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I.V. Kolokolov, A.S. Yelkhovsky

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НОВОСИБИРСК

Schwinger Terms as a Source of Gauge Anomaly in Hamiltonian Approach

I.V. Kolokolov, A.S. Yelkhovsky

Institute of Nuclear Physics
630090, Novosibirsk, USSR

ABSTRACT

Connection between Schwinger terms in commutators of fermionic current components and gauge non-invariance of the transition amplitude for the system of Weyl fermions in an external vector field is established in the successive Hamiltonian approach. Then we investigate properties of a gauge group representation in the space of fermionic field states.

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1. INTRODUCTION

For the first time the complete expression of the gauge anomaly in the theory with Weyl fermions was obtained by W. Bardeen [1], who explicitly evaluated Feynman diagrams. Afterwards more elegant technigues were developed for evaluation of the local gauge anomaly in the path-integral formalism [2—4] and its connection with various forms of the index theorem has been clarified [5].

The works [6, 7] have revived an interest to the Hamiltonian interpretation of anomalies and Schwinger terms in the commutators of the generators of gauge transformations. The algebraic methods of [6—8] enable one to obtain these commutators starting with the known expression for the anomaly.

On the other hand, it is interesting to find a reverse way: from Schwinger terms to the anomaly in the framework of the successive Hamiltonian approach. In this way features of the space of quantized fermionic field states and of the evolution operator, acting in this space, become more clear. Our paper is devoted to such approach.

We study the system of left Weyl fermions in the external gauge vector field. In Sect. 2 we evaluate the gauge variation of the evolution operator and commutators of different components of the fermionic current thus arising. Our results disagree with that of other authors [6, 7, 9, 10] obtained earlier. Sect. 3 is devoted to the study of features of the states space for the Weyl Fermi-field. The results of our work and their relation to that of other authors are discussed in Sect. 4.

2. ANOMALOUS COMMUTATORS AND ANOMALY

Consider the system of left Weyl fermions in the external c-number field $A_\mu(t,\vec{x})$, described by the secondly-quantized Hamiltonian

$$H(t)=\int d\vec{x}\,\psi^+(\vec{x})[i\vec{\sigma}\vec{\nabla}-A_0(t,\vec{x})-\vec{\sigma}\vec{A}(t,\vec{x})]\,\psi(\vec{x}). \tag{2.1}$$

The transition amplitude of an initial state vector of the fermionic field $|i\rangle$ to a final one $|f\rangle$ during a time T equals

$$S_{fi}=\langle f|\,T\exp\Big(-i\int_0^T dt\,H(t)\Big)\,|i\rangle\equiv\langle f|S|i\rangle, \tag{2.2}$$

where T..., as usual, stands for chronological product.

In order to find out how does the transition amplitude (2.2) change under an infinitesimal local gauge transformation of the external field and initial and final state vectors we make the identity transformation of S_{fi}

$$S_{fi}=\langle f|\,U_v^{-1}(T)\,U_v(T)\,SU_v^{-1}(0)\,U_v(0)|i\rangle \tag{2.3}$$

with the help of the unitary operator

$$U_v(t)\approx 1+i\int d\vec{x}\,v^a(t,\vec{x})\rho^a(\vec{x})\equiv 1+i\int\rho v(t), \tag{2.4}$$

where $\rho^a(\vec{x})=\psi^+(\vec{x})\,t^a\psi(\vec{x})$ is the charge density operator, t^a are colour matrices, normalized by the condition $\mathrm{Tr}\,t^a t^b=\frac{1}{2}\delta^{ab}$ (in Abelian case $v^a\to v$, $\rho^a(\vec{x})\to\psi^+(\vec{x})\,\psi(\vec{x})$). Expanding (2.3) into the power series in v, we have:

$$i\langle f|S\int\rho v(0)|i\rangle-i\langle f|\int\rho v(T)\,S|i\rangle+$$
$$+i\int_0^T dt\langle f|\,T\,(iS\,[H(t),\int\rho v(t)]+S\int\rho\dot{v}(t))\,|i\rangle=0 \tag{2.5}$$

(we denote time derivative by a point). The first and the second terms in the left-hand side of (2.5) correspond to the variation of the initial and final state vectors under gauge transformation. As for the third and the fourth ones, the formal use of the canonical anticommutation relations

$$\{\psi(\vec{x}),\psi(\vec{y})\}=\{\psi^+(\vec{x}),\psi^+(\vec{y})\}=0,$$

$$\{\psi_\alpha^+(\vec{x}),\psi_\beta(\vec{y})\}=\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{y}) \tag{2.6}$$

for evaluation of the commutator $[H,\int\rho v]$ turns these terms into the gauge variation of the external field

$$A_\mu(t,\vec{x})\to A_\mu^v(t,\vec{x})=A_\mu(t,\vec{x})+D_\mu\,v(t,\vec{x})$$

in the Hamiltonian (2.1). Then the transition amplitude (2.2) proves to be gauge invariant:

$$\langle f|\,S[A_\mu]\,|i\rangle=\langle f^v|\,S[A_\mu^v]\,|i^v\rangle.$$

Show now that the need to regularize local products of the singular operators $\psi(\vec{x})$ and $\psi^+(\vec{x})$ leads to violation of the last equation.

