems with the large shape fluctuations.

It is the boson phenomenology that is used widely last for description of energy spectra and transition probabilities in the soft spherical nuclei [1—5]. The original hypothesis is that it is possible to describe the collective excitations in terms of new quasiparticles, namely, bosons, and their interactions. Then the microscopic theory has to map fermion dynamics into the boson space and to derive the effective boson Hamiltonian and multipole operators.

Such a method was originated from the work by Belyaev and Zelevinsky [6, 7], where the collective bosons have been introduced by the regular method based on the random phase approximation (RPA) for the system of interacting fermions in upper nuclear whells.

For the soft quadrupole mode being of greatest interest the boson Hamiltonian $H_c$ contains the harmonic part $H_0 = \sum_{\mu} d^+_{\mu} d_{\mu}$ and interaction terms $H_{int}$.

The RPA-pnonons are different in many respects from the «liquid-drop» ones considered in the pioneering paper by A. Bohr [1] but for the phenomenological consideration only the symmetry and corresponding quantum numbers are essential rather then the microscopic structure of phonons. It is the formal universality of the boson Hamiltonian that has given rise the great amount of hypothesis concerning the intrinsic structure of appropriate collective variables $d$, $d^+$. This inspires the variety of boson models because the detail of the interaction $H_{int}$ being determined by the higher fermion correlations, by the Pauli principle and by the coupling with the noncollective degrees of freedom should depend on physical hypothesis.

There are two basic ways to treat interaction in the phenomenological boson Hamiltonian.

The former, called in works [8, 9] as «old phenomenology» starts from the adiabaticity of the collective motion and postulated the anharmonic corrections to the potential energy being the dominating in $H_{int}$. These corrections are strong enough to lead to essential quadrupole boson number nonconservation. Such an approach was justified in refs [7, 10], where the anharmonic terms of $H$ appear as higher corrections to RPA from the fermionic loops. It is equivalent to the calculation of anharmonic corrections from the leading terms of boson expansion (BE) of pair fermionic operators. The method have been developed later and successfully used for description of the spherical and deformed nuclei in refs [5, 22, 23], although the specific technique used by the authors was extremely complicated and influenced by shortcomings decreasing the theoretical precision of the results [22, 3].

The second approach is the «new phenomenology» or IBM (Interacting Boson Model) [2, 11, 18, 29]. It takes into account the processes of boson scattering but includes additionally the new monopole $s$-boson, so that the total boson number $N=N_d+N_s$ is is strictly conserved. It is possible by use of the Holstein—Primakoff-type representation [2] to exclude the $s$-boson and to obtain a pure quadrupole Hamiltonian with $N_d$-nonconservation effects. The result is equivalent to the usual Hamiltonian with anharmonic terms in the $N\to\infty$ limit only. However, such a limit is incompatible with the standard interpretation of IBM which treats $s$- and $d$-bosons as images of fermion (hole) pairs in the valence shells coupled to the total angular momenta $L=0$ and $L=2$ correspondingly. The total boson number $N$ for a given nuclide is fixed by the occupation of fermion shells bein of order $\lesssim 10$ for typical soft nuclei. Thus, the «old phenomenology» and the IBM aren't reducible to each other for the standard interpretation of the IBM-bosons.

Without a dwelling on the conceptial difficulties of the IBM (see, for example, refs [4, 5, 24]) it is likely to remark, keeping in mind the phenomenological aspect of the IBM, that in spite of abundance of fitted parameters (especially for the IBM-2 taking into account so called proton and newtron bosons separately) the IBM does not provide the quantitative description of fine characteristics of nuclear spectra [25, 26, 4]; the basic prediction of the IBM, namely, the «cut-off» of the collective effects for $N_d \geqslant N$ looks like an artefact of the theory. The contraction of the collective space near the



magic or nearmagic nuclei is not so sharp as it should be in IBM. One need in that case to refuse the boson number conservation which is the cornerstone of the IBM-concept.

Meanwhile the «old phenomenology» based on the reliable microscopic grounds allows one to describe, by use of few (and in many cases less than that of the IBM) parameters smoothly varying from one nucleus to another, to understand a large amount of nuclear data (energies and $B$(E2)-values) the quality of description being comparable and often better than obtained in the IBM. The aim of the present paper is to prove these statements.

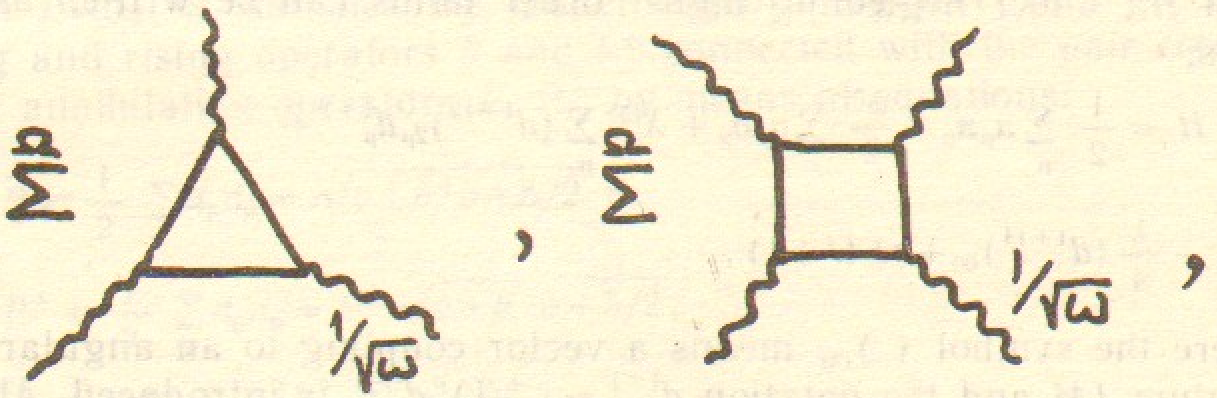
After the justification of the simple Hamiltonian containing the strong quartic and weak cubic anharmonicity as well as corrections to kinetic energy, the algorithm for obtaining its eigenvalues will be described. The $B$(E2)-transition probabilities and quadrupole moment expectation values will be calculated.

As an example of the application of the method, we fit parameters and compare the model results with the exsperimental data for the chains of Pd and Ru isotopes where the IBM-1 and IBM-2 calculations are also available. In general one finds the best agreement of the present model which allows us to «rehabilitate» the old phenomenology as a perspective and «competible» approach.

The first part of the paper contains the formulation of the model and computional expressions for observed quantities. The comparison with data for specific nuclei will be presented in the second part.

## 2. THE MAIN ANHARMONIC TERMS

It is easy to estimate, following [7, 4], the anharmonic terms in the Hamiltonian $H_{int}$, corresponding to the loop-graphs of type



where the phonon $\sim\sim$ is considered to be coherently formed by $\Omega \sim A^{2/3} \gg 1$ single-particle excitations so each vertex contains the normalization factor $\Omega^{-1/2}$. For the soft quadrupole mode, when $\omega \ll 2\bar{E}$ ($2\bar{E}$ is the doubled cooper pair breaking energy being the typical single-particle excitation energy scale) the sum over the fermion internal states leads to the estimate [4]

$$H^{(n)} \sim \omega \Omega^{-n/2+1} (2\bar{E}/\omega)^{n/2+1} (d^{(+)n}), \tag{1}$$

where the amplitudes $d^{(+)}$, $d^{(-)}$ are connected with the boson creation and annihilation operators $d^+$, $d$ and with the collective coordinates $\alpha$ and momenta $\pi$ by means of equations

$$\alpha_\mu = \frac{1}{\sqrt{2\omega}} (d_\mu + (-1)^\mu d^+_{-\mu}) \equiv \frac{1}{\sqrt{2\omega}} d^{(+)}_\mu, \tag{2a}$$

$$\pi_\mu = i\sqrt{\frac{\omega}{2}} (d_\mu - (-1)^\mu d^+_{-\mu}) \equiv i\sqrt{\frac{\omega}{2}} d^{(-)}. \tag{2b}$$

Thus, the $H^{(4)} \sim \omega$ term becomes the main one for the adiabatic region where $2\bar{E}/\omega \sim \Omega^{1/3}$ (which is approximately the case for real nuclei), the cubic term $H^{(3)}$ as well as all the graphs with the odd number of intrinsic fermion lines are significally suppressed due to the particle-hole symmetry with respect to the Fermi surface in nuclei far from the magic ones (similarly to the Furry theorem)

A substitution of the pair of amplitudes $d^{(+)}$ by the momentum combinations $d^{(-)}$ brings in the additional smallness $\omega^2/(2\bar{E})^2$, so that the corrections to the kinetic energy are relatively small in the adiabatic limit. However, one of those is four-phonon component proportional to $\sigma \hat{J}^2 = \sigma J(J+1)$ which is of independent interest corresponding to the virtual rotation of the mean field with the slowly vibrating deformation. Phenomenological fit [16] gives a typical value of the coefficient $\sigma$ of order $10^{-2}$. Finally, the collective Hamiltonian $H_c$, under neglecting higher order terms can be written as follows:

$$H_c = \frac{1}{2} \sum_\mu \pi_\mu \pi_{\bar\mu} + \frac{\omega^2}{2} \sum_\mu \alpha_\mu \alpha_{\bar\mu} + X^{(3)} \sum_\mu (d^{(+)2})_{2\mu} d^{(+)}_{\bar\mu} +$$

$$+ \frac{\lambda}{4} (d^{(+)4})_{00} + \sigma J (J+1), \tag{3}$$

where the symbol $(\ )_{JM}$ means a vector coupling to an angular momentum $JM$ and the notation $d^{(\pm)}_{\bar\mu} = (-1)^\mu d^{(\pm)}_{-\mu}$ is introduced. All the variables are choosen in eq. (3) dimensionless for a convenience.



The cubic interaction coupling constant $X^{(3)}$ later on is considered as small one and is taken into account as a perturbation.

Note that Hamiltonian (3) coincides by its form with that of the Bohr—Mottelson model (except of the virtual rotation term) but the main nonlinear term $H^{(4)}$ here is not treated as a small one and the pure nonlinear limit $\mathring{\omega}^2 \ll \lambda$ should be considered quite seriously as well as the harmonic one $\mathring{\omega}^2 \gg \lambda$.

It should be pointed out that the $\sigma J(J+1)$ term in (3) is in some sense conventional since this diagonal correction being originated from the averaging over the single-particle quasirotational admixtures to the collective wave functions does not vanish for all states besides the ground state and first excited $2^+_1$ which could be choosen purely collective.

### 3. DIAGONALIZATION OF THE PRINCIPAL PART OF HAMILTONIAN

The most important part of eq. (3) is the quartic anharmonic Hamiltonian

$$H = \frac{1}{2} \sum_\mu \pi_\mu \pi_{\bar\mu} + \frac{\mathring{\omega}^2}{2} \sum_\mu \alpha_\mu \alpha_{\bar\mu} + \lambda \Big( \sum_\mu \alpha_\mu \alpha_{\bar\mu} \Big)^2, \tag{4}$$

where the structure of $H^{(4)}$ is determined uniquely [20]. In spite of the arbitrary strength of the quartic interaction, $H$ possess the five-dimentional rotational invariance O(5) and does conserve the corresponding Casimir operator $v(v+3)$, where $v$ coincides with the seniority, the number of bosnons which aren't coupled to the angular momentum $J=0$:

$$N_d = \sum_\mu d^+_\mu d_\mu = v + 2n \equiv 2P_0 - 5/2. \tag{5}$$

The operator of «condensate» boson pair number with (those with $J=0$), $n$, satisfies the ladder algebra together with the lowering and rising operators $b$ and $b^+$ connected with the pair creation and annihilation operators $P$, $P^+$ by means of equations:

$$P \equiv \frac{1}{2} \sum_\mu d_\mu d_{\bar\mu} = \sqrt{v + b^+ b + 5/2}\ b,$$

$$P^+ \equiv \frac{1}{2} \sum_\mu d_\mu d_{\bar\mu} = b^+ \sqrt{v + b^+ b + 5/2},$$

$$[b, n] = b, \qquad [b^+, n] = -b^+.$$

The triplet of operators $P$, $P^+$, $P_0$ conserving the seniority $v$ obeys the commutation relations of the O(2,1)-group.

