ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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STATISTICAL PROPERTIES OF THE QUASI-ENERGY SPECTRUM OF A SIMPLE INTEGRABLE SYSTEM

PREPRINT 87-31

НОВОСИБИРСК

Statistical Properties of the Quasi-Energy Spectrum of a Simple Integrable System

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ABSTRACT

Numerical results about the statistical properties of a number sequence generated by zero-entropy map of a 2d torus are presented. The problem is related to localization theory with a pseudorandom potential. The validity of Poisson statistics for the corresponding integrable quantum model is analyzed.

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The investigation of the properties of conservative quantum systems that are chaotic in the classical limit has led to inquire about the statistical properties of their energy spectra. It is now well established that, in the case of fully developed classical chaos (as, for example, for the Sinai billiard) the Wigner — Dyson statistics provides a very good description for the fluctuation properties of such spectra [1]. A similar result was obtained for quasi-energy spectrum of the simple models [2—3].

In the opposite case of a completely integrable classical system, it is generally accepted that the statistics of the quantum energy levels should be generally Poisson. This was first shown by Berry and Tabor [4] who theoretically derived the exponential Poisson law for the distribution function of spacings between neighbouring levels of a generic conservative systems.

Nevertheless, the expectation that the Poisson statistics might also describe high-order statistical properties in the integrable case was contradicted by results of Ref. [5], in which the Δ_3-statistics for a rectangular incommensurate billiard was found to significally deviates from the Poisson theoretical dependence. The behaviour of the Δ_3-statistics was then theoretically analyzed by Berry [6], and it is now understood that the statistical properties of semiclassical integrable spectra depend on which scale is being used in order to analyze them.

In contrast to the large attention which has been devoted to the energy spectra in the integrable case, much less is known about quasi-energy spectra; the important difference in this case being that quasi-energy eigenvalues are considered to lie in some finite interval, thus providing additional mixing.

In this paper we carefully investigate the statistics of quasi-energies in a very simple case, which is related to the well-known quantum rotator (see, e. g. [7]). The sequence of quasi-energy eigenvalues we consider is given by the simple formula:

$$\lambda_n=\left\{\lambda_0+n\theta_0+\frac{\tau}{2}n(n-1)\right\}=\left\{\lambda_0+\frac{\tau}{2}n^2+\left(\theta_0-\frac{\tau}{2}\right)n\right\}. \qquad (1)$$

where $\{\ \}$ denotes fractional part, θ_0 and τ are fixed irrational numbers. In the particular case $\lambda_0=0$, $\theta_0=\tau/2$, we get

$$\lambda_n=\left\{\tau\frac{n^2}{2}\right\}. \qquad (2)$$

which is just the quasi-energy spectrum of a rotator in the limit of vanishing perturbation. The statistical properties of a number sequence closely related to (2) also play an important role in establishing a connection between the rotator problem and Anderson localization for a quantum particle on a 1-d lattice