It is convenient to regularize the operator of the charge density with a spherically-symmetric smooth function $Q_\varepsilon(r)$:

$$\rho^a(\vec{x})=\int d\vec{r}\,Q_\varepsilon(r)\,\psi^+\Big(\vec{x}-\frac{\vec{r}}{2}\Big)\,t^a\,\psi\Big(\vec{x}+\frac{\vec{r}}{2}\Big),$$
$$\lim_{\varepsilon\to 0}Q_\varepsilon(r)=\delta(\vec{r}) \tag{2.7}$$

(subscript ε is dropped in what follows). Evaluating the commutator of $\int\rho v$ with the free Hamiltonian, we obtain:

$$\Big[\int d\vec{x}\,\psi^+(\vec{x})\,i\vec{\sigma}\vec{\nabla}\psi(\vec{x}),\ \int d\vec{y}\int d\vec{r}\,Q(r)\,\psi^+\Big(\vec{y}-\frac{\vec{r}}{2}\Big)\,v(\vec{y})\psi\Big(\vec{y}+\frac{\vec{r}}{2}\Big)\Big]=$$
$$=-i\int d\vec{x}\,v^a(\vec{x})\,\vec{\nabla}\int d\vec{r}\,Q(r)\psi^+\Big(\vec{x}-\frac{\vec{r}}{2}\Big)\,\vec{\sigma}\,t^a\psi\Big(\vec{x}+\frac{\vec{r}}{2}\Big).$$

We see that it is natural to use the same function $Q(r)$ for the regularization of the current density operator $\vec{j}^a(\vec{x})=-\psi^+(\vec{x})\,\vec{\sigma}\,t^a\psi(\vec{x})$. Then the last expression takes the form

$$i\int d\vec{x}\,v^a(\vec{x})\,\vec{\nabla}\,\vec{j}^a(\vec{x}),$$

relating to the Abelian part of the gauge variation of $\vec{A}(t,\vec{x})$.

Turn now to the commutator of the interaction Hamiltonian with the gauge transformation generator:

$$-[\int d\vec{x}\,A_\mu^a(t,\vec{x})\,j_\mu^a(\vec{x}),\ \int d\vec{y}\,v^b(t,\vec{y})\,\rho^b(\vec{y})]=$$
$$=-\int d\vec{x}\,A_\mu^a(t,\vec{x})\,v^b(t,\vec{x})\,i\,f^{abc}\,j_\mu^c(\vec{x})+i\,W_v(t). \tag{2.8}$$

Its non-canonical part

$$W_v=i\int d\vec{r}\,Q(r)\int d\vec{r}\,'\,Q(r')\int d\vec{x}\Big[\psi^+\Big(\vec{x}-\frac{\vec{r}+\vec{r}\,'}{2}\Big)\hat{A}\Big(\vec{x}-\frac{\vec{r}\,'}{2}\Big)v\Big(\vec{x}+\frac{\vec{r}}{2}\Big)\times$$
$$\times\psi\Big(\vec{x}+\frac{\vec{r}+\vec{r}\,'}{2}\Big)-\psi^+\Big(\vec{x}-\frac{\vec{r}}{2}\Big)\hat{A}(\vec{x})\,v(\vec{x})\,\psi\Big(\vec{x}+\frac{\vec{r}}{2}\Big)-(\hat{A}\rightleftarrows v)\Big]=$$
$$=i\int d\vec{x}\int d\vec{y}\,v^b(\vec{y})\,(A_0^a(\vec{x})[\rho^a(\vec{x}),\rho^b(\vec{y})]_{an}-A_i^a(\vec{x})\,[j_i^a(\vec{x}),\rho^b(\vec{y})]_{an}) \qquad (2.9)$$

is expressed in terms of anomalous commutators of the charge and current densities. In (2.9) we use $\hat{A}$ for $A_0+\vec{\sigma}\vec{A}$ and omit the time argument for brevity. Note that the last expression is equal to zero in the naive local limit. Now eq. (2.5) can be represented in the form

$$\langle f^v\,|\,S[A_\mu^v]\,|\,i^v\rangle-\langle f\,|\,S[A_\mu]\,|\,i\rangle=i\int_0^T dt\langle f\,|\,T(SW_v(t))\,|\,i\rangle. \qquad (2.10)$$

