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The Gravity on Regge Lattice of a Special Kind

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ABSTRACT

We study the possibility to quantize gravity on Regge lattice periodic in a given coordinate system. The latter is fixed by imposing gauge conditions on the tetrad: $e^a_\mu = 0$ at $a < \mu$, $e^a_\mu = 1$ at $a = \mu$. In this gauge any usual local gravity action (say the Einsteinian one) is polinomial in e^a_μ, $a > \mu$. We study the functional integral measure in these variables and find it's entropy. The measure turns out to be singular at $e^a_\mu = 0$ thus leading to an unstable theory.

INTRODUCTION

The problem of a consistent definition of the functional integral for the Einstein gravity [1] is complicated by nonrenormalizability of the theory. In this connection Regge calculus [2] is of interest for it allows one to work with discrete set of field variables. This approach deals with the piecewise-flat Riemannian manifolds built of flat simplices. Geometry is defined by specifying the edge lengths L_i, the curvature being represented by the appearance of deficit angles. Regge calculus proves to be useful for solving the problems of classical general relativity (see references in papers [3—10]). Recently some important problems of quantum theory also have been analysed within this approach, namely, quantization of the string on the lattice [3, 4], studying the gravity theories with higher derivatives [5], simplicial minisuperspace [6], Monte-Carlo simulations [7], dynamical generation of symmetries [10] and so on. Summing over discrete manifolds using a functional integral quantization was suggested in Ref. [11], and systematic functional integral formulation of Regge gravity was given in Ref. [12]: to construct the measure the DeWitt metric on the space of infinitesimal variations of metric tensor $\|\delta g\|^2$ [13, 14] is used. Metric tensor is defined by the lengths L_i and four functions $\xi^\mu(x)$ parametrizing gauge transformation. Then the following expression for $\|\delta g\|^2$ in terms of δL_i and $\delta\xi^\mu$ is obtained [12]:

$$\|\delta g\|^2 = \delta L_i K_{ij} \delta L_j + \delta\xi^\mu G_{\mu\nu} \delta\xi^\nu .$$

It allows one to write the measure of interest as

$$\int d\mu(L) = \int \prod_i dL_i \sqrt{\operatorname{Det} K}\, \sqrt{\operatorname{Det} G}\, .$$

The advantages of Regge approach allowing one to approximate the continuous Riemannian manifold by still continuous, not the discrete one with any desired accuracy and to get approximate description of the system immediately in terms of invariants make the idea [10] on the simplicial structure of real space-time quite attractive one. The usual smooth metric arises as the result of some averaged description at large scales. We can fix the coordinate system and construct the measure on the space of equivalence classes of metric by computing corresponding Faddeev—Popov factor. In the continuum case the latter suffers from UV divergences but it can be consistently defined on Regge lattice. The full solution of the problem would involve averaging over random lattice (as discussed in Ref. [10]). This seems to be quite difficult task. It turns possible, however, to calculate the Faddeev—Popov factor on the lattice periodic in a definite coordinate system. In this paper the coordinate system is chosen by imposing a definite gauge conditions on the smoothed large scale metric. As a result, we get the measure singular in the vicinity of flat background which leads to an unstable theory. The paper is organized as follows. In Sect. 2 the gauge is introduced and considered. In Sect. 3 the formal Faddeev—Popov determinant is written and in Sect. 4 it's Regge lattice analog is considered. In Sect. 5 we calculate the measure on the two-dimensional Regge lattice of the type considered in Ref. [8]. We show that the main effect of introducing Regge lattice develops through the appearance of IR unstable diagrams in the perturbative expansion for the Faddeev—Popov factor. We first sum up the leading IR singularities. In the final answer the IR cutoff decouples and enters the overall normalization of the functional integral. Instead of IR divergence we get nonanalytic behaviour of the entropy of the measure in the vicinity of flat background. In Sect. 6 we study the structure of corrections of the next orders in e^a_μ and ε (ε is the lattice spacing) to the result obtained and also observe decoupling the cutoff. In Sect. 7 the Regge lattice of an arbitrary dimension is considered. Then we conclude.

2. THE GAUGE

Let us specify the tetrad e^a_μ corresponding to the metric $g_{\mu\nu} = e^a_\mu e^a_\nu$ by imposing the following conditions:

$$e^a_\mu = 1,\ a=\mu; \quad e^a_\mu = 0,\ a<\mu; \quad a,\mu = 1,\dots,n\,. \tag{1}$$

Hereafter we refer to this as to the «triangular» gauge. The contravariant components are

$$g^{\mu\nu} = e^{a\mu} e^{a\nu},$$
$$e^{a\mu} = 1,\ a=\mu; \quad e^{a\mu} = 0,\ a>\mu\,, \tag{2}$$
$$e^{k,k+l} = -e^{k+l}_k + \sum_{k<m<k+l} e^{k+l}_m e^m_k - \sum_{k<m<n<k+l} e^{k+l}_n e^n_m e^m_k + \dots$$

with $e^{a\mu}$ being polinomial in e^a_μ of the degree $n-1$. In terms of $g_{\mu\nu}$ the gauge conditions read

$$g_{nn} = 1, \quad \det \begin{vmatrix} g_{n-1,n-1} & g_{n-1,n} \\ g_{n,n-1} & g_{nn} \end{vmatrix} = 1, \dots, \quad g \equiv \det g_{\mu\nu} = 1. \tag{3}$$