The Hamiltonian (4) could be expressed in terms of the $P$, $P^+$, $P_0$ only:

$$H=\frac{1}{2}(1+\mathring{\omega}^2)\,2P_0+\frac{1}{2}(\mathring{\omega}^2-1)(P+P^+)+\lambda(2P_0+P+P^+)^2, \tag{6}$$

which allows one to diagonalize it analytically with a good accuracy by means of the canonical transformation of the operators $P$, $P^+$, $P_0$ to the new ones $\tilde{P}$, $\tilde{P}^+$, $\tilde{P}_0$ [15, 16]. The «dangerous graphs» compensation condition [15] gives an equation to determine the frequencies $\omega_v>0$ corresponding to the true excitations of the system:

$$\omega_v^3-\mathring{\omega}^2\omega_v=4\,(v+7/2)\,\lambda. \tag{7}$$

Fig. 1 illustrates the behaviour of the solution of the eq. (7) versus the parameters $\mathring{\omega}$, $\lambda$.

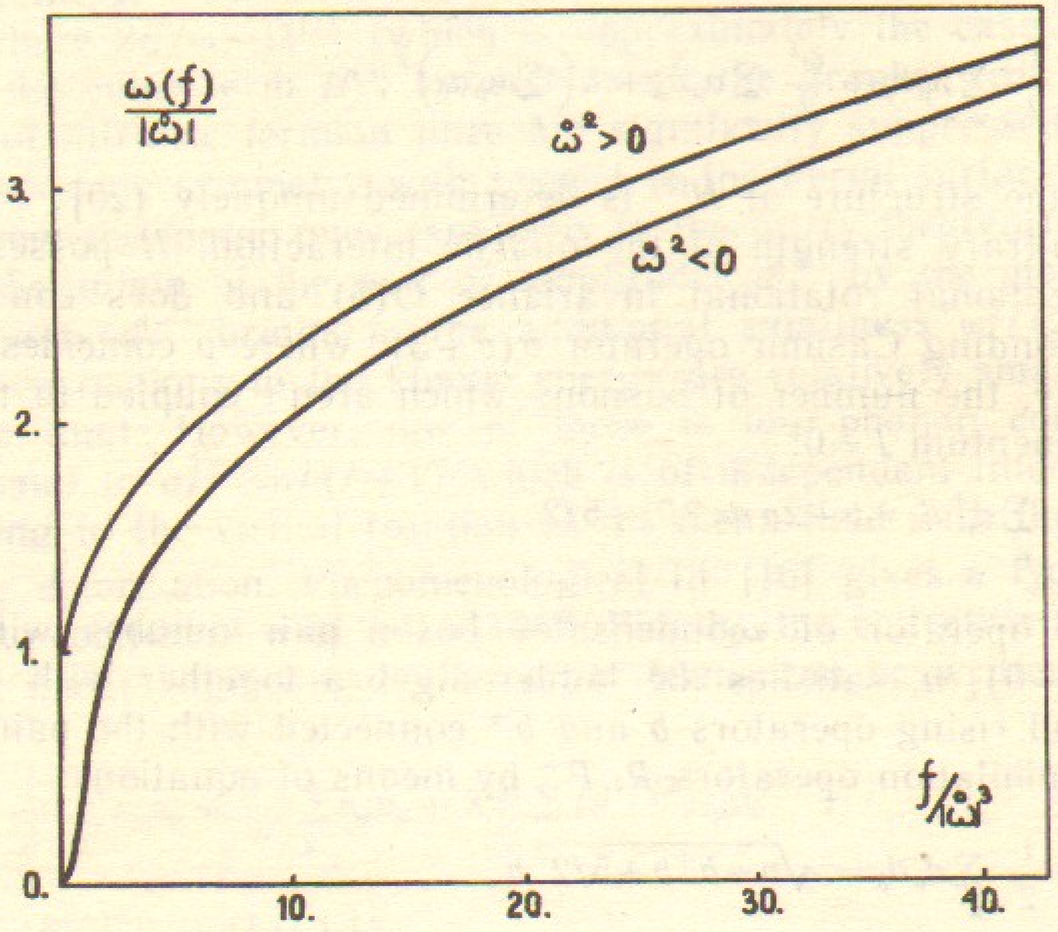



*Fig. 1.* Solution of the equation (7) for the renormalized frequency $\omega_v$ (in units of $|\mathring{\omega}|$) as a function of $f=4\left(v+\frac{7}{2}\right)\lambda$.



The Hamiltonian, after the canonical transformation, becomes to a sum of the *c*-number part $E_{v,n}$ and the operator term describing renormalized interaction of the new excitations,

$$H=E_{v,n}+\omega_v\tilde{\lambda}_v\,[4\,(\tilde{n}\tilde{P}+\tilde{P}^+\tilde{n})+\tilde{P}^2+\tilde{P}^{+2}]\,, \tag{8}$$

As a result of the transformation, the renormalized interaction is strongly depressed, since

$$\tilde{\lambda}_v=\lambda/\omega_v^3=\frac{1-\mathring{\omega}^2/\omega_v^2}{4\,(v+7/2)}\leqslant\frac{1}{14}\,,\qquad \text{for } \mathring{\omega}^2\geqslant 0. \tag{9}$$

The dependence of $\tilde{\lambda}_v$ and $E_{v,n}$ on the parameter $\mathring{\omega}^2/\lambda^{2/3}$ is shown in Fig. 2. one can see that the residual interaction contributing to the energy in the second order of the perturbation theory in the new parameter $\tilde{\lambda}_v$ can be neglected at least as soon as $\mathring{\omega}^2\geqslant 0$. The energy spectrum is described with a good accuracy by the simple formula

$$E_{v,n}=\frac{1}{4}\left(3\omega_v+\frac{\mathring{\omega}^2}{\omega_v}\right)\left(v+\frac{5}{2}\right)+2\omega_v n\left\{1+\tilde{\lambda}_v\left[3\,(\tilde{n}-1)+v+\frac{7}{2}\right]\right\}. \tag{10}$$

At $\mathring{\omega}^2/\lambda^{2/3}\leqslant 0$ it is necessary to take into account the residual interaction in (8); the similar formulae for second and third orders of perturbation theory are given in Appendix (A1—3).

The method used breaks down at $\mathring{\omega}^2/\lambda^{2/3}\lesssim-(6\div 8)$, when the residual interaction ceases to be weak. In this case the wave function is localized not in the vicinity of the origin zeroth point but near a new minimum of the potential energy $U$

$$U(\beta)=\frac{1}{2}\,\mathring{\omega}^2\beta^2+\lambda\beta^4,\qquad \beta^2\equiv\sum_\mu\alpha_\mu\alpha_{\bar{\mu}}\,. \tag{11}$$

Such a situation supposedly takes place in some isotopes of Xe, Ba and Pt, where the level $0_2^+$ from the two-phonon triplet lies too high in the energy scale. Then the modification of the method is necessary by introducing of an additional variational parameter taking into account the displacement of the wave function.

For the typical soft nuclei as Ru and Pd, it is possible to adopt $\mathring{\omega}^2/\lambda^{2/3}\sim 0$. Then the energies of states with quantum numbers $v$ and $n$ are given, for the cubic interaction being neglected by the expression (10) with the rotational correction $J(J+1)$ (see sec. 1).

$$E\,(v,n,J)=E_{v,n}+\sigma J\,(J+1)\,. \tag{12}$$

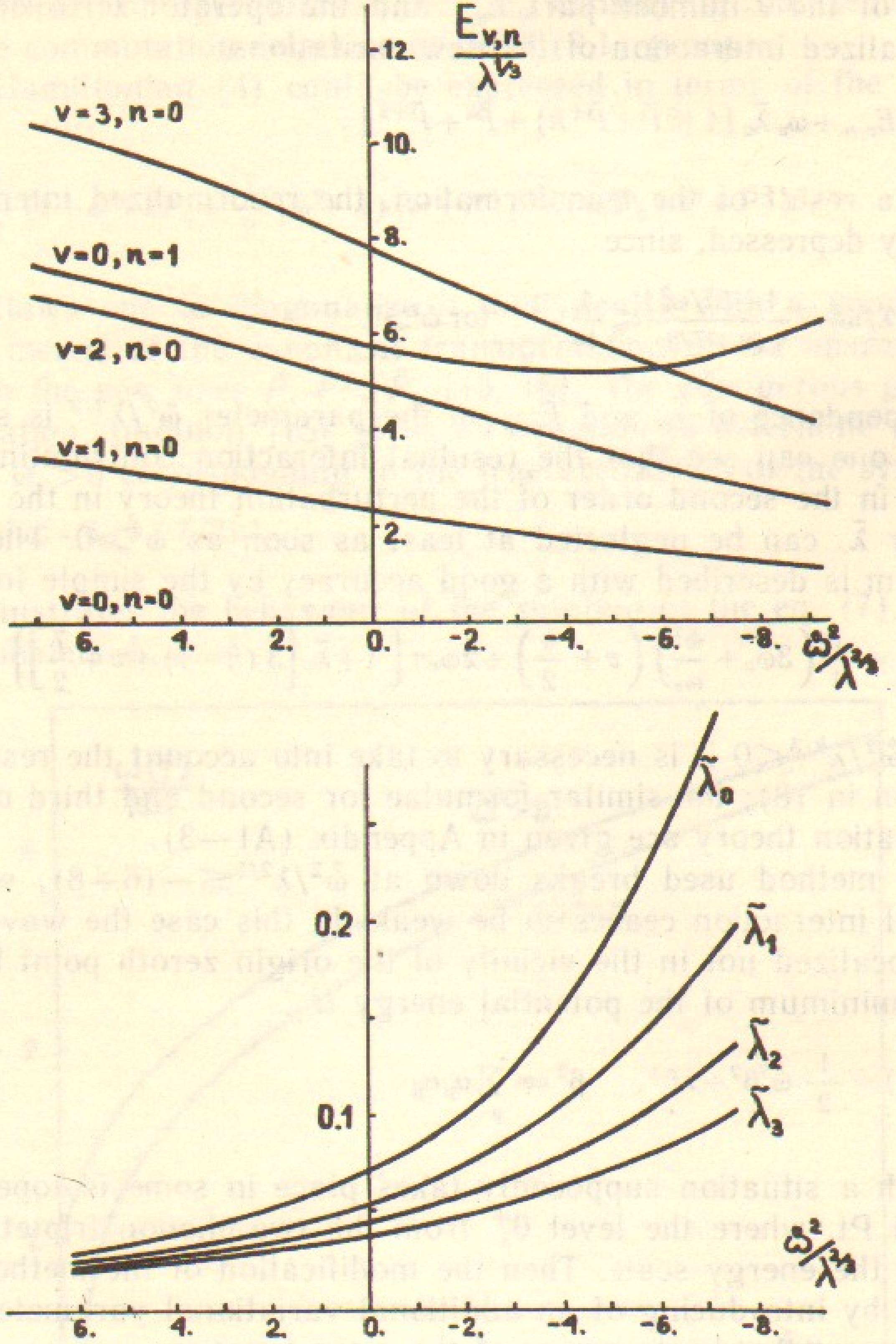



*Fig. 2.* *a*—Energies $E_{v,n}$ (eq. (10)) in units of $\lambda^{1/3}$, measured from the ground state energy versus the parameter $\mathring{\omega}^2/\lambda^{2/3}$ for small $v$ and $\tilde{n}$. Note the crossing of the «two-phonon» $0_2^+$ level and the level with $v=3$ at $\mathring{\omega}^2/\lambda^{2/3}<-6$. *b*—Dependence of the reduced coupling constant $\tilde{\lambda}_v$ (eq. (9)) on the parameter $\mathring{\omega}^2/\lambda^{2/3}$ for $v=0$, 1, 2, 3.