Right-hand side of (2.10) can be different from zero due to singularities only, which arise for close $\vec{x}$ and $\vec{y}$ in the matrix element

$$\langle f\,|\,TS\psi_\beta^+(\vec{y})\,\psi_\alpha(\vec{x})\,|\,i\rangle\equiv\langle f\,|\,S(T,t)\,\psi_\beta^+(\vec{y})\,\psi_\alpha(\vec{x})S(t,0)\,|\,i\rangle,$$

where α, β stand for Lorentz and group indices. This matrix element can be expressed as the product of S_{fi} and the limit

$$-i\lim_{\tau\to-0}G_{\alpha\beta}(t+\tau,\vec{x};\;t,\vec{y})\cdot S_{fi}$$

of the causal Green function of the Weyl fermion in the external field:

$$G_{\alpha\beta}(t+\tau,\vec{x};\;t,\vec{y})=$$
$$=\frac{1}{i}\theta(\tau)\,S_{fi}^{-1}\langle f\,|\,S(T,t+\tau)\,\psi_\alpha(\vec{x})\,S(t+\tau,t)\,\psi_\beta^+(\vec{y})\,S(t,0)\,|\,i\rangle-$$
$$-\frac{1}{i}\theta(-\tau)\,S_{fi}^{-1}\langle f\,|\,S(T,t)\,\psi_\beta^+(\vec{y})\,S(t,t+\tau)\,\psi_\alpha(\vec{x})\,S(t+\tau,0)\,|\,i\rangle. \qquad (2.11)$$

One can be directly convinced that (2.11) satisfies the equation (indices are omitted)

$$\Big(i\frac{\partial}{\partial\tau}-i\vec{\sigma}\frac{\partial}{\partial\vec{x}}\Big)\,G(t+\tau,\vec{x};\;t,\vec{y})+$$
$$+\int d\vec{r}\,Q(r)\hat{A}\Big(t+\tau,\vec{x}+\frac{\vec{r}}{2}\Big)\,G(t+\tau,\vec{x}+\vec{r};\;t,\vec{y})=\delta(\tau)\delta(\vec{x}-\vec{y}). \qquad (2.12)$$

It is worthy to emphasize once more that we are interested in the part of the Green function which is singular when $\vec{x}\to\vec{y}$. Just this part can be evaluated using the equation (2.12). We shall search for its solution by the standard iterational procedure as the series in powers of the external field

$$G=G^{(0)}+G^{(1)}+G^{(2)}+\dots \qquad (2.13)$$

From the dimensional analysis it is obvious that singularities are contained in $G^{(n)}$ with $n\leqslant 2$.

Free part $G^{(0)}$ of the Green function in the limit of $\tau\to-0$ is, by the definition (2.11)

$$G^{(0)}(\tau\to-0,\vec{x}-\vec{y})=$$
$$=\int\frac{d^4p}{(2\pi)^4}e^{-ip_0\tau+i\vec{p}(\vec{x}-\vec{y})}\,\frac{p_0-\vec{\sigma}\vec{p}}{p^2+i0}\to-\frac{1}{2\pi^2}\,\frac{\vec{\sigma}(\vec{x}-\vec{y})}{(\vec{x}-\vec{y})^4}. \qquad (2.14)$$

Substituting (2.14) in the expression for $G^{(1)}$

$$G^{(1)}(t+\tau,\vec{x};\;t,\vec{y})=-\int d\vec{r}\,Q(r)\int dt'\,d\vec{z}\,G^{(0)}\Big(t+\tau-t',\vec{x}+\frac{\vec{r}}{2}-\vec{z}\Big)\times$$
$$\times\hat{A}(t',\vec{z})\,G^{(0)}\Big(t'-t,\vec{z}-\vec{y}+\frac{\vec{r}}{2}\Big), \qquad (2.15)$$

and picking out the singular part, we obtain:

$$G^{(1)}(t-0,\vec{x};\;t,\vec{y})\to\frac{1}{2\pi^2}\int d\vec{r}\,Q(r)\Big(i\frac{(\vec{\sigma}\vec{n})(\vec{A}\vec{n})}{(\vec{x}-\vec{y}+\vec{r})^2}+\frac{\vec{\sigma}(\vec{n}\times\vec{e})+\vec{n}\vec{b}}{4|\vec{x}-\vec{y}+\vec{r}|}\Big), \qquad (2.16)$$

where $\vec{n}=\dfrac{\vec{x}-\vec{y}+\vec{r}}{|\vec{x}-\vec{y}+\vec{r}|}$; $\vec{e}=-\dot{\vec{A}}-\vec{\nabla}A_0$ and $\vec{b}=\vec{\nabla}\times\vec{A}$ are the Abelian parts of the electric and magnetic fields. The vector-potential and its derivatives in (2.16) are taken at the point $\Big(t,\dfrac{\vec{x}+\vec{y}}{2}\Big)$.