Conversely, with (3) being fulfilled the tetrad is reducable by means of local Lorentzian rotations to the form (1). In turn, $g'_{\mu\nu}$ can be reduced to the form (3) by transformation $x'^\mu = y^\mu(x)$ obeying the equations

$$g'_{\mu\nu}\, y^\mu_{,n}\, y^\nu_{,n} = 1 \quad \det \begin{vmatrix} g'_{\mu\nu} y^\mu_{,n-1} y^\nu_{,n-1} & g'_{\mu\nu} y^\mu_{,n-1} y^\nu_{,n} \\ g'_{\mu\nu} y^\mu_{,n} y^\nu_{,n-1} & g'_{\mu\nu} y^\mu_{,n} y^\nu_{,n} \end{vmatrix} = 1, \dots,$$
$$(\det g'_{\mu\nu})\,(\det y^\mu_{,\nu})^2 = 1\,. \tag{4}$$

Eqs (4) allow us to express $y^\mu_{,n}$ in terms of $y^\mu_{,\nu}$, $\nu<n$, and solve by iterations. This gives, at least formally, a solution of the Cauchy problem in the vicinity of an initial hyperplane $x^n = \text{const}$.

The Einstein action takes the form

$$\int G \sqrt{g}\, d^n x = \int d^n x \Big\{ g^{\mu\nu} \Big[e^{a\lambda} e^a_{\mu,\rho} e^{b\rho} e^b_{\lambda,\nu} + \frac{1}{2} e^{a\lambda} e^a_{\mu,\rho} e^{b\rho} e^b_{\nu,\lambda} +$$
$$+ \frac{1}{2} e^a_{\mu,\nu} e^a_{\lambda,\rho} (g^{\mu\rho} g^{\lambda\nu} - g^{\mu\lambda} g^{\nu\rho}) \Big] \Big\}\,. \tag{5}$$

It is polinomial in e^a_μ, and this property also holds for cosmological term and higher derivatives included. Finally, let us write out the field propagators. These can be read off from the solution of the Einstein equations

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=T_{\mu\nu}\,,\qquad \mu\neq\nu\,. \tag{6}$$

In the linear approximation

$$e_1^2=...+\frac{1}{2}\,\frac{n-3}{n-2}\left(\frac{\partial_2}{\partial_1}+\frac{\partial_1}{\partial_2}\right)^2\Delta^{-1}T_{12}+$$
$$+\frac{1}{2}\left[\frac{n-3}{n-2}\,\frac{\partial_2\partial_3}{\partial_1^2}\Delta^{-1}-\frac{1}{n-2}\,\frac{\partial_1^2+\partial_2^2+\partial_3^2}{\partial_2\partial_3}\Delta^{-1}\right]T_{13}-$$
$$-\frac{1}{2}\,\frac{1}{n-2}\,\frac{(\partial_1^2+\partial_2^2)(\partial_3^2+\partial_4^2)}{\partial_1\partial_2\partial_3\partial_4}\Delta^{-1}T_{34}+\,...,\,... \tag{7}$$

$$\Delta=\sum_{\mu=1}^{n}\partial_\mu^2$$

(unwritten terms and formulas correspond to all possible permutations of indexes 1, 2, ..., n). At $n=2$ the Einstein action is a topological invariant and at $n=3$ the wave propagator Δ^{-1} is cancelled: the three-dimensional general relativity is locally trivial [15].

3. THE FORMAL FADDEEV—POPOV DETERMINANT

Let us write out the local gauge-invariant measure on the tetrad space,

$$d\mu(e)=\prod_x\,[\det e_\mu^a(x)]^{-n}\prod_{a,\mu}de_\mu^a(x)\,. \tag{8}$$

An element δe_μ^a of the tangent space can be parametrized by $n(n-1)/2$ elements $\Lambda_b^a=-\Lambda_a^b$ of the local Lorentz group, n elements η^μ of the general covariant group $(x'^\mu=x^\mu-\eta^\mu(x))$ and $n(n-1)/2$ elements θ_μ^a $(a>\mu)$ corresponding to the physical degrees of freedom:

$$\delta e_\mu^a=\theta_\mu^a+e_\nu^a\,\eta^\nu_{,\mu}+e_\mu^b\,\Lambda_b^a\,. \tag{9}$$

For $n=2$, e. g.,

$$\delta e_\mu^a=\begin{pmatrix} e\Lambda+\eta^1_{,1} & \Lambda+\eta^1_{,2}\\ \theta+e\eta^1_{,1}+\eta^2_{,1}-\Lambda & e\eta^1_{,2}+\eta^2_{,2}\end{pmatrix}$$

$$\Lambda=\Lambda_2^1=-\Lambda_1^2\,,\qquad e=e_1^2\,,\qquad \theta=\theta_1^2\,.$$