In such a model, all calculations are extremely simple to be carried out with the aid of a pocket calculator. In ref. [16] the predictions of the model (12) were compared with experimental data for the chain of $_{46}$Pd isotopes and the agreement was achieved by fitting only one parameter $\sigma$ better than with the use of three-parameter formulae of IBM. As an example, the energy levels of $^{110}_{46}$Pd from ref. [28] as well as results of calculations by use of formulae (12) for $\mathring{\omega}^2/\lambda^{2/3}=5.2$ and $\sigma=0.026$ are shown in Fig. 3. What is striking is a good agreement between the experimental values and theoretical ones for only two parameters fitted (spectrum was normalized according to requirement $E(2_1^+)_{th}=E(2_1^+)_{exp}$. The nuclei $^{110}$Pd and $^{104}$Ru (see below) appear to be typical ones displaying the developed boson bands; those are labelled in figure by letters $Y$, $X$, $Z$, $\Delta$, $\beta$ according to the notions adopted in [2]. The O(5)-multiplets of states with the same $v$ splitted by the correction $\sigma J(J+1)$ are well pronounced. The SU(5)-multiplets with same $N_d=v+2\tilde{n}$ are destructed by the interaction $H^{(4)}$; that is seen clearly in the case of $^{104}$Ru for which the $0_2^+$ and $2_3^+$-states are shifted up in the energy. It should be mentioned that the almost pure O(5)-symmetry with a strong nonconservation of boson number (Hamiltonian of type (4)) is apparently realized in pure quality in some nuclei [27].

The eigenfunctions of the Hamiltonian $H$ with quantum numbers $v$, $\tilde{n}$ and angular momentum $(JM)$ can be expanded over the nonperturbed eigenstates $|v,n,JM\rangle$ of the harmonic approximation with the same $v$ but different $n$,

$$|v,\tilde{n},JM\rangle=\sum_{n'}A^v_{n,n'}\,|v,n',JM\rangle\,. \tag{13}$$

In our approximation, the overlap factors $A^v_{n,n'}(\omega_v,\mathring{\omega})$ are given by the formula [16]

$$A^v_{n,n'}(\omega_v,\mathring{\omega})=\frac{(-1)^{n'}}{\Gamma(v+5/2)}\left[\frac{\Gamma(v+n+5/2)\,\Gamma(v+n'+5/2)}{n!\,n'!}\right]^{1/2}\left(\frac{2\sqrt{\omega_v|\mathring{\omega}|}}{\omega_v+|\mathring{\omega}|}\right)^{v+5/2}\times$$
$$\times\left(\frac{\omega_v-|\mathring{\omega}|}{\omega_v+|\mathring{\omega}|}\right)^{n+n'}F\left(-n',-n,v+5/2,\frac{-4\omega_v|\mathring{\omega}|}{(\omega_v-|\mathring{\omega}|)^2}\right), \tag{14}$$

where $F$ is a hypergeometric function. It is evident from the expression (12) that the superposition (13) has a large width $\Delta n'$ for a strong anharmonicity $\mathring{\omega}^2/\lambda^{2/3}<1$. At the weak anharmonicity limit $\mathring{\omega}^2/\lambda^{2/3}\gg1$, formulas are equivalent to those of perturbation theory.

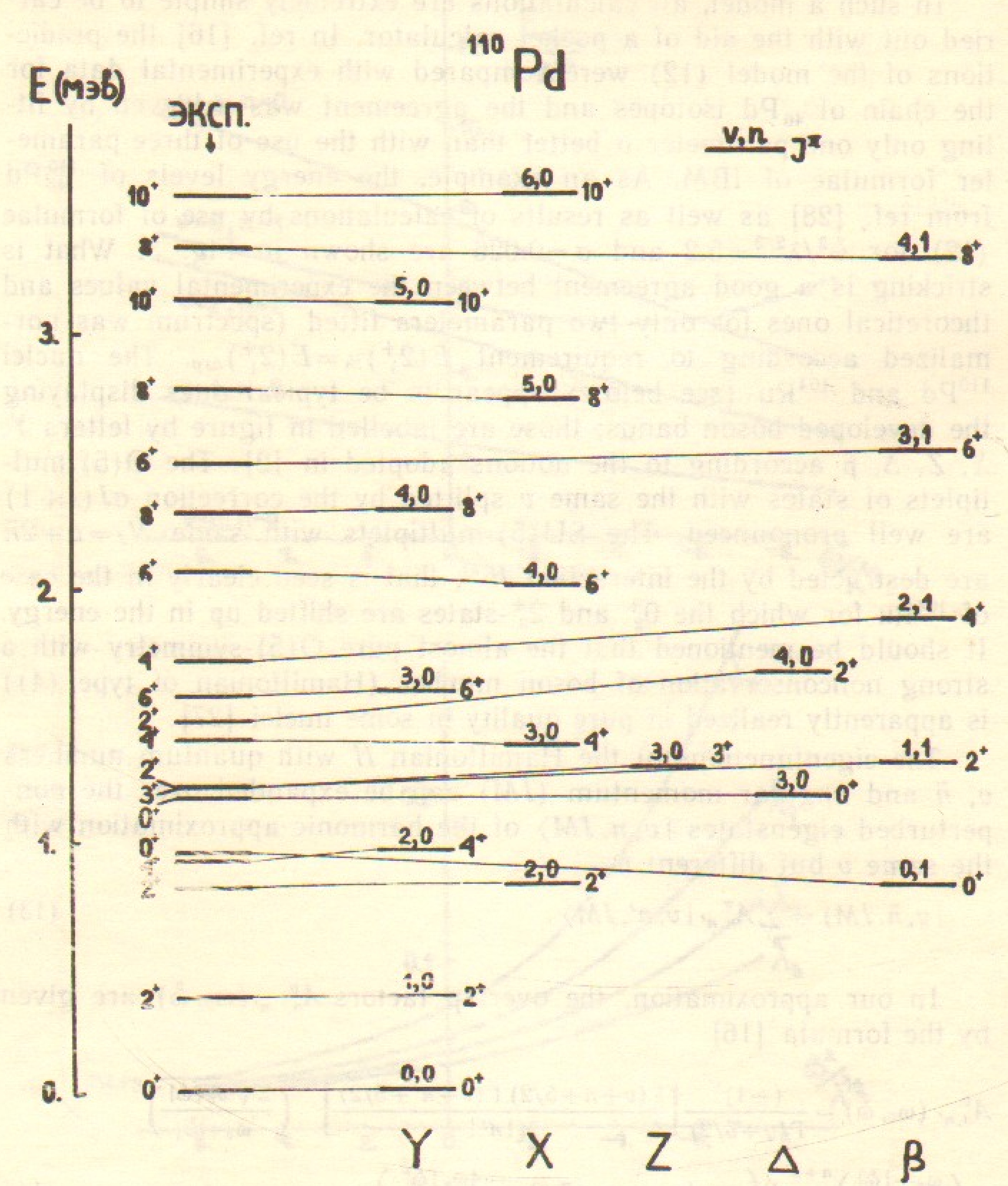



*Fig. 3.* Energy levels in $^{110}$Pd: the experimental data on the left and the results of calculations in the present model for $\hbar\omega^2/\lambda^{2/3}=5.2$ and $\sigma=0.026$. The values of $v$ and $\bar{n}$ are marked above levels.



## 4. QUADRUPOLE TRANSITION PROBABILITIES AND STATIC PROPERTIES

To calculate the collective E2-transitions, one expresses usually the phenomenological quadrupole operator expressed in terms of the collective variables only. For the microscopical reason, one should choose it in the form

$$T_\mu^{(E2)} = d_\mu^{(+)} + \varkappa\,((d^{(+)2})_0\, d^{(+)})_{2\mu} + q\,(d^{(+)2})_{2\mu} + q'\,(d^{(-)2})_{2\mu} + \frac{k}{2}\,([J, d^{(-)}]_+)_{2\mu}\,, \qquad (15)$$

where the new operator $\varkappa(d^{(+)3})_{2\mu}$ is introduced besides the main term $d_\mu^{(+)}$ and the additional $q(d^{(+)2})_{2\mu}+q'(d^{(-)2})_{2\mu}$ term to describe weak cross-over transitions and quadrupole moment expectation values. The term $\varkappa(d^{(+)3})_{2\mu}$ arises from the same fermionic loop as $H^{(4)}$ via the boson expansion of fermion operators. According to the microscopic estimates it should be $\varkappa\lesssim 0$ and $|\varkappa|\sim 0.1\div 0.3$, which is an agreement with fitting results for many nuclei [16, 27]. The two main effects causing by the new operator are the attenuation of the enhanced transition probabilities with $|\Delta N_d|=1$ (especially for transitions from the β-band like $0_2^+\to 2_1^+$, see Fig. 4) and the enhancement of forbidden transitions $2_3^+\to 0_1^+$ with $\Delta N_d=3$. Such probabilities are observed experimentally being extremely sensitive to the value of $\varkappa$.

One should note that it is possible to describe the beaviour of the transition probabilities along the yrast-band in dependence on value $J$ in the broad range of nuclei by variation of only one parameter $\varkappa$ in $T^{(E2)}$ from 0. to $-0.40$ (see Fig. 5).

In accordance with the adiabaticity the momentum amplitudes enter into (15) by the «minimal» manner and coefficients $q'$ and $k$ should be rather small. This is the case as can be seen from the results of fitting (for the most nuclei $q'$, $k\sim 10^{-2}$, see [16] and below).

The last term in (15) containing the angular momentum operator has the same origin as the quasirotational term in eq. (12) being essential for the description of fine characteristics of the $B$(E2)-transition pattern, such as «splitting» of values $B(4_1^+\to 2_1^+)$ and $B(2_2^+\to 2_1^+)$.