Similarly, singular part of $G^{(2)}$

$$G^{(2)}(t+\tau,\vec{x};\;t,\vec{y})=-\int d\vec{r}\,Q(r)\int dt'\,d\vec{z}\,G^{(0)}\Big(t+\tau-t',\vec{x}+\frac{\vec{r}}{2}-\vec{z}\Big)\times$$
$$\times\hat{A}(t',\vec{z})\,G^{(1)}\Big(t',\vec{z}+\frac{\vec{r}}{2};t,\vec{y}\Big) \qquad (2.17)$$

equals

$$G^{(2)}(t-0,\vec{x};\ t,\vec{y}) \to \frac{1}{8\pi^2}\int d\vec{r}\,Q(r)\int d\vec{r}\,'\,Q(r')\left(\frac{i\vec{n}(\vec{A}\times\vec{A}')}{|\vec{x}-\vec{y}+\vec{r}+\vec{r}\,'|}+\right.$$

$$\left.+\frac{i\vec{\sigma}(\vec{A}\times\vec{n})A_0'-i A_0\vec{\sigma}(\vec{A}'\times\vec{n})}{|\vec{x}-\vec{y}+\vec{r}+\vec{r}\,'|}+2\frac{(\vec{\sigma}\vec{n})(\vec{A}\vec{n})(\vec{A}'\vec{n})}{|\vec{x}-\vec{y}+\vec{r}+\vec{r}\,'|}\right), \qquad (2.18)$$

where

$$\vec{n}=\frac{\vec{x}-\vec{y}+\vec{r}+\vec{r}\,'}{|\vec{x}-\vec{y}+\vec{r}+\vec{r}\,'|},\quad A_\mu=A_\mu\left(t,\frac{1}{2}\left(\vec{x}+\vec{y}+\frac{\vec{r}-\vec{r}\,'}{2}\right)+\frac{1}{6}\left(\vec{x}-\vec{y}-\frac{\vec{r}+\vec{r}\,'}{2}\right)\right),$$

$$A_\mu'=A_\mu\left(t,\frac{1}{2}\left(\vec{x}+\vec{y}+\frac{\vec{r}-\vec{r}\,'}{2}\right)-\frac{1}{6}\left(\vec{x}-\vec{y}-\frac{\vec{r}+\vec{r}\,'}{2}\right)\right).$$

Before substitution of the obtained expressions into (2.9), (2.10) note, that anomalous commutators contain two kinds of terms. Those ones, which do not violate parity, exist in the vector theory too. For the first time the existence of non-canonical terms in the current commutators was pointed out in the pioneer work by J. Schwinger [11]. The explicit form and physical origin of the vector Schwinger terms were discussed then in works [12, 13]. In the following we are interested in the terms only, which are specific for the chiral theory.

In (2.16), (2.18) such terms contain absolutely antisymmetric tensor ε_{ijk}. Substituting them into (2.9), (2.10) and taking the local limit we arrive at the result, independent on an explicit form of the function $Q(r)$:

$$[\rho(\vec{x}),\rho(\vec{y})]_{an}=-\frac{i}{18\pi^2}\vec{\mathscr{B}}\vec{\nabla}\,\delta(\vec{x}-\vec{y}), \qquad (2.19)$$

$$[\,j_i(\vec{x}),\rho(\vec{y})\,]_{an}=-\frac{i}{18\pi^2}\varepsilon_{ijk}\mathscr{E}_j\,\partial_k\,\delta(\vec{x}-\vec{y}) \qquad (2.20)$$

in the Abelian case, and

$$[\rho^a(\vec{x}),\rho^b(\vec{y})\,]_{an}=-\frac{i}{72\pi^2}d^{abc}\varepsilon_{ijk}\left(\partial_i A_j^c-\frac{3}{8}f^{cde}A_i^d A_j^e\right)\partial_k\delta(\vec{x}-\vec{y})+$$

$$+i f^{abc}\frac{1}{576\pi^2}d^{cde}\varepsilon_{ijk}\partial_i A_j^d A_k^e\,\delta(\vec{x}-\vec{y}), \qquad (2.21)$$

$$[\,j_i^a(\vec{x}),\rho^b(\vec{y})\,]_{an}=\frac{i}{72\pi^2}d^{abc}\varepsilon_{ijk}\left(\dot{A}_j^c+\partial_j A_0^c-\frac{3}{4}f^{cde}A_j^d A_0^e\right)\partial_k\delta(\vec{x}-\vec{y})-$$