Then we rewrite (8) in terms of θ_μ^a, η^μ, Λ_b^a. In what follows we identify $D\theta_\mu^a$ with De_μ^a in the triangular gauge. The $D\Lambda_b^a$, $D\eta^\mu$-integrations are cancelled by dividing by the volumes of the corresponding groups as far as gauge invariant physical quantities are considered within the functional integral approach:

$$d\mu(e)=\Phi(e)\;D\theta\;D\Lambda\;D\eta^1\;D\eta^2,\qquad D[\,...\,]\equiv\prod_x d[\,...\,]\,,$$

$$\Phi(e)=\frac{D(\delta e_1^1,\delta e_2^1,\delta e_2^2)}{D(\Lambda,\eta^1,\eta^2)}=\operatorname{Det}L\,, \tag{11}$$

$$L\eta=L\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix}=\begin{pmatrix}\partial_1-e\partial_2 & 0\\ e\partial_2 & \partial_2\end{pmatrix}\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix}. \tag{12}$$

Det means the determinant in the operator sence.

4. THE MEASURE ON A REGGE LATTICE

The determinant (11) requires the UV regularization and we would like to use a Regge lattice as the natural regulator. Now on the lattice the definitions (8), (11), (12) should be modified because some averaging over simplices attached to any given point is to be done. A more general way to introduce the measure is that using the Gaussian normalization [16]:

$$\int\exp\left(-\frac{1}{2}\,\|\delta e\|^2\right)d\mu(e)=1\,. \tag{13}$$

Here the infinitesimal metric on the tetrad space similar to that of DeWitt [13] is introduced:

$$\|\delta e\|^2=\int\Big[g^{\mu\nu}\,\delta e_\mu^a\,\delta e_\nu^a+\frac{C}{4}(g^{\mu\nu}\,\delta g_{\mu\nu})^2+$$
$$+C'\Big(g^{\mu\nu}\,\delta e_\mu^a\,\delta e_\nu^a-\frac{1}{4}g^{\mu\nu}\,g^{\lambda\rho}\,\delta g_{\mu\lambda}\,\delta g_{\nu\rho}\Big)\Big]g^{1/2}\,d^nx\,. \tag{14}$$

Up to an overall factor it is the most general invariant without the derivatives. Now we can substitute (9), (11) into (13), (14) and perform simple Gaussian integration over $D\theta$, $D\Lambda$:

$$\Phi(e)^{-1}=\int \exp\left\{-\frac{1}{2}\int (\det e_\mu^a)\, d^2x[\,(L\eta)^2+C(\eta^\mu_{,\mu})^2]\right\} D\eta^1\, D\eta^2. \qquad (15)$$

Naively, this leads to (12). At the same time (15) can be easily ae-fined on a Regge lattice, the expression under the exponential sign being the sum over simplices. Working with the class of the piece-wise-flat geometries we adopt the gauge group to be that of the piecewise-linear transformations. Therefore the derivatives $\eta^\mu_{,\nu}$ are uniquelly defined as finite differences once $\eta^\mu(x)$ at the lattice sites are known. In the case of a periodic Regge lattice the eq. (15) gives a recipe of averaging Det L over simplices of the elementary cell; namely, the operator L should be understood in the sence of the root-mean-squared value, $(\overline{L^+L})^{1/2}$.

For arbitrary n we find

$$\Phi(e)^{-1}=\int \exp\left\{-\frac{1}{2}\int\left[\sum_a (e_\mu^a e^{a\nu}\eta^\mu_{,\nu})^2+C(\eta^\mu_{,\mu})^2\right]\times\right.$$

$$\left.\times (\det e_\mu^a)\, d^nx\right\} D\eta^1 \ldots D\eta^n. \qquad (16)$$

Note that (15), (16) are valid for e_μ^a written in a more general form than (1): the diagonal elements e_μ^a are not required to be unity. In fact, using the freedom connected with the local Lorentz rotations we can always choose the tetrad $e_\mu^a(\sigma)$ in each simplex σ in this form. As for the metric $g_{\mu\nu}(\sigma)$ it can differ from the smoothed one $g_{\mu\nu}$ obeying the property (3) by the terms $O(\varepsilon)$. Thus

$$e_\mu^a(\sigma)=e_\mu^a+O(\varepsilon)\,,$$
$$e_\mu^a(\sigma)=0\,, \quad a<\mu\,. \qquad (17)$$

(Here we imply the following construction [17]: the edge lengths are chosen to be the geodesic lengths in the smoothed Riemannian manifold.) In the leading approximation in ε the tetrad field does not depend on the simplex chosen in a given cell and obeys the condition (1).

The main effect of introducing Regge lattice arises, as we shall see, due to the differences between the expressions for the differentiation operator $\partial_\mu(\sigma)$ inside the different simplices σ in the cell. These expressions are equivalent up to the terms $O(\varepsilon)$ only on the set of sufficiently smooth functions. However, the diagrams of the expansion of (15), (16) in terms of e_μ^a are UV divergent. This

means that the functional integrals (15), (16) are saturated by the functions $\eta^\mu(x)$ which vary strongly from site to site on the lattice. Therefore the dependence of $\partial_\mu(\sigma)$ on σ is of importance here.