Using the matrix elements of operator $T^{(E2)}$ between the oscillator states $|v, n, J\rangle$ (see Appendix B) and eq. (14), it is easy to compute the probabilities of transitions with $\Delta N_d=1$ and $\Delta N_d=3$

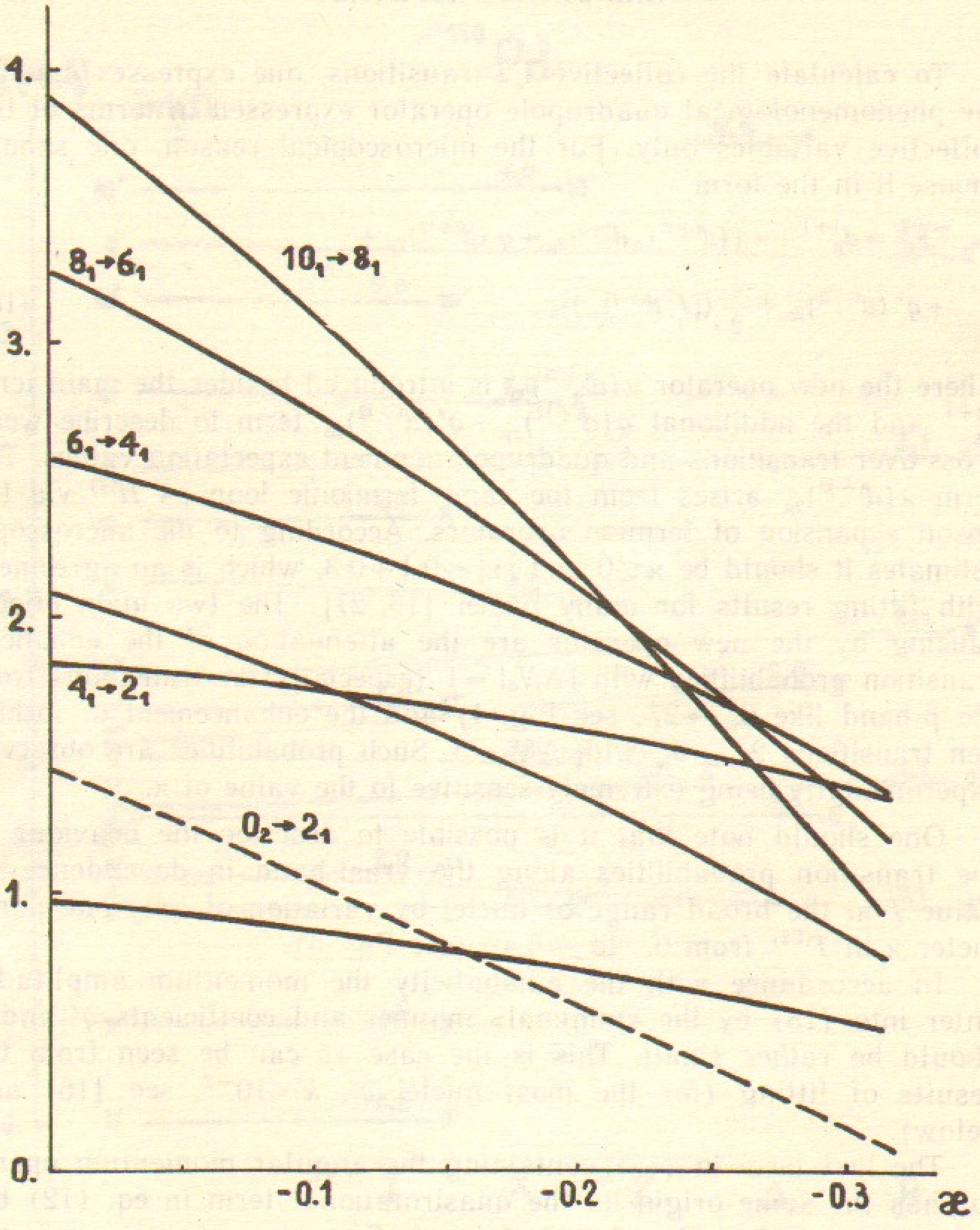



*Fig. 4.* E2-transition probabilities with $\Delta N_d = 1$ in units of $B(\mathrm{E2};\ 2_1^+ \to 0_1^+)$ induced by the $T^{(\mathrm{E2})}$ operator (eq. (15)) in the simplest one-parameter variant $\varkappa \neq 0$ and for the pure quartic interaction, $\mathring{\omega}^2/\lambda^{2/3} = 0$ as functions of $\varkappa$.



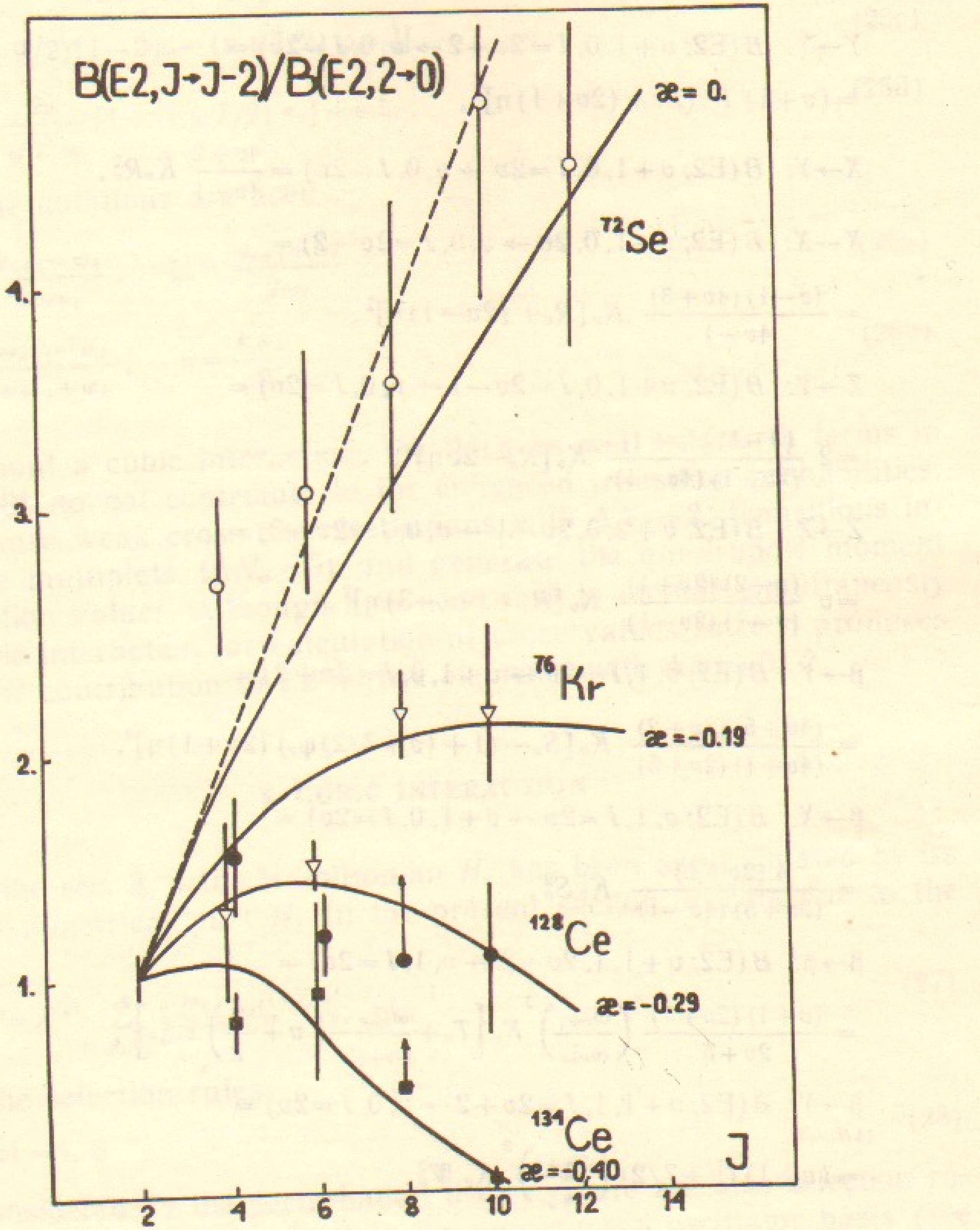



*Fig. 5.* Experimental values of E2-transition probabilities inside the yrast-band in units of $B(\mathrm{E2};\ 2_1^+ \to 0_1^+)$ for some isotopes of Se, Kr and Ce as compared with the model prediction in the simplest case $\mathring{\omega}^2/\lambda^{2/3} = 0$ with the only parameter $\varkappa$ in $T^{(\mathrm{E2})}$ (15). The experimental data for the Se and Kr isotopes are taken from ref. [29], and for Ce from ref. [30].

(bands connected by the transitions marked on the left)

$Y \to Y$: $B(\mathrm{E2}; v+1, 0, J=2v+2 \to v, 0, J=2v) =$

$$= (v+1)\, K_v [R_v + (2v+1)\eta]^2, \tag{16}$$

$X \to Y$: $$B(\mathrm{E2}; v+1, 0, J=2v \to v, 0, J=2v) = \frac{4v+2}{4v-1} K_v R_v^2, \tag{17}$$

$X \to X$: $B(\mathrm{E2}; v+1, 0, 2v \to v, 0, J=2v-2) =$

$$= \frac{(v-1)(4v+3)}{4v-1} K_v [R_v + (2v-1)\eta]^2, \tag{18}$$

$Z \to Y$: $B(\mathrm{E2}; v+1, 0, J=2v-1 \to v, 0, J=2v) =$

$$= 2\, \frac{(v-1)(4v+1)}{(2v-1)(4v-1)} K_v [R_v - 2v\eta]^2, \tag{19}$$

$Z \to Z$: $B(\mathrm{E2}; v+2, 0, 2v-1 \to v, 0, J=2v-3) =$

$$= v\, \frac{(v-2)(2v+1)}{(v-1)(2v-1)} K_v [R_v + (4v-3)\eta]^2, \tag{20}$$

$\beta \to Y$: $B(\mathrm{E2}; v, 1, J=2v \to v+1, 0, J=2v+2) =$

$$= \frac{(4v+5)(2v+2)}{(4v+1)(2v+5)} K_v [S_v - (1 + (v+7/2)\varphi_v)(2v+1)\eta]^2, \tag{21}$$

$\beta \to X$: $B(\mathrm{E2}; v, 1, J=2v \to v+1, 0, J=2v) =$

$$= \frac{4(2v+1)}{(2v+5)(4v-1)} K_v S_v^2, \tag{22}$$

$\beta \to \beta$: $B(\mathrm{E2}; v+1, 1, 2v+2 \to v, 1, J=2v) =$

$$= \frac{(v+1)(2v+7)}{2v+5} \left(\frac{\omega_{v+1}}{\omega_v \xi_v}\right)^2 K_v \left[T_v + \frac{\omega_v \xi_v}{\omega_{v+1}} - \left(v + \frac{9}{2}\right) \varepsilon_v \xi_v\right]^2, \tag{23}$$

$\beta \to Y$: $B(\mathrm{E2}; v+1, 1, J=2v+2 \to v, 0, J=2v) =$
$(\Delta N_d = 3)$

$$= (v+1)(v+7/2) \left(\frac{\omega_{v+1}}{\omega_v \xi_v}\right)^2 K_v W_v^2, \tag{24}$$

here the factors $R_v$, $S_v$, $T_v$ and $W_v$ are determined by formulas

$$R_v = 1 + \frac{2\varkappa}{\sqrt{5}} \cdot \frac{2v+7}{\omega_v + \omega_{v+1}}, \tag{25a}$$

$$S_v = 1 - \left(v + \frac{7}{2}\right)\varphi_v + \frac{4\varkappa}{\sqrt{5}} \cdot \frac{2v+7}{\omega_v + \omega_{v+1}} [1 - \varphi_v (v+9/2)/2], \tag{25b}$$



$$T_v = \xi_v - \left(v + \frac{9}{2}\right) \varepsilon_v \xi_v + \frac{2\varkappa}{\sqrt{5}\ \omega_v} [2 + (2v+7)\varepsilon_v + (v+9/2)[1 - 2\varphi_v - (v+7/2)\varphi_v \varepsilon_v]], \tag{25c}$$

$$W_v = \frac{2\varkappa}{\sqrt{5}\ \omega_v} [1 + (v+7/2)\varepsilon_v] + \varepsilon_v \xi_v, \tag{25d}$$

where the notations are used

$$\varepsilon_v = \frac{\omega_{v+1} - \omega_v}{2\omega_{v+1}}, \quad \xi_v = \frac{\omega_{v+1} + \omega_v}{2\omega_v}, \tag{26a}$$

$$\varphi_v = \frac{\omega_{v+1} - \omega_v}{\omega_{v+1} + \omega_v}, \qquad \eta = \frac{k\,\omega_v}{\sqrt{6}}. \tag{26b}$$

Without a cubic interaction, the third as well as fourth terms in $T^{(\mathrm{E2})}$ [16] do not contribute to the enhanced transition probabilities. They cause weak cross-over transitions with $\Delta N_d = 2$, transitions inside the multiplets ($\Delta N_d = 0$) and generate the quadrupole moment expectation values. Although it is necessary to include simultaneosly the cubic interaction for calculation of those values since it produces the same contribution to E2-matrix elements with $\Delta N_d = 0$, 2.