$$-i f^{abc}\frac{1}{576\pi^2}d^{cde}\varepsilon_{ijk}\,(\partial_j A_0^d A_k^e-A_0^d\partial_j A_k^e)\delta(\vec{x}-\vec{y}) \qquad (2.22)$$

in the non-Abelian one. We imply here that anomalous commutators are equal to $\frac{\langle f\,|\,T(S[\ldots,\rho^b(\vec{y})]_{an})|\,i\,\rangle}{\langle f\,|\,S\,|\,i\,\rangle}$, where $[\ldots,\rho^b(\vec{y})]_{an}$ are defined by (2.9). In (2.19) — (2.22) $\vec{\mathscr{E}}=-\dot{\vec{A}}-\vec{\nabla}A_0$ and $\vec{\mathscr{B}}=\vec{\nabla}\times\vec{A}$ are the electric and magnetic fields in the Abelian theory; d^{abc} are symmetric constants of the joint representation normalized in the standard way,

$$d^{abc}=2\,\mathrm{Tr}\,t^a\{t^b,t^c\}.$$

Vector-potential and its derivatives in (2.19) — (2.22) are taken at the point $\left(t,\frac{\vec{x}+\vec{y}}{2}\right)$. Note, that right-hand side of (2.21), (2.22) includes terms of the canonical structure $(i f^{abc} j_\mu^c\delta(\vec{x}-\vec{y}))$. They can be absorbed in the redefinition of the current operators in the Hamiltonian (2.1):

$$\rho^a(\vec{x})\to\rho_A^a(\vec{x})=\rho^a(\vec{x})+\frac{1}{576\pi^2}d^{abc}\varepsilon_{ijk}\,\partial_i A_j^b A_k^c, \qquad (2.23)$$

$$j_i^a(\vec{x})\to j_{iA}^a(\vec{x})=j_i^a(\vec{x})-\frac{1}{576\pi^2}d^{abc}\varepsilon_{ijk}\,(\partial_j A_0^b A_k^c-A_0^b\,\partial_j A_k^c), \qquad (2.24)$$

It is equivalent to the addition of the local counterterm

$$-\int d\vec{x}\,[A_0^a(\rho_A^a-\rho^a)-\vec{A}^a(\vec{j}_A^{\,a}-\vec{j}^{\,a})]=0 \qquad (2.25)$$

to the Hamiltonian of the system. Then the anomalous commutators for the non-Abelian theory take the form

$$[\rho_A^a(\vec{x}),\rho^b(\vec{y})]_{an}=-\frac{i}{72\pi^2}d^{abc}\varepsilon_{ijk}\left(\partial_i A_j^c-\frac{3}{8}f^{cde}A_i^d A_j^e\right)\partial_k\delta(\vec{x}-\vec{y}), \qquad (2.26)$$

$$[\,j_{iA}^a(\vec{x}),\rho^b(\vec{y})]_{an}=\frac{i}{72\pi^2}d^{abc}\varepsilon_{ijk}\left(\dot{A}_j^c+\partial_j A_0^c-\frac{3}{4}f^{cde}A_j^d A_0^e\right)\partial_k\delta(\vec{x}-\vec{y}). \qquad (2.27)$$

Our result for $[\rho^a(\vec{x}),\rho^b(\vec{y})]_{an}$ does not agree with any one obtained earlier [7—10]. Possible origin of this disagreement is discussed in Sect. 4.

The compact expression for the right-hand side of the equation (2.10),

$$i\int_0^T dt\,\langle f\,|\,T(SW_v(t))|\,i\,\rangle=$$

$$= \frac{i}{24\pi^2} \int d^4x \, \varepsilon^{\mu\nu\alpha\beta} \operatorname{Tr} v \, \partial_\mu \Big(A_\nu \partial_\alpha A_\beta - \frac{i}{2} A_\nu A_\alpha A_\beta \Big) \cdot S_{fi} , \tag{2.28}$$

which agrees with the well-known expression for the consistent local anomaly [1, 14], can be obtained from (2.21), (2.22) (or (2.19), (2.20) in the Abelian case) after the addition of the local counterterm

$$- \frac{1}{72\pi^2} \int d\vec{x} \operatorname{Tr} A_0 \Big\{ \vec{A}, \vec{b} + \frac{3}{4} i \vec{A} \times \vec{A} \Big\} , \tag{2.29}$$

is made.

Thus, for the local in the time and space gauge transformation $(v(t, |\vec{x}| \to \infty) = v(0, \vec{x}) = v(T, \vec{x}) = 0)$ we have

$$\langle f | S[A_\mu^v] | i \rangle - \langle f | S[A_\mu] | i \rangle = i \mathscr{A}_v \langle f | S | i \rangle , \tag{2.30}$$

where

$$\mathscr{A}_v = \frac{1}{24\pi^2} \int d^4x \, \varepsilon^{\mu\nu\alpha\beta} \operatorname{Tr} v \partial_\mu \Big(A_\nu \partial_\alpha A_\beta - \frac{i}{2} A_\nu A_\alpha A_\beta \Big) , \tag{2.31}$$

and the evolution operator contains, except the fermionic Hamiltonian (2.1), the local polynome of the external field (2.29).