5. THE TWO-DIMENSIONAL CASE

Consider the periodic Regge lattice described in Ref. [8]: the elementary cell is formed by links connecting a given vertex of hypercube with other 2^n-1 vertices. Thus the hypercube is represented as the union of $n!$ hypertetrahedra. At $n=2$ we have the two triangles σ_1 and σ_2 (Fig. 1). Let T_μ be translation in the x^μ direction by ε, $T_\mu^+=T_\mu^{-1}$, $\delta_\mu=T_\mu-1$. Infinitesimal metric $\|\delta e\|^2$ takes the form (we put $C=0$ for simplicity):

$$\int (L\eta)^2(\det e_\mu^a)\, d^2x=\frac{1}{2}\sum_x[\,(T_2\delta_1\eta^1-e\delta_2\eta^1)^2+(\delta_1\eta^1-eT_1\delta_2\eta^1)^2+$$

$$+(\delta_2\eta^2+e\delta_2\eta^1)^2+(T_1\delta_2\eta^2+eT_1\delta_2\eta^1)^2]+O(\varepsilon)\,. \qquad (18)$$

After Gaussian integration over $D\eta^2$ in the leading approximation in ε (18) reduces to

$$\frac{1}{2}\sum_x[\,(\delta_1\eta^1+e\delta_2^+\eta^1)^2+(\delta_1\eta^1-eT_1\delta_2\eta^1)^2]\,. \qquad (19)$$

Let us temporarily omit all the terms in the square brackets in (19) with the exception of the first one (i. e. consider the case in which averaging over simplices is absent). Then we have*)

$$\ln\Phi=\mathrm{Tr}\ln(\delta_1+e\delta_2^+)=\mathrm{const}+$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\mathrm{Tr}\,(e\delta_2^+\delta_1^{-1})^n=\mathrm{const}+\sum_x \ln(1+e(x))\,,$$

$$\Phi=\mathrm{Det}\,(\delta_1+e\delta_2^+)=\mathrm{const}\cdot\prod_x(1+e(x))\,. \qquad (20)$$

---

*) To invert the derivative the boundary conditions should be specified; we define δ as follows: $(\delta\eta)_n=\eta_{n+1}-\eta_n$ (at $n=1, 2, \ldots, N-1$), $(\delta\eta)_N=-\eta_N$. That is, we fix $\eta=0$ outside some large although finite interval 1, 2, ..., N. Then $(\delta^{-1}\eta)_n=-\eta_n-\eta_{n+1}-\ldots-\eta_N$.

This expansion can be interpreted as the sum of the diagrams for the functional integral representation for Det $L(\sigma_1)$ in terms of anticommuting variables θ, θ^+. There is also another expansion for this determinant which follows when expanding the original Gaussian integral over $D\eta$ as power series in e.

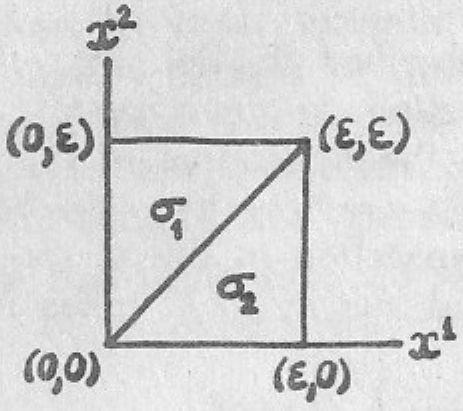

Fig. 1. The cell of the simplest two-dimensional Regge lattice.

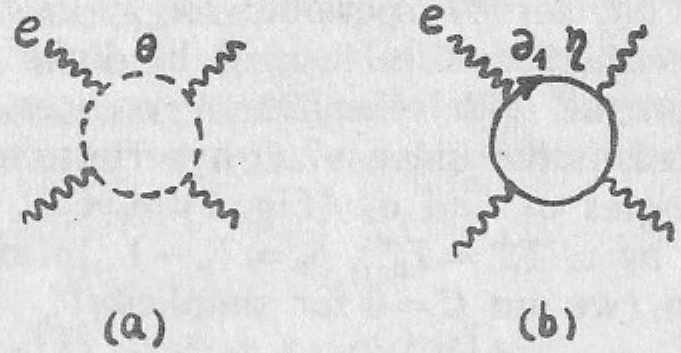

Fig. 2. The diagrams with anticommuting ghost fields (*a*) and corresponding diagrams with scalar ghosts (*b*). The arrow denotes the derivative.

Let us compare both expansions. There is the correspondence between the diagrams with Fermi ghost fields θ, θ^+ (see Fig. 2,*a*) and the diagrams of Fig. 2,*b* with the scalar field η^1. Let us denote the derivative ∂_1 at a vertex by an arrow. Then the diagrams of Fig. 2,*b* are those with all the arrows similarly oriented. Of course, the diagrams having at least two neighbouring vertices with oppositively oriented arrows are also possible, see Fig. 3,*a*. However, these two vertices form an effective local vertex (the propagator of η^1 is $(\delta_1^+\delta_1)^{-1}$) which is cancelled by the vertex $-e^2(\partial_2\eta^1)^2/2$ (see Fig. 3,*b*). Therefore, in the naive discretization the sum of all the scalar diagrams reduces to the sum of the diagrams of Fig. 2,*b*.