## 5. CUBIC INTERACTION

In the sec. 3, 4 the Hamiltonian $H_c$ has been approximated by its O(5)-symmetrical part $H$. In the present section, effects due to the term

$$H^{(3)} = X^{(3)} \sum_{\mu_1, \mu_2, \mu_3} C^{2\mu_3}_{2\mu_1\, 2\mu_2}\, d^{(+)}_{\mu_1}\, d^{(+)}_{\mu_2}\, d^{(+)}_{\bar{\mu}_3}, \tag{27}$$

with the selection rules

$$|\Delta v| = 1,\ 3 \tag{28}$$

are considered by the perturbation theory. There are also selection rules $|\Delta n| \leqslant 3$ and $|\Delta N_d| = 1$, 3 in the unperturbed oscillator basis (see Appendix, eqs A7—14), which are, rigorously speaking, violated in the physical basis $|v, n, J)$ of eigenvectors of $H$ (see eqs (13, 14)). As a consequence of (28), $H^{(3)}$ does not conserve $v$ and violates the O(5)-symmetry. Thus, the eigenstates of the full Hamiltonian $H_c = H + H^{(3)}$ have not definite values of the seniority $v$. Nevertheless, for $H^{(3)} \ll H$ it is possible to label the wave functions corrected according to the perturbation theory by values of $v$. Later the states containing the correction from $H^{(3)}$ are marked by asterisks in distinction of those from O(5)-scheme $|v, n, J)$:

$$|v, \tilde{n}, J)^* = \sum_{v', n'} C^{vnJ}_{v'n'} |v', n', J) \,. \qquad (29)$$

Because of (28) $H^{(3)}$ hasn't diagonal matrix elements so that the first nonvanishing correction to energy appears in the second order of perturbation theory only. In the same approximation, the eigenvalues $E^*(v, \tilde{n}, J)$ of the Hamiltonian $H_c$ are given by

$$E^*(v, \tilde{n}, J) = E(v, \tilde{n}, J) + \Delta E(v, \tilde{n}, J) \,, \qquad (30)$$

expressions for the corrections $\Delta E(v, \tilde{n}, J)$ are written down in Appendix (A30—34).

The relative smallness of the cubic corrections is determined by ratio $\Delta E(v, \tilde{n}, J)/E^{(\cdot}, \tilde{n}, J) \sim X^{(3)2}/\omega_0^5$, since the canonical transformation (sec. 3) corr. ponds to the coordinates and momentum rescaling

$$\tilde{d}^{(+)}_\mu = \frac{1}{\sqrt{\omega_v}} d^{(+)}_\mu; \qquad d^{(-)}_\mu = \sqrt{\omega_v}\, d^{(-)}_\mu \,. \qquad (31)$$

So that the cubic coupling constant enters in all observed quantities as $X^{(3)}/\omega_0^{5/2}$. It is useful to choose the value of

$$\chi = X^{(3)2}/\omega_0^5 \qquad (32)$$

as an universal measure of the cubic anharmonicity; all the large quantities (energy shifts and transition probabiliies with $|\Delta N_d| = 1$) are linear in $\chi$ in the main order.

In Fig. 6, the energies $E^*(v, \tilde{n}, J)$ measured from the ground state energy are plotted in units of the $E_{2_1^+}$ for the three typical ratios of $\mathring{\omega}$ and $\lambda$. In the absence of quartic interaction ($\lambda = 0$) the degenerated vibrational multiplets are observed being splitted by inclusion of $\chi \neq 0$. The middle part of Fig. 6 corresponds to the pure quartic asymptotic for $\mathring{\omega}^2/\lambda^{2/3} = 0$. Here the effects by cubic interaction are superimposed on the pattern of conserved O(5)-multiplets of states with same $v$ (the $\sigma J(J+1)$ correction is not included here). The right part of Fig. 6 corresponds to the case $\mathring{\omega}^2/\lambda^{2/3} = -4$ where the potential energy (11) has a pronounced minimum at the nonzero deformation parameter $\beta$. Here the cubic anharmonicity tends to



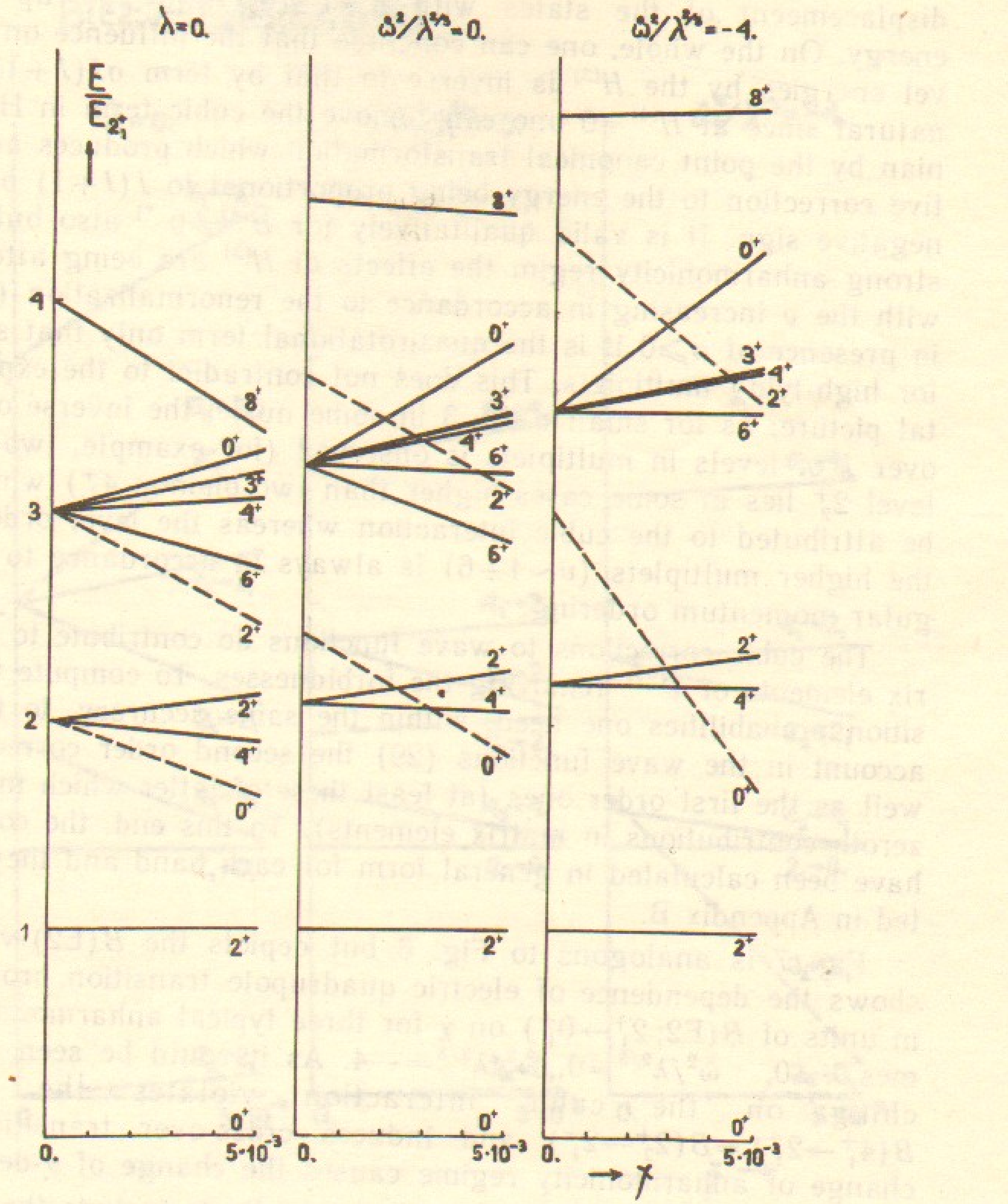



*Fig. 6.* Energy levels with the quantum numbers $v$, $\tilde{n}$, $J$ (in units of $E_{2_1^+}$) calculated with the account of the $H^{(3)}$ for $\sigma = 0$ versus the strength of the cubic interaction $\chi = X^{(3)2}/\omega_0^5$ for three cases: without any quartic anharmonicity, in the strong quartic anharmonicity limit and for $\mathring{\omega}^2 < 0$ (from left to right).

compensate effects of the strong quartic anharmonicity inducing the displacemeent of the states with $\tilde{n}=1$ ($0_2^+$ and $2_3^+$) up in the energy. On the whole, one can conclude that the influence on the level energies by the $H^{(3)}$ is inverse to that by term $\sigma J(J+1)$. It is natural since at $H^{(4)}=0$ one can remove the cubic term in Hamiltonian by the point canonical transformation which produces an effective correction to the energy being proportional to $J(J+1)$ but with negative sign. It is valid qualitatively for $H^{(4)}\neq 0$ *) also but in the strong anharmonicity regim the effects of $H^{(3)}$ are being attenuated with the $v$ increasing in accordance to the renormalization (31), so in presence of $\sigma\geqslant 0$ it is the quasirotational term only that survives for high-lying multiplets. This does not contradict to the experimental picture: as for small $v\sim 2, 3$ in some nuclei the inverse ordering over $J$ of levels in multiplets is observed (for example, two-phonon level $2_2^+$ lies in some cases higher than two-phonon $4_1^+$) which may be attributed to the cubic interaction whereas the level ordering in the higher multiplets ($v\sim 4\div 6$) is always in accordance to the angular momentum ordering.

The cubic corrections to wave functions do contribute to all matrix elements of $T^{(E2)}$ removing the forbidnesses. To compute the transition probabilities one need, within the same accuracy, to take into account in the wave functions (29) the second order corrections as well as the first order ones (at least these of latter which make nonzeroth contributions in matrix elements). To this end, the coefficients have been calculated in general form for each band and they are listed in Appendix B.

Fig. 7 is analogous to Fig. 6 but depicts the $B(E2)$-values. It shows the dependence of electric quadrupole transition probabilities in units of $B(E2; 2_1^+\to 0_1^+)$ on $\chi$ for three typical anharmonicity regimes $\lambda=0$, $\mathring{\omega}^2/\lambda^{2/3}=0$, $\mathring{\omega}^2/\lambda^{2/3}=-4$. As it could be seen the switching on the cubic interaction violates the equality $B(4_1^+\to 2_1^+)=B(2_2^+\to 2_1^+)$ and induces cross-over transitions. The change of anharmonicity regime causes the change of $\chi$-dependence of $B(E2)$ values. One sees also the necessity to include the $\varkappa(d^{(+)3})_2$ term into $T^{(E2)}$-operator (plots are obtained for $\varkappa=0$) since the experimental values don't exceed usually 1.5—2.0 for a quantity $B(0_2^+\to 2_1^+)/B(2_1^+\to 0_1^+)$ being of order 0.5—0.9.

*) The same procedure couldn't be made up at the strong quartic anharmonicity, which makes doubtful results obtained by means of boson expansions [22], because the anharmonicity appeared to be strong.