3. SCHWINGER TERMS AND SPACE OF FERMIONIC FIELD STATES

In the Schrödinger representation infinities in matrix elements of local operators and Schwinger terms arise due to the singular character of the wave functionals $\Phi[\psi^+(\vec{x})]$ (see, e. g., [15]). For fermions these singularities relate to infinite number of filled states in the Dirac sea. For example, the vacuum functional $\Phi_A[\psi^+(\vec{x})]$ of the fermionic field in a fixed external field $A_\mu(\vec{x})$ has the form

$$\Phi_A[\psi^+(\vec{x})] = \prod_n b_n , \qquad b_n = \int d\vec{x} \, \psi^+(\vec{x}) \chi_n(\vec{x}) , \tag{3.1}$$

where $\chi_n(\vec{x})$ is the set of negative-energy eigenfunctions for the Weyl Hamiltonian in the external field. The action of some local operator, say $\rho(\vec{x})$, on Φ_A,

$$\rho(\vec{x}) \Phi_A[\psi^+(\vec{x})] = \lim_{M, N \to \infty} \sum_{n, n'}^{N} \chi_n^+(\vec{x}) \chi_{n'}(\vec{x}) \, b_n \frac{\delta}{\delta b_{n'}} \prod_m^M b_m , \tag{3.2}$$

is not well defined, and a regularization is necessary either for $\rho(\vec{x})$, or for $\Phi_A[\psi^+(\vec{x})]$. We regularize operators (first $M \to \infty$, then $N \to \infty$).

In our opinion, the origin of the anomalous commutator $[\rho^a(\vec{x}), \rho^b(\vec{y})]_{an}$ is in the fact that the vector Φ_{A^g}, where $g(\vec{x}) \sim \delta(\vec{x} - \vec{x}_0)$ is the parameter of the point gauge transformation, cannot be represented as a normalizable linear combination of vectors from the Fock space built over Φ_A. Really, the gauge variation of field coordinates a_n^+, b_n does not decrease when n increases. (The situation is similar to the ordinary spontaneous symmetry breaking, when, e. g., a homogeneous rotation of all spins in the infinite ferromagnet to an arbitrary small angle leads to the state, which is orthogonal to the initial one. In our case the infinite number of negative-energy states with arbitrary large momenta is crucial. (See also [16]). In fact, the point gauge transformation is homogeneous just for these degrees of freedom.) A regularization of the gauge transformation operator (say, a restriction of its action to finite number of modes, as in our case) changes properties of the gauge group representation in the space of states. Now the gauge-transformed state is a normalizable superposition of states from the initial Fock space. (Similar approach is developed in [17].) The phenomenon of such kind exists even in the quantum mechanics. Really, any finite reduction of creation and annihilation operators a and a^+ would spoil the commutation relation $[a, a^+] = 1$, since $\operatorname{Tr}[A, B] = 0$ for any two finite-dimensional matrices A and B. (See also [18].)

Consider now, how does the operator U_v transform a fermionic state vector $|\ \rangle_A$. Take for simplicity the vacuum state vector $|0\rangle_A$ in a given external field. (All the following is valid for any finite-particle state built over $|0\rangle_A$.)

In an interacting theory the operator U_v transfers the vacuum $|0\rangle_A$ into some superposition of states over $|0\rangle_{A^v}$. To return to the initial configuration of the external field, we pass along the infinitesimal loop in the gauge group with the help of the operator

$$I(u, v) = U_{[u,v]/i} \, U_u^{-1} \, U_v^{-1} \, U_u \, U_v . \tag{3.3}$$

Remind that we are resricted now to a fixed moment of time.

If the vacuum state $|0\rangle_A$ is completely determined by the external field and hence is the only for the given external field (up to an arbitrary phase), then

$$I(u,v)\,|0\rangle_A = |0\rangle_A\,. \tag{3.4}$$

Rewritting (3.3) in the form

$$I(u,v) = 1 - [\textstyle\int \rho u\,,\ \int \rho v]_{an}\,, \tag{3.5}$$

where the operator $[\int \rho u\,,\ \int \rho v]_{an}$ is defined by (2.9), we see, that in the chiral theory the action of $I(u,v)$ on the vacuum is non-trivial:

$${}_A\langle 0|\,I(u,v)\,|0\rangle_A = 1 + i\omega(u,v\,|\,A)\,. \tag{3.6}$$

Here $\omega(u,v\,|\,A) = i\,{}_A\langle 0|[\int \rho u\,,\ \int \rho v]_{an}|0\rangle_A$ does not vanish in general case and is determined by the right-hand side of eqs (2.19) or (2.21). It means that the fermionic vacuum in the chiral theory is not a functional of the external field [7]. It can be easily shown, that in the non-Abelian theory the operator $I(u,v)$ transfers one vacuum to another:

$$I(u,v)|0\rangle_A = e^{i\omega(u,v|A)}|0\rangle'_A\,. \tag{3.7}$$

In fact, consider the vacuum expectation value of the charge density operator:

$${}_A\langle 0|\rho^a(\vec{x})\,|0\rangle_A = {}_A\langle 0|I^{-1}(u,v)\,I(u,v)\,\rho^a(\vec{x})\,I^{-1}(u,v)\,I(u,v)|0\rangle_A\,.$$

Since

$$I(u,v)\,\rho^a(\vec{x})\,I^{-1}(u,v) = \rho^a(\vec{x}) + [\rho^a(\vec{x}), [\textstyle\int \rho u\,,\ \int \rho v]_{an}],$$

and for the non-Abelian theory

$${}_A\langle 0|[\rho^a(\vec{x}), [\textstyle\int \rho u\,,\ \int \rho v]_{an}]\,|0\rangle_A =$$

$$= -\frac{i}{48\pi^2}\,\mathrm{Tr}\Big\{\Big(\vec{b} + \frac{4i}{5}\vec{A}\times\vec{A}\Big)\Big[t^a, \frac{\{u,\vec{\nabla}v\} - \{v,\vec{\nabla}u\}}{2}\Big] +$$

$$+ \frac{2i}{5}\{t^a,[u,v]\}\vec{\nabla}(\vec{A}\times\vec{A}) + \vec{\nabla}\Big(\Big(\frac{1}{6}\vec{b} + \frac{i}{5}\vec{A}\times\vec{A}\Big)\{t^a,[u,v]\}\Big)\Big\}\,, \tag{3.8}$$

the operator $I(u,v)$ has to transfer an initial state vector to another one, with different vacuum charge density,

$${}_A\langle 0|'\,\rho^a(\vec{x})\,|0\rangle'_A = {}_A\langle 0|\rho^a(\vec{x})\,|0\rangle_A - {}_A\langle 0|[\rho^a(\vec{x}), [\textstyle\int \rho u\,,\ \int \rho v]_{an}]\,|0\rangle_A\,,$$

but with the same energy

$${}_A\langle 0|'\,H|0\rangle'_A = {}_A\langle 0|H|0\rangle_A\,.$$

In other words, there are several physically non-equivalent fermionic vacua in the given external field. Discussion before shows that they can be distinguished by singular operators only.

The explicit calculation gives in the Abelian case:

$${}_A\langle 0|[\rho(\vec{x}), [\rho(\vec{y}), \rho(\vec{z})]]\,|0\rangle_A = 0\,. \tag{3.9}$$

Therefore the eq. (3.6) in the Abelian theory can be written down more directly:

$$I(u,v)|0\rangle_A = e^{i\omega(u,v|A)}|0\rangle_A\,. \tag{3.10}$$

Making now the identity transformation in the scalar product ${}_A\langle 0|0\rangle_{A'}$,

$${}_A\langle 0|0\rangle_{A'} = {}_A\langle 0|II^{-1}|0\rangle_{A'} = \exp i[\omega(u,v\,|A) - \omega(u,v\,|A')]_A\langle 0|0\rangle_{A'}\,,$$

and using the explicit form of the phase $\omega(u,v\,|A)$, we come to conclusion that fermionic vacua of the Abelian theory in different magnetic fields are orthogonal.

In two dimensions the expressions for the charge and current densities coincide, and the anomaly arises due to non-zero commutator of the charge densities. In order to compare with the four-dimensional case, and as a simple illustration of a somewhat different approach, consider the two-dimensional Abelian theory with Weyl fermions. Put $A_0 = 0$ and restrict ourselves to time-independent gauge transformations.

The Hamiltonian has the form

$$H_A = \int dx\,(\psi^+(x)\,i\,\partial_x\psi(x) - A(x)\,\rho(x))\,,$$
$$\rho(x) = \psi^+(x)\,\psi(x)\,. \tag{3,11}$$

The Schwinger term in the commutator

$$[\rho(x), \rho(y)] = \frac{i}{2\pi}\,\delta'(x-y) \tag{3.12}$$

does not depend on the external field. Adding the local counterterm $\Delta H_A = \int dx\,A^2(x)$ to (3.11) (or, equivalently, taking in place of $\rho(x)$ its normally-ordered counterpart $\rho_A(x) = \rho(x) - A(x)$ [19]), we manage to obtain:

$$U_v H_A U_v^{-1} = H_{A^v},$$
$$U_v = 1 + i \int dx\, \rho(x)\, v(x), \quad A^v = A + \partial_x v. \tag{3.13}$$

Therefore, the anomaly can display itself in the transformation of the fermionic state vector only. Fixing some field configuration $\tilde{A}(x)$, we define the phase of the vacuum vector $|0\rangle_A$ along the gauge orbit by the relation

$$|0\rangle_A = \exp\left(\frac{i}{4\pi} \int dx\, v(x) A(x)\right) U_v |0\rangle_{\tilde{A}} \tag{3.14}$$

for $A = \tilde{A}^v$. The gauge transformation is now accompanied by the phase:

$$U_v |0\rangle_A = e^{i\,\omega(v|A)} |0\rangle_{A^v}, \tag{3.15}$$

where

$$\omega(v|A) = -\frac{1}{4\pi} \int dx\, v(x) A(x). \tag{3.16}$$

This phase has appeared due to the two-cocycle α_2 in the gauge group representation [7]:

$$U_g U_v = e^{i\,\alpha_2(g,v)} U_{gv},$$
$$\alpha_2 = \frac{1}{4\pi} \int dx\, g(x)\, \partial_x v(x). \tag{3.17}$$

Thus, the transition amplitude and its gauge-transformed counterpart are connected by the relation

$$\langle f^v | S[A^v] | i^v \rangle = e^{i\mathscr{A}_v} \langle f | S[A] | i \rangle,$$

where the anomaly

$$\mathscr{A}_v = \frac{1}{4\pi} \left(\int dx\, v(x) A(T,x) - \int dx\, v(x) A(0,x) \right) =$$
$$= \frac{1}{4\pi} \int dt\, dx\, v(x) \dot{A}(t,x) \tag{3.18}$$

coincides with the known expression.

The two-dimensional theory differs crucially from that in four dimensions. The point is that there is no a local functional of the

external field in four dimensions whose addition to the Hamiltonian would provide the validity of the four-dimensional analog of eq. (3.13). The anomaly in four dimensions cannot be reduced to some time-boundary terms even for time-independent gauge transformation. The similar phenomenon is discussed in the recent work [20]. Earlier close issues were considered in [21].

4. DISCUSSION

The straightforward calculations of Sect. 2 show that the Schwinger terms (2.19—2.22) lead, in the sence of eq. (2.5), to the consistent gauge anomaly. However, the anomalous commutators themselves are regularization-dependent.

In fact, let consider for simplicity the Abelian theory. The expressions for the commutators together with the vector terms are

$$[\rho(\vec{x}), \rho(\vec{y})] = -\frac{i}{18\pi^2} \vec{\mathscr{B}} \vec{\nabla} \delta(\vec{x} - \vec{y}),$$
$$[j_i(\vec{x}), \rho(\vec{y})] = -\frac{i}{\varepsilon^2} \partial_i \delta(\vec{x} - \vec{y}) - \frac{i}{18\pi^2} \varepsilon_{ijk} \mathscr{E}_j \partial_k \delta(\vec{x} - \vec{y}) + \ldots, \tag{4.1}$$

where

$$\frac{1}{\varepsilon^2} \equiv \frac{1}{3\pi^2} \int d\vec{r}\, d\vec{r}\,' \frac{Q(r) Q(r')}{(\vec{r} + \vec{r}\,')^2}. \tag{4.2}$$

They contain quadratically-divergent term [11] (dots stand for inessential finite vector terms). It is easily to see that the anomalous terms in commutators of the operators

$$\tilde{\vec{j}}(\vec{x}) = \vec{j}(\vec{x}) - \frac{\varepsilon^2}{18\pi^2} \vec{\mathscr{E}} \times \vec{j}(\vec{x}),$$
$$\tilde{\rho}(\vec{x}) = \rho(\vec{x}) + \frac{\varepsilon^2}{36\pi^2} \vec{\mathscr{B}} \vec{j}(\vec{x}), \tag{4.3}$$

which coincide with $\vec{j}(\vec{x})$ and $\rho(\vec{x})$ in the local limit, vanish. The origin of the gauge anomaly in this regularization is the difference between $\tilde{\vec{j}}(x)$ and the current in eq. (2.8), and the explicit dependence of the operator $\tilde{\rho}$ on the external field. Note, that for the evaluation of anomalous commutators and anomaly one can use any possible regularization. But in any case the result will be completely determined by an explicit expression of regularized operators through the field operators. If such expressions absent, the successive analysis in the Hamiltonian framework is impossible.

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I.V. Kolokolov, A.S. Yelkhovsky

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А.С. Елховский, И.В. Колоколов

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