If averaging over simplices of the cell is present, the sum of the diagrams of Fig. 3 (*a*, *b*) is proportional to $\bar{\partial}_2^+\bar{\partial}_2 - \overline{\partial_2^+\partial_2}$ (superlining means the averaging) and differs from zero (it is a negatively defined bilinear in terms of $\partial_2(\sigma_i)-\partial_2(\sigma_j)$, $i, j=1,\ldots,n!$). For the considered Regge lattice the graphs of Fig. 3 result in the effective vertex

$$-\frac{1}{2}\left(e\,\frac{T_1-T_2^+}{2}\,\delta_2\eta^1\right)^2. \qquad (21)$$

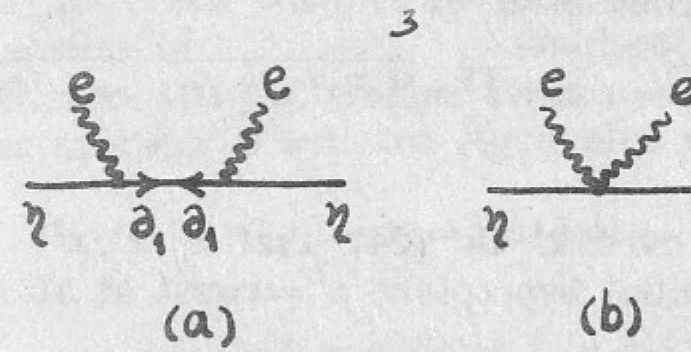

Fig. 3. The elements which are absent in the diagrams of Fig. 2*b* (they are mutually cancelled in the naive lattice discretization).

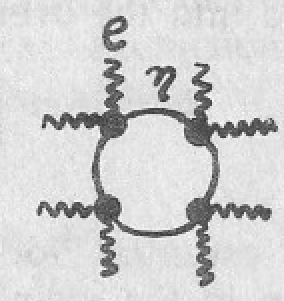

Fig. 4. Infrared divergent diagrams arising in the Regge discretization (the blob denotes the sum of the graphs of Figs 3,*a* and 3,*b*).

As a result, the IR divergent diagrams arise (see Fig. 4). The cutoff can be introduced by imposing boundary conditions on η^1 at large x^1. In the x-representation the propagator $\langle\eta^1\eta^1\rangle = (\delta_1^+\delta_1)^{-1}$ takes the form

$$\langle \eta^1(k^1,k^2)\,\eta^1(l^1,l^2)\rangle = \frac{1}{\varepsilon^2}\,\delta_{k^2l^2}\left(\frac{N}{4}-\frac{1}{2}\,|k^1-l^1|\right). \qquad (22)$$

Here k^μ, l^μ are the coordinates in units of ε, N is the IR cutoff — the size of an interval in the x^1 direction outside which we put $\eta^1=0$. The resulting dependence on N is fictitious: it is connected with nonanalyticity of the measure. Let us sum up the most divergent diagrams. The latter are those which have $2n$ e-legs joined in pairs to form the blocks (21). We take into account only N-term in the propagator (22) connecting different such blocks (21) (then T_1 can be substituted by unity in (21)). We find that (for sufficiently small Ne^2) these diagrams are summed up to give

$$\ln\Phi(e) = \sum_{x^2}\int_0^{2\pi}\frac{dp_2}{2\pi}\,\frac{1}{2}\ln\left[1+\frac{N}{4}\,\Gamma_0(p_2+i\varepsilon\partial_2)\sum_{x^1}e^2(x^1,x^2)\,\Gamma_0(p_2+i\varepsilon\partial_2)\right]\Psi_0,$$

$$\Gamma_0(p) = -\frac{1}{2}\,\delta^+(p)\,\delta(p) = -\frac{1}{2}\,(e^{-ip}-1)\,(e^{ip}-1). \qquad (23)$$

Here $\Psi_0(x^2)=1$ is the constant function on which the standing from the left operator expression $O(x^2, p_2+i\varepsilon\partial_2)$ acts. For arbitrary e^2 (23) is understood in the sence of the analytic continuation.

In the continuum limit $\varepsilon\to 0$, $N\to\infty$ and N-dependence is included into the overall normalization of Φ *):

$$\Phi = \text{const} \cdot \prod_{x^2} \sqrt{\sum_{x^1} e^2(x^1, x^2)} = \text{const} \cdot \prod_{x^2} \sqrt{\int e^2(x^1, x^2)\, dx^1}\ . \qquad (24)$$

It is essential for summation over x^1 in (24) that $e^2(x^1, x^2)$ is known in the vicinity of some initial hyperplane $x^2 = \text{const}$ at all x^1 (see the discussion following (4)).

6. THE STRUCTURE OF THE CORRECTIONS

Here we show that taking into account the less divergent diagrams or the terms $O(N^{-1})$ and $O(\varepsilon)$ in the already considered diagrams does not break the decoupling of the IR cutoff and gives at small e^a_μ the small corrections to the result obtained (24).