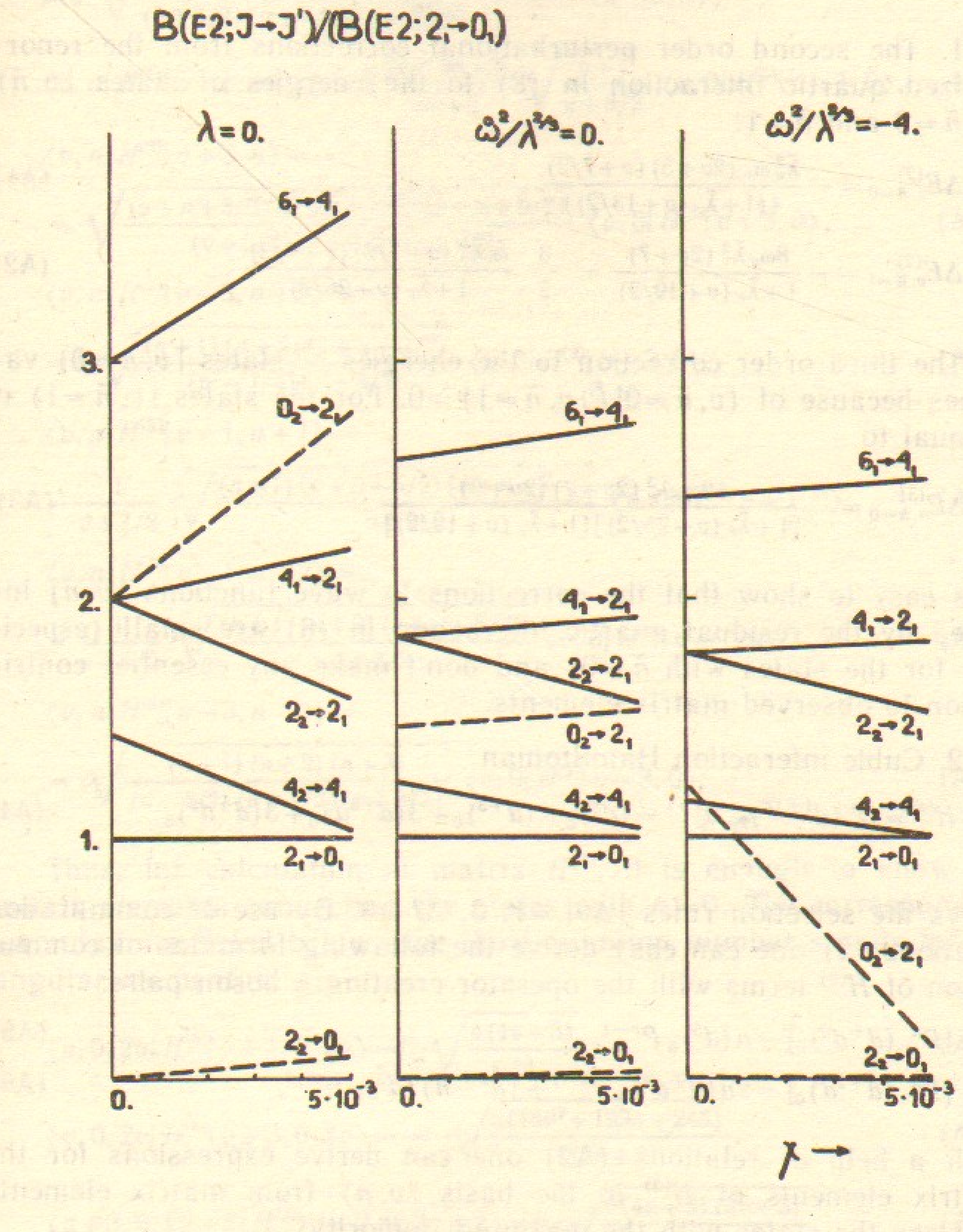



*Fig. 7.* Dependence of $B(E2)$-values calculated with account of the main term $d^{(+)}$ in $T^{(E2)}$ (15) only on the strength of the cubic interaction $\chi=X^{(3)2}/\omega_0^5$ for the three anharmonicity regimes: $\lambda=0$, $\mathring{\omega}^2/\lambda^{2/3}=0$ and $\mathring{\omega}^2/\lambda^{2/3}=-4$.

## APPENDIX A

1. The second order perturbational corrections from the renormalized quartic interaction in (8) to the energies of states $|v,\tilde{n})$ for $\tilde{n}=0$ and $\tilde{n}=1$:

$$\Delta E^{(2)}_{v,\tilde{n}=0} = -\frac{\tilde{\lambda}_v^2\,\omega_v\,(2v+5)\,(v+7/2)}{4\,(1+\tilde{\lambda}_v\,(v+13/2))}\,, \tag{A1}$$

$$\Delta E^{(2)}_{v,\tilde{n}=1} = -\frac{8\omega_v\,\tilde{\lambda}_v^2\,(2v+7)}{1+\tilde{\lambda}_v\,(v+19/2)} - \frac{3}{2}\,\frac{\omega_v\tilde{\lambda}_v^2\,(v+7/2)\,(v+9/2)}{1+\tilde{\lambda}_v\,(v+25/2)}\,. \tag{A2}$$

The third order correction to the energies of states $|v,n=0)$ vanishes because of $(v,\tilde{n}=0|\tilde{P}|\,v,\tilde{n}=1)=0$. For the states $|v,\tilde{n}=1)$ it is equal to

$$\Delta E^{(3)}_{v,\tilde{n}=0} = -\frac{12\omega_v\,\lambda_v^3\,(2v+7)\,(2v+9)}{[1+\tilde{\lambda}_v\,(v+25/2)][1+\tilde{\lambda}_v\,(v+19/2)]}\,. \tag{A3}$$

It is easy to show that the corrections to wave functions $|v,n)$ induced by the residual quartic interaction in (8) are small (especially for the states with $\tilde{n}=0$) and don't make any essential contribution to observed matrix elements.

2. Cubic interaction Hamiltonian

$$H^{(3)} = \sum_{\mu} (d^{(+)2})_{2\mu}\, d^{(+)}_{\bar{\mu}} = (d^3)_0 + (d^{+3})_0 + 3(d^{+2}d)_0 + 3(d^{+}d^2)_0 \tag{A4}$$

obeys the selection rules $|\Delta v|=1,\ 3,\ \Delta J=0$. By use of commutation relations (2) one can easy derive the following formulas of commutation of $H^{(3)}$ terms with the operator creating $n$ boson pairs:

$$[P^n, (d^+d^2)_0] = n\,(d^3)_0\,P^{n-1}, \tag{A5}$$

$$[P^n, (d^{+2}d)_0] = 2n\,(d^+d^2)_0\,P^{n-1} + (n^2-n)\,(d^3)_0\,P^{n-2}, \tag{A6}$$

with a help of relations (A2) one can derive expressions for the matrix elements of $H^{(3)}$ in the basis $|v,n\rangle$ from matrix elements between the states with the maximum seniority:

$$\langle v,n|\,H^{(3)}|\,v+1,n\rangle = \frac{v+3n+7/2}{v+7/2}\sqrt{\frac{v+n+5/2}{v+5/2}}\,\langle v,0|\,H^{(3)}|\,v+1,0\rangle, \tag{A7}$$

$$\langle v,n|\,H^{(3)}|\,v+3,n-1\rangle =$$



$$=3\sqrt{\frac{n\,(v+n+5/2)\,(v+n+7/2)}{(v+5/2)\,(v+7/2)\,(v+9/2)}}\,\langle v,0|\,H^{(3)}|\,v+3,0\rangle, \tag{A8}$$

$$\langle v,n|\,H^{(3)}|\,v-1,n+1\rangle = 2\,\frac{2v+3n+5}{2v+5}\sqrt{\frac{n+1}{v+3/2}}\,\langle v,0|\,H^{(3)}|\,v-1,0\rangle, \tag{A9}$$

$$\langle v,n|\,H^{(3)}|\,v+3,n\rangle =$$

$$=\sqrt{\frac{(v+n+5/2)\,(v+n+7/2)\,(v+n+9/2)}{(v+5/2)\,(v+7/2)\,(v+9/2)}}\,\langle v,0|\,H^{(3)}|\,v+3,0\rangle, \tag{A10}$$

$$\langle v,n|\,H^{(3)}|\,v-3,n+2\rangle =$$

$$=3\sqrt{\frac{(n+1)\,(n+2)\,(v+n+3/2)}{(v^2-1/4)\,(v+3/2)}}\,\langle v,0|\,H^{(3)}|\,v-3,0\rangle, \tag{A11}$$

$$\langle v,n|\,H^{(3)}|\,v+1,n+1\rangle =$$

$$=\frac{1}{v+7/2}\sqrt{\frac{(n+1)\,(v+n+5/2)\,(v+n+7/2)}{v+5/2}}\,\langle v,0|\,H^{(3)}|\,v+1,0\rangle, \tag{A12}$$

$$\langle v,n|\,H^{(3)}|\,v-1,n+2\rangle =$$

$$=\frac{1}{v+5/2}\sqrt{\frac{(n+1)\,(n+2)\,(v+n+5/2)}{v+3/2}}\,\langle v,0|\,H^{(3)}|\,v-1,0\rangle, \tag{A13}$$

$$\langle v,n|\,H^{(3)}|\,v-3,n+3\rangle =$$

$$=\sqrt{\frac{(n+1)\,(n+2)\,(n+3)}{(v-1/2)\,(v+1/2)\,(v+3/2)}}\,\langle v,0|\,H^{(3)}|\,v-3,0\rangle, \tag{A14}$$

Thus, for calculation of matrix $H^{(3)}$, it is enough to know the matrix elements connecting the states with $n=0$. The corresponding formulas are listed below (the third quantum number stands for the angular momentum $J$):

$$\langle v,0,2v|\,H^{(3)}|\,v+1,0,2v\rangle = 6\sqrt{\frac{v\,(4v+3)}{14}}, \tag{A15}$$

$$\langle v,0,2v|\,H^{(3)}|\,v+3,0,2v\rangle = = \sqrt{\frac{6\,(18v^2+199v+245)}{7\,(2v+7)}}, \tag{A16}$$

$$\langle v+3,0,2v+2|\,H^{(3)}|\,v+2,0,2v+2\rangle = 6\sqrt{\frac{v\,(4v+9)\,(2v+3)}{7\,(2v+7)}}, \tag{A17}$$

$$\langle v,0,2v-2|\,H^{(3)}|\,v+3,0,2v-2\rangle = \sqrt{\frac{6\,(36v^3+944v^2+2175v+1024)}{7\,(2v+5)\,(2v+7)}}, \tag{A18}$$

$$\langle v+3,0,2v+3|\,H^{(3)}|\,v+4,0,2v+3\rangle = 6\sqrt{\frac{v\,(4v+9)}{14}}, \tag{A19}$$

$$\langle v+3, 0, 2v+3 | H^{(3)} | v+6, 0, 2v+3\rangle = \sqrt{\frac{6(54v^2+739v+1859)}{7(2v+13)}}, \tag{A20}$$

$$\langle v+3, 0, 2v | H^{(3)} | v+2, 0, 2v\rangle =$$

$$= -12\sqrt{\frac{6v(v-1)(4v+3)(4v+5)(4v+2)}{7(2v+5)(2v+7)(18v^2+199v+245)}}, \tag{A21}$$

$$\langle v+2, 0, 2v-2 | H^{(3)} | v+3, 0, 2v-2\rangle =$$

$$= \frac{6(2v-1)}{2v+5}\sqrt{\frac{(v-1)(4v-1)(36v^3+944v^2+2175v+1024)}{14(2v+7)(18v^2+163v+64)}}, \tag{A22}$$

3. Wave functions of states $|v, 0, J, M=J\rangle$.

$$Y\text{-band:}\ |v, 0, J=2v\rangle = \frac{1}{\sqrt{v!}}(d_2^+)^v|0\rangle, \tag{A23}$$

$$X\text{-band:}\ |v, 0, J=2v-2\rangle =$$

$$= \frac{1}{\sqrt{(4v-1)(v-2)!}}\left(2d_2^+d_0^+ - \sqrt{\frac{3}{2}}\,d_1^{+2}\right)(d_2^+)^{v-2}|0\rangle =$$

$$= \sqrt{\frac{7}{2(4v-1)(v-2)!}}(d^{+2})_{22}(d_2^+)^{v-2}|0\rangle = \tag{A24}$$

$$X'\text{-band:}\ |v, 0, J=2v-4\rangle = \frac{1}{\sqrt{(4v-5)(4v-3)(2v-3)}}\times$$

$$\times\left\{\frac{7}{2}\sqrt{\frac{(2v+1)}{2(v-4)!}}((d^{+2})_{22})^2(d_2^+)^{v-4}|0\rangle -\right.$$

$$\left. -4\sqrt{(v-3)(v-2)}\ |v-2, 1, J=2v-4\rangle\right\}, \tag{A25}$$

$$\Delta\text{-band:}\ |v, 0, J=2v-6\rangle = \frac{1}{\sqrt{18v^2+91v-190}}\times$$

$$\times\left\{\sqrt{\frac{7(2v+1)}{6(v-3)!}}(d^{+3})_{00}(d_2^+)^{v-3}|0\rangle -\right.$$

$$\left. -\sqrt{6(v-3)(4v-9)}\ |v-2, 1, J=2v-6\rangle\right\}, \tag{A26}$$

$$\Delta'\text{-band:}\ |v, 0, J=2v-8\rangle = \frac{1}{\sqrt{36v^3+620v^2-4461v+2023}}\times$$

$$\times\left\{\sqrt{\frac{7(4v^2-1)}{6}}\ (d^{+3})_0|v-3, 0, 2v-8\rangle -\right.$$



$$-2\sqrt{\frac{3(v-5)(4v-11)(2v-7)(2v-1)}{2v-3}}\ |v-2, 1, 2v-8\rangle -$$

$$\left. -2\sqrt{\frac{6(v-4)(4v-13)(2v+1)}{2v-3}}\ |v-4, 2, 2v-8\rangle\right\}, \tag{A27}$$

$$Z\text{-band:}\ |v, 0, J=2v-3\rangle = \frac{1}{\sqrt{(v-1/2)(v-1)(v-3)!}}\times$$

$$\times\left(\sqrt{\frac{3}{2}}\,d_2^+d_1^+d_0^+ - \frac{1}{2}d_1^{+3} - d_2^{+2}d_{-1}\right)(d_2^+)^{v-3}|0\rangle, \tag{A28}$$

$$Z'\text{-band:}\ |v, 0, J=2v-5\rangle = \frac{1}{\sqrt{(v-2)(v-3/2)(4v-7)(v-5)!}}\times$$

$$\times\left(\sqrt{\frac{3}{2}}\,d_2^+d_1^+d_0^+ - \frac{1}{2}d_1^{+3} - d_2^{+2}d_{-1}^{\pm}\right)\times$$

$$\times\left(2d_2^+d_0^+ - \sqrt{\frac{3}{2}}\,d_1^{+2}\right)(d_2^+)^{v-5}|0\rangle. \tag{A29}$$

Wave functions of states with $n\neq 0$ might be derived from those by using the relation $|v, n\rangle = \frac{b^{+n}}{\sqrt{n!}}|v, 0\rangle$.

4. Corrections to energies of $Y$-, $X$-, $Z$-, $\Delta$- and $\beta$-bands from the cubic interaction in the second order of the perturbation theory:

$$Y:\ \Delta E\,(v, \tilde{n}=0, J=2v) =$$

$$= -\frac{6}{7(2v+7)}\frac{X^{(3)2}}{\omega_v^3}\sum_{n=0}^{\infty}\left\{\frac{18v^2+199v+245}{E_{v+3,n}-E_{v,0}}\ |A_{n,0}^{v+3}(\omega_{v+3}, \omega_v)|^2 +\right.$$

$$\left. + \frac{6v(4v+3)}{E_{v+1,n}-E_{v,0}}\ |A_{n,1}^{v+1}(\omega_{v+1}, \omega_v) + \sqrt{v+\frac{7}{2}}\,A_{n,0}^{v+1}(\omega_{v+1}, \omega_v)|^2\right\}. \tag{A30}$$

$$X:\ \Delta E\,(v, \tilde{n}=0, J=2v-2) =$$

$$= -\frac{6}{7}\frac{X^{(3)2}}{\omega_v^3}\sum_{n=0}^{\infty}\left\{\frac{3(v-1)(4v-1)}{E_{v-1,n}-E_{v,0}}\ |A_{n,0}^{v-1}(\omega_{v-1}, \omega_v) +\right.$$

$$+ \frac{2}{\sqrt{v+3/2}}\,A_{n,1}^{v-1}(\omega_{v-1}, \omega_v) + \frac{2}{\sqrt{(v+3/2)(v+5/2)}}\,A_{n,2}^{v-1}(\omega_{v-1}, \omega_v)|^2 +$$

$$+\frac{6(v-2)(4v+1)(2v-1)}{(2v+3)(E_{v+1,n}-E_{v,0})}\ |A_{n,0}^{v+1}(\omega_{v+1},\omega_v)+\frac{1}{\sqrt{v+7/2}}A_{n,1}^{v+1}(\omega_{v+1},\omega_v)|^2+$$

$$+\frac{36v^3+944v^2+2175v+1024}{(2v+5)(2v+7)(E_{v+3,n}-E_{v,0})}\ |A_{n,0}^{v+3}(\omega_{v+3},\omega_v)|^2\Big\},\qquad \text{(A31)}$$

$\Delta$: $\Delta E\ (v=3, \tilde{n}=0, J=0)=$

$$=-\frac{1}{7}\frac{X^{(3)2}}{\omega_3^3}\sum_{n=0}^{\infty}\Big\{\frac{1144\,|A_{n,0}^6(\omega_6,\omega_3)|^2}{E_{6,n}-E_{3,0}}+$$

$$+\frac{1}{E_{0,n}-E_{3,0}}\,|4\sqrt{2}\ A_{n,3}^0(\omega_0,\omega_3)+6\sqrt{21}\,A_{n,1}^0(\omega_0,\omega_3)+$$

$$12\sqrt{3}\ A_{n,2}^0(\omega_0,\omega_3)+\sqrt{210}\,A_{n,0}^0(\omega_0,\omega_3)|^2\Big\},\qquad \text{(A32)}$$

$Z$: $\Delta E\ (v, \tilde{n}=0, J=2v-3)=$

$$=-\frac{6}{7(2v+7)}\frac{X^{(3)2}}{\omega_v^3}\sum_{n=0}^{\infty}\Big\{\frac{6(v-3)(4v-3)}{E_{v+1,n}-E_{v,0}}\times$$

$$\times\Big|\sqrt{v+\frac{7}{2}}A_{n,0}^{v+1}(\omega_{v+1},\omega_v)+A_{n,1}^{v+1}(\omega_{v+1},\omega_v)\Big|^2+$$

$$+\frac{54v^2+415v+128}{E_{v+3,n}-E_{v,0}}\ |A_{n,0}^{v+3}(\omega_{v+3},\omega_v)|^2\Big\},\qquad \text{(A33)}$$

$\beta$: $\Delta E\ (v, \tilde{n}=1, J=2v)=-\frac{6}{(2v+5)(2v+7)}\frac{X^{(3)2}}{\omega_v^3}\times$

$$\times\sum_{n=0}^{\infty}\Big\{\frac{3v(4v+3)}{E_{v+1,n}-E_{v,1}}\ \Big|4\sqrt{v+\frac{7}{2}}A_{n,0}^{v+1}(\omega_{v+1},\omega_v)+$$

$$+(2v+13)\ A_{n,1}^{v+1}(\omega_{v+1},\omega_v)+2\sqrt{2v+9}\ A_{n,2}^{v+1}(\omega_{v+1},\omega_v)\Big|^2+$$

$$+\frac{2(18v^2+199v+245)}{E_{v+3,n}-E_{v,1}}\ |3A_{n,0}^{v+3}(\omega_{v+3},\omega_v)+$$

$$+\sqrt{v+\frac{11}{2}}A_{n,1}^{v+3}(\omega_{v+3},\omega_v)|^2\Big\}.\qquad \text{(A34)}$$

Eqs (A30—34) contain formally sums over $n$ from zero up to infinity since the cubic operator hasn't rigorous selection rules for $v$, $n$ in the basis of physical states in contrast to the unperturbed oscillator basis. Although it is enough for the practical calculations to adopt an approximation $A_{nn'}^v\simeq\delta_{n,n'}$ since coefficient in front of the small quantity $\chi$ is considered. It could be evidently seen in the example of the ground-state. The corresponding correction is equal

$\Delta E\ (v, \tilde{n}=0, J=0)=$

$$=-\frac{X^{(3)2}}{\omega_0^3}\sum_{n=0}^{\infty}\frac{|A_{n,0}^3(\omega_3,\omega_0)|^2}{E_{3,n}-E_{0,0}}\ |\langle 3,0,0|\,(d^{(+)3})_{00}|0,0,0\rangle|^2=$$

$$=-\frac{X^{(3)2}}{\omega_0^3}\ |\langle 3,0,0|\,(d^{(+)3})_{00}|0,0,0\rangle|^2\times$$

$$\times\Big\{\frac{|A_{0,0}^3(\omega_3,\omega_0)|^2}{E_{3,0})-E_{0,0}}+\sum_{n\neq 0}^{\infty}\frac{|A_{n,0}^3(\omega_3,\omega_0)|^2}{E_{3,n}-E_{0,0}}\Big\}.\qquad \text{(A35)}$$

A following estimate is valid for a sum in curly brackets:

$$\sum_{n\neq 0}^{\infty}\frac{|A_{n,0}^3(\omega_3,\omega_0)|^2}{E_{3,n}-E_{0,0}}\leqslant\frac{1}{E_{3,0}-E_{0,0}}\sum_{n\neq 0}^{\infty}|A_{n,0}^3(\omega_3,\omega_0)|^2=$$

$$=\frac{1-|A_{0,0}^3(\omega_3,\omega_0)|^2}{E_{3,0}-E_{0,0}},\qquad \text{(A36)}$$

and since $A_{nn}^v(\omega_v,\omega_{v'})\simeq 1$ with a good accuracy (see (14)) a sum over $n\neq 0$ may be neglected.

**APPENDIX B**

1. The quadrupole transition operator $T^{(E2)}$ (15) acting in the space of O(5) representations $\{|v,n,J)\}$ has the selection rules $|\Delta v|=0,1,2$. It might be divided into even $T_e$ part and odd $T_o$ one with the selection rules $|\Delta N_d|=2k$ and $|\Delta N_d|=2k+1$, $|\Delta v|=1$ correspondingly:

$$T^{(E2)}=T_e+T_o\ ,\qquad \text{(B1a)}$$

$$T_o=d^{(+)}+\varkappa(d^{(+)3})_2=d^{(+)}+\frac{2\varkappa}{\sqrt{5}}(2P_0+P+P^+)\,d^{(+)},\qquad \text{(B1b)}$$

$$T_e=q\,(d^{(+)2})_2+q'(d^{(-)2})_2.\qquad \text{(B1c)}$$

Here the last term in $T^{(E2)}$ is not accounted for small corrections induced this term have been already calculated and included into (16) – (24). One can easy obtain the recursion relations for reduced matrix elements similar to those from the book [1] (p.605)

$$\langle v,n\|d^{(\pm)}\|v+1,n\rangle=\sqrt{\frac{v+n+5/2}{v+5/2}}\,\langle v,0\|d^{(+)}\|v+1,0\rangle, \tag{B2}$$

$$\langle v-1,n+1\|d^{(\pm)}\|v,n\rangle=\pm\sqrt{\frac{n+1}{v+3/2}}\,\langle v-1,0\|d^{(+)}\|v,0\rangle, \tag{B3}$$

here the matrix elements of the operator $(d^{(+)3})_2$ between unperturbed states $|v,n\rangle$ are expressed through the corresponding quantities for the states with the maximum seniority $|v,0\rangle$:

$$\langle v,n\|(d^{(+)3})_2\|v+1,n\rangle=$$
$$=(v+3n+7/2)\sqrt{\frac{v+n+5/2}{v+5/2}}\,\langle v,0\|d^{(+)}\|v+1,0\rangle, \tag{B4}$$

$$\langle v,n\|(d^{(+)3})_2\|v+1,n+1\rangle=$$
$$=\sqrt{\frac{(v+n+5/2)(v+n+7/2)(n+1)}{v+5/2}}\,\langle v,0\|d^{(+)}\|v+1,0\rangle, \tag{B5}$$

$$\langle v,n\|(d^{(+)3})_2\|v+1,n-1\rangle=$$
$$=(2v+3n+4)\sqrt{\frac{n}{v+5/2}}\,\langle v,0\|d^{(+)}\|v+1,0\rangle, \tag{B6}$$

$$\langle v,n\|(d^{(+)3})_2\|v+1,n-2\rangle=$$
$$=\sqrt{\frac{n(n-1)(v+n+3/2)}{v+5/2}}\,\langle v,0\|d^{(+)}\|v+1,0\rangle. \tag{B7}$$

A general formulae for the matrix elements of $T^{(E2)}$ between an arbitrary states with $|\Delta v|=1$ follows from eqs (B4–7):

$$(v,n\|T^{(E2)}\|v+1,n')\equiv(v,n\|T_o\|v+1,n')=\frac{\langle v,0\|d^{(+)}\|v+1,0\rangle}{\sqrt{(v+5/2)\omega_v}}\times$$

$$\times\Big\{A^{v+1}_{n',n}(\omega_{v+1},\omega_v)\sqrt{v+n+\frac{5}{2}}+\sqrt{n}\,A^{v+1}_{n',n-1}(\omega_{v+1},\omega_v)+$$

$$+\frac{Q}{\omega_v}\Big[\Big(v+3n+\frac{7}{2}\Big)\sqrt{v+n+\frac{5}{2}}\,A^{v+1}_{n',n}(\omega_{v+1},\omega_v)+$$

$$+\sqrt{(n+1)\Big(v+n+\frac{5}{2}\Big)\Big(v+n+\frac{7}{2}\Big)}\,A^{v+1}_{n',n+1}(\omega_{v+1},\omega_v)+$$



$$+(2v+3n+4)\sqrt{n}\,A^{v+1}_{n',n-1}(\omega_{v+1},\omega_v)+$$

$$+\sqrt{n(n-1)\Big(v+n+\frac{3}{2}\Big)}\,A^{v+1}_{n',n-2}(\omega_{v+1},\omega_v)\Big]\Big\}, \tag{B8}$$

where the notation is used:

$$Q=\frac{2}{\sqrt{5}}\,\varkappa. \tag{B9}$$

Coefficients $A^{v+1}_{n',n''}(\omega_{v'},\omega_{v''})$ are determined in the text (14). The next formulae are valid for the $T_e$ matrix elements with the boson number changes $|\Delta N_d|=0,\ 2$ which correspond to the quadrupole moment expectation values and cross-over transitions:

$$(v,n\|T^{(E2)}\|v,n')=\langle v,n\|T_e\|v,n'\rangle, \tag{B10}$$

$$\langle v,n\|T_e\|v,n\rangle=2\,\frac{v+2n+5/2}{v+5/2}\Big(\frac{q}{\omega_v}+q'\omega_v\Big)\,\langle v,0\|(d^+d)_2\|v,0\rangle, \tag{B11}$$

$$\langle v,n\|T_e\|v,n+1\rangle=2\,\frac{\sqrt{(n+1)(v+n+5/2)}}{v+5/2}\times$$
$$\times\Big(\frac{q}{\omega_v}+q'\omega_v\Big)\,\langle v,0\|(d^+d)_2\|v,0\rangle, \tag{B12}$$

$$(v,n\|T^{(E2)}\|v+2,n')=\frac{\langle v,0\|(d^2)_2\|v+2,0\rangle}{\sqrt{(v+5/2)(v+7/2)}}\times$$

$$\times\Big(\frac{q}{\omega_v}+q'\omega_v\Big)\Big[\sqrt{n(n-1)}\,A^{v+2}_{n',n-2}(\omega_{v+2},\omega_v)+$$

$$+\sqrt{(v+n+5/2)(v+n+7/2)}\,A^{v+2}_{n',n}(\omega_{v+2},\omega_v)+$$

$$+\sqrt{n(v+n+5/2)}\,A^{v+2}_{n',n-1}(\omega_{v+2},\omega_v)\Big]. \tag{B13}$$

Below we write down the summary of equations for the matrix elements of the $T^{(E2)}$ operator between the states $|v,n,J)$:

$$\langle v,0,2v\|d^{(+)}\|v+1,0,2v+2\rangle=\sqrt{v+1}, \tag{B14}$$

$$\langle v+1,0,2v\|d^{(+)}\|v+2,0,2v+2\rangle=\sqrt{\frac{v(4v+7)}{4v+3}}, \tag{B15}$$

$$\langle v+3,0,2v\|d^{(+)}\|v+2,0,2v+2\rangle=7\sqrt{\frac{3(4v+7)}{(2v+7)(18v^2+199v+245)}}, \tag{B16}$$

$$\langle v+3,0,2v\|d^{(+)}\|v+4,0,2v+2\rangle=$$

$$=\sqrt{\frac{(v+1)(2v+7)(18v^2+235v+462)}{(2v+9)(18v^2+199v+245)}}, \qquad \text{(B17)}$$

$$\langle v+2,0,2v\|d^{(+)}\|v+1,0,2v+2\rangle=$$

$$=-4\sqrt{\frac{v(v-1)(v+1)}{(v+5/2)(2v+1)(4v+5)(4v+3)}}, \qquad \text{(B18)}$$

$$\langle v+3,0,2v\|d^{(+)}\|v+4,0,2v+2\rangle=$$

$$=\sqrt{\frac{(4v+2)(36v^3+1052v^2+4171v+4179)}{(4v-1)(2v+9)(18v^2+199v+245)}}, \qquad \text{(B19)}$$

$$\langle v,0,2v\|d^{(+)}\|v+1,0,2v\rangle=\sqrt{\frac{4v+2}{4v-7}}, \qquad \text{(B20)}$$

$$\langle v+1,0,2v\|d^{(+)}\|v+2,0,2v\rangle=$$

$$=-4\sqrt{\frac{2v(v-1)(2v+7)}{(2v+5)(4v+5)(4v+3)(4v-1)}}, \qquad \text{(B21)}$$

$$\langle v+3,0,2v\|d^{(+)}\|v+2,0,2v\rangle=$$

$$=-8\sqrt{\frac{3(v-1)(2v+5)(4v+5)}{(2v+7)(4v-1)(18v^2+199v+245)}}, \qquad \text{(B22)}$$

$$\langle v+3,0,2v\|d^{(+)}\|v+4,0,2v\rangle=$$

$$=\sqrt{\frac{(4v+2)(36v^3+1052v^2+4171v+4179)}{(4v-1)(2v+9)(18v^2+199v+245)}}, \qquad \text{(B23)}$$

$$\langle v,0,2v\|(d^2)_2\|v+2,0,2v+2\rangle=2\sqrt{\frac{4v+7}{14}}, \qquad \text{(B24)}$$

$$\langle v+1,0,2v\|(d^+d)_2\|v+1,0,2v+2\rangle=2\sqrt{\frac{2v(v+1)}{7(4v+3)}}, \qquad \text{(B25)}$$

$$\langle v+3,0,2v\|(d^{+2})_2\|v+1,0,2v+2\rangle=$$

$$=(6v+49)\sqrt{\frac{6(v+1)}{7(2v+7)(18v^2+199v+245)}}, \qquad \text{(B26)}$$

$$\langle v,0,2v\|(d^+d)_2\|v,0,2v\rangle=\sqrt{\frac{v(2v+1)(4v+3)}{7(4v-1)}}, \qquad \text{(B27)}$$

$$\langle v+1,0,2v\|(d^+d)_2\|v+1,0,2v\rangle=\sqrt{\frac{(2v+1)(4v+3)}{7v(4v-1)}}\,\frac{4v^2-v-6}{4v+3}, \qquad \text{(B28)}$$

$$\langle v+2,0,2v\|d^{(+)}\|v+1,0,2v\rangle=$$



$$=2(2v+3)\sqrt{\frac{(v-1)(4v+5)}{v(4v-1)(4v+3)(2v+5)}}, \qquad \text{(B29)}$$

2. The coefficients $C^{v,n,J}_{v',n'}$ in the corrections to wave functions (29) in the first and second order of the perturbation theory for $H^{(3)}$ are given by formulae*):

$$C^{(1)\,v,n,J}_{v',n'}=\frac{X^{(3)}}{\omega_v^{3/2}}\,\frac{\langle v,n,J|\,H^{(3)}|\,v',n',J\rangle}{E_{v,n}-E_{v',n'}},\quad v'\neq v,\ n'\neq n,$$

$$C^{(2)\,v,n,J}_{v',n'}=\frac{X^{(3)2}}{\omega_v^{3/2}\,(E_{v,n}-E_{v',n'})}\times$$

$$\times\sum_{v'',n''}\frac{\langle v,n,J|\,H^{(3)}|\,v'',n'',J\rangle\,\langle v'',n'',J|\,H^{(3)}|\,v',n',J\rangle}{\omega_{v''}^{3/2}\,(E_{v,n}-E_{v'',n''})},$$

$$C^{(2)\,v,n,J}_{v,n}=1-\frac{X^{(3)2}}{2\omega_v^3}\sum_{v'',n''}\frac{|\,\langle v,n,J|\,H^{(3)}|\,v'',n'',J\rangle\,|^2}{(E_{v,n}-E_{v'',n''})^2},$$

where the sums are over all the intermediate $v''$, $n''$ satisfying the selection rules $|\Delta v|=1,3$; $|\Delta n|\leqslant 3$; $|\Delta N_d|\leqslant 3$, and the matrix elements of $H^{(3)}$ are calculated by use of eqs (A7—22) calculated by use of formulas (A7—22) in the harmonic oscillator basis.

The observed matrix elements of operator $T^{(E2)}$ between the physical states $|\,v,n,J)^*$ are determined (with the cubic corrections included) according to

$$^*(v',n',J'\|T^{(E2)}\|v,n,J)^*=$$

$$=(C^{(2)\,v,n,J}_{v,n}+C^{(2)\,v',n',J'}_{v',n'}-1)\,(v',n',J'\|T^{(E2)}\|v,n,J)+$$

$$+\sum_{v_1,v_2,n_1,n_2}C^{(1)\,v,n,J}_{v_1,n_1}\,C^{(1)\,v',n',J'}_{v_2,n_2}\,(v_2,n_2,J'\|T^{(E2)}\|v_1,n_1,J)+$$

$$+\sum_{v_1,n_1}C^{(2)\,v,n,J}_{v_1,n_1}\,(v',n',J'\|T^{(E2)}\|v_1,n_1,J)+$$

$$+\sum_{v_1,n_1}C^{(2)\,v,n,J}_{v_1,n_1}\,(v_1,n_1,J'\|T^{(E2)}\|v,n,J),$$

where the values $(v',n',J'\|T^{(E2)}\|v,n,J)$ are determined by formulas (B8—12).

*) Here the same approximation is assumed as in the calculation of energies $A^{v}_{n,n'}(\omega_v,\omega)\simeq\delta_{n,n'}$ (see Appendix A).

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**Collective Quadrupole Excitations of Spherical Nuclei in the Framework of Nonlinear Vibrations Model I: The Theory**

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**Коллективные квадрупольные возбуждения сферических ядер в модели нелинейных колебаний I. Теория**

Ответственный за выпуск С.Г.Попов

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