There are corrections in e and ε. The former can be taken into account by adding to (21) the blocks with more than two e-legs. These blocks are constructed of a number of vertices $e\eta\eta$ and $ee\eta\eta$ by «partial» Wick pairing of all but two fields η^1 intended to form external lines, only finite parts of propagators of η^1 being taken into account. On the contrary, different blocks are paired by propagators with only «infinite» $\sim N$ parts left (therefore the derivative ∂_1 acting on the external η^1 leg of such the block gives zero). Summation of these diagrams results in the expression of the type (23) with the terms like $\sim eeeN, \ldots$ taken into account under the logarithm sign:

$$\ln \Phi = \sum_{x^2} \int_0^{2\pi} \frac{dp_2}{2\pi} \frac{1}{2} \ln\left(1 + \frac{N}{4}\Gamma\right) \Psi_0,$$

$$\Gamma = \Gamma_0(p_2 + i\varepsilon\partial_2) \sum_{x^1} e^2(x^1, x^2)\, \Gamma_0(p_2 + i\varepsilon\partial_2) + O([e]^3)\,. \qquad (25)$$

Disentangling the derivatives under the logarithm sign in (25) leads to the corrections in ε (there are also corrections connected with the terms $O(\varepsilon)$ in $\int (L\eta)^2$ arising, e. g., due to the more accurate determination of $e^a_\mu(\sigma)$, see (17); taking into account them reduces simply to a modification of the effective vertex Γ in (25)). Let us check whether the limit $N \to \infty$ exists for them. Consider expansion of (25) over commutators. There is the expansion of a function $f(a+b)$ of operators a, b acting on the eigenvector Ψ_0 of the operator a with the eigenvalue a_0 [18]:

$$f(a+b) = f(a_0+b) - \frac{1}{2} f''(a_0+b)[a, b] +$$

$$+ \frac{1}{6} f'''(a_0+b)[a-b, [a,b]] + \frac{1}{8} f^{IV}(a_0+b)[a,b]^2 + \ldots \qquad (26)$$

Now $f(a+b) = \ln(a+b)$, $b = 1 + N\Gamma(x,p)/4$, $a = N[\Gamma(x, p+i\varepsilon\partial) - \Gamma(x,p)]/4$, $\Psi_0 = 1$, $a_0 = 0$. A term of the expansion containing k operators a and l operators b contains also the factor $f^{(k+l)}(a_0+b) \sim b^{-k-l}$. Therefore the dependence on N at $N \to \infty$ is cancelled at $\Gamma \neq 0$, but the integral over dp_2 can become divergent since $\Gamma(x,p) = O(|\delta(p)|^2) \to 0$ at $p \to 0$. At finite N this integral is effectively cut off when $\Gamma \sim 1/N$. If $\Gamma(x,p)|_{p\to 0} = \text{const} \cdot p^\alpha$ it is not difficult to see that the answer is proportional to $\varepsilon^m N^q$, $q \leqslant (m-1)/\alpha$. Thus, the repeated limit $\lim_{\varepsilon\to 0} \lim_{N\to\infty}$ does not exist. If, however, one passes to the limit $\varepsilon \to 0$ at the fixed lattice size $N\varepsilon = \text{const}$, the considered corrections in ε vanish.

7. THE CASE OF ARBITRARY DIMENSION

The above analysis can be readily generalized to the case of arbitrary dimension n. For example, at $n=3$ we have

$$\int (L\eta)^2 = \int (\det e^a_\mu)\, d^3x\, [\, (\eta^3_{,3} + e^3_2 \eta^2_{,3} + e^3_1 \eta^1_{,3})^2 +$$

$$+ (\eta^2_{,2} + e^{23}\eta^2_{,3} + e^2_1 \eta^1_{,2} + e^2_1 e^{23} \eta^1_{,3})^2 + (\eta^1_{,1} + e^{12}\eta^1_{,2} + e^{13}\eta^1_{,3})^2]\,. \qquad (27)$$

Integration over $D\eta^3$ in the leading order in ε is trivial. Integrating over $D\eta^2$, $D\eta^1$ we temporarily omit η^1 in the second term in [...] in (27) as if the naive shift of the integration variable were done (the analysis below shows that this shift is indeed possible in the leading approximation in ε, e^a_μ). Then these integrations are factorized and things go parallel to the two-dimensional case. Now averaging is to be made over six tetrahedra of the cubic cell. The answer for $\ln \Phi$ in the main approximation reads

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*) We omit the terms like (20) in $\ln \Phi$ which are nonsingular at $e^a_\mu = 0$.

$$\ln\Phi = \frac{1}{2}\varepsilon^{-2}\int dx^1\,dx^3\ \ln\int (e^{23})^2\,dx^2 +$$

$$+\frac{1}{2}\varepsilon^{-2}\int dx^2\,dx^3 \int_0^{2\pi}\frac{dp_2}{2\pi}\frac{dp_3}{2\pi}\ln\left\{\frac{1}{6}\sum_\sigma |D^1(\sigma)|^2 - \left|\frac{1}{6}\sum_\sigma D^1(\sigma)\right|^2\right\}_{p_1=0},$$

$$D^1(\sigma) = [\,e^{12}\partial_2(\sigma) + e^{13}\partial_3(\sigma)\,]\,\frac{\delta_1}{\partial_1(\sigma)}, \tag{28}$$

the derivatives being taken in the momenta representation in which the translations are written as $T_\mu = \exp(ip_\mu)$.

Now let us show that the naive shift of the variable performed when integrating over $D\eta^2$ above is possible in the leading approximation in ε, e^a_μ. For this purpose we consider the diagrams in which the field η^1 from the second term in [...] in (27) is present. These diagrams can be classified into three types according to which propagators, $\langle\eta^1\eta^1\rangle$, $\langle\eta^2\eta^2\rangle$ or both these ones are substituted by

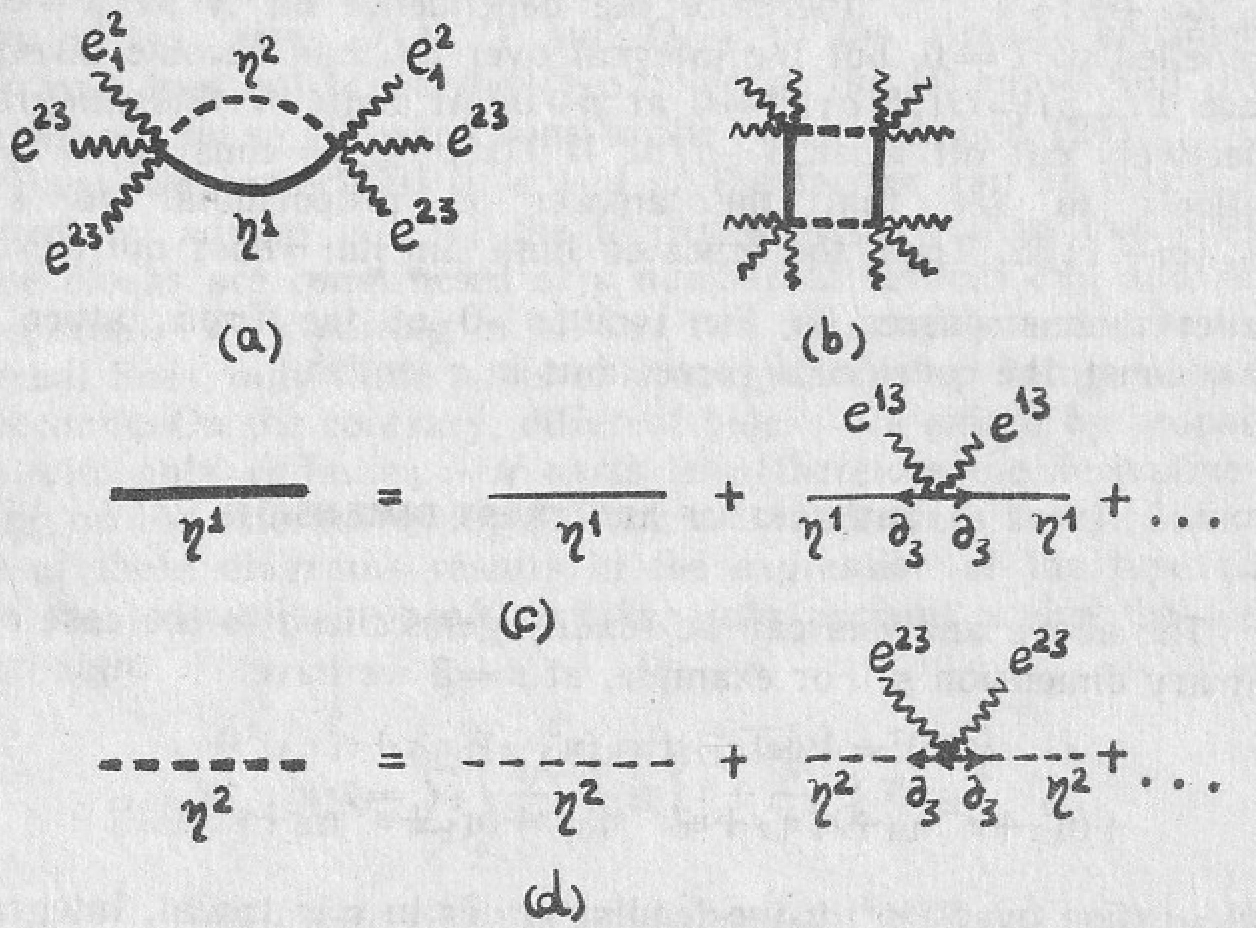

Fig. 5. Examples of the diagrams (a, b) in which both η^1 and η^2 propagators are substituted by their divergent parts; solid lines denote the propagators in which only the diagonal insertions $\eta^1\eta^1$ and $\eta^2\eta^2$ are taken into account (c, d).

their IR divergent parts N_1 and N_2. The diagrams of the first type (i. e. those proportional to N_1^m) contain at least one effective vertex of the type

$$\frac{1}{2}\eta^1\left[\,\overline{\partial_2^+ e_1^2\partial_2}\,\frac{1}{\delta_2^+\delta_2}\,\overline{\partial_2^+ e_1^2\partial_2} - \overline{\partial_2^+(e_1^2)^2\partial_2}\,\right]\eta^1 + O([e]^3). \tag{29}$$

Since $[\,\delta_\mu, e^a_\nu] = O(\varepsilon)$ it is $O([e]^3)$ in the leading approximation in ε. Analogously, if IR divergence is produced by N_2-piece of $\langle\eta^2\eta^2\rangle$ we get the effective vertex of the type

$$\frac{1}{2}\eta^2\ \overline{\partial_2^+ e_1^2\partial_2}\,\frac{1}{\delta_1^+\delta_1}\,\overline{\partial_2^+ e_1^2\partial_2}\ \eta^2. \tag{30}$$

Due to the occurence of the derivative ∂_2 acting on the external η^2-leg the diagram vanishes. Finally, the diagrams proportilnal to $N_1^l N_2^m$ contain $2k$ vertices of the type $\eta^1\eta^2$, see Fig. 5 (a,b) for $k=1,2$ respectively; the solid line denotes partially dressed propagators in which insertions of the type $\eta^1\eta^1$ and $\eta^2\eta^2$ are taken into account, see Fig. 5 (c, d). Explicit expressions for these propagators can be obtained analogously to the expression (23) for $\ln\Phi$: the difference consists in some changing of combinatorics due to which the diagrams are summed to the inverse function, not to the logarithm. As a result, the contribution of the diagram 5,a to $\ln\Phi$ is

$$-\frac{1}{2}\varepsilon^{-1}\int d^3x\,(e_1^2)^2(e^{23})^4\left[\,\int (e^{13})^2\,dx^1\right]^{-1}\left[\,\int (e^{23})^2\,dx^2\right]^{-1}. \tag{31}$$

The sum of all such the diagrams can be written as

$$\frac{1}{2}\,\mathrm{Tr}\ln(1-\hat{O}) = O(\varepsilon^{-1}) \tag{32}$$

where $\hat{O}$ is the following operator on the space of the functions of x^2, x^3:

$$(\hat{O}\Psi)(x^2,x^3) = \int g(x^2,y^2,x^3)\ \Psi(y^2,x^3)\,dy^2,$$

$$g(x^2,y^2,x^3) = \int f(x^1,x^2,x^3)\ f(x^1,y^2,x^3)\,dx^1,$$

$$f(x^1,x^2,x^3) = e_1^2(e^{23})^2(x^1,x^2,x^3)\left[\,\int (e^{13})^2(y^1,x^2,x^3)\,dy^1\right]^{-1/2}\times$$

$$\times\left[\,\int (e^{23})^2(x^1,y^2,x^3)\,dy^2\right]^{-1/2}. \tag{33}$$

In general case in the leading in e^a_μ, ε approximation $\ln\Phi$ is represented as a sum of $n-1$ terms, the a-th one depending only on $e^{a\mu}$, $\mu=1,...,a-1$:

$$\ln\Phi=\frac{1}{2}\sum_{a=1}^{n-1}\sum_{x^1,...,\widehat{x^a},...,x^n}\int_0^{2\pi}\frac{dp_1}{2\pi}...\frac{\widehat{dp_a}}{2\pi}...\frac{dp_n}{2\pi}\times$$

$$\times\ln\left(\frac{1}{n!}\sum_\sigma|D^a(\sigma)|^2-\left|\frac{1}{n!}\sum_\sigma D^a(\sigma)\right|^2\right)_{p_a=0}, \tag{34}$$

$$D^a(\sigma)=\sum_{\mu>a}e^{a\mu}\,\partial_\mu(\sigma)\frac{\delta_a}{\partial_a(\sigma)}. \tag{35}$$

The cap in (34) means that the corresponding symbol is omitted; $\partial_\mu(\sigma)$ are functions of $p_1,...,p_n$ (some polinomials of $T_\nu=\exp(ip_\nu)$). It is important that $e^{a\mu}$ enter this expression by means of a positively defined at almost all $p_1,...,\widehat{p}_a,...,p_n$'s quadratic form under the logarithm sign. Indeed, this form is zero only if $D^a(\sigma_1)=D^a(\sigma_2)$ for any two of $n!$ simplices σ_1, σ_2. With taking into account (35) this gives an overcompleted uniform linear system for $e^{a\mu}$ having the unique solution $e^{a\mu}=0$ provided the momenta $p_1,...,\widehat{p}_a,...,p_n$ do not belong to some set of zero measure (this set is singled out by triviality of the translations in some directions, $T_\mu T_\nu...T_\lambda=\exp i(p_\mu+p_\nu+...+p_\lambda)=1$). Expression (34) is of the order of ε^{-n+1} and is logarithmically singular at $e^a_\mu\to 0$.

8. CONCLUSION

Thus, the entropy on the measure of the considered Regge lattice is nonanalytic at the Euclidean point $e^a_\mu=0$ and results in the instability of the theory. Analogous conclusions can be made also for general form of the metric on the tetrad space at $C=0$, see (16). Besides that, we can generalize our result to any Regge lattice periodic in the considered coordinate system. In this case we still get expression like (34) for the entropy in which $n!$ is substituted by the number of simplices of the elementary cell and the derivatives $\partial_\mu(\sigma)$ are determined by the simplicial structure of the cell. A possibility to get physically acceptable theory can only be connected with breaking the periodicity and making use of the random Regge lattice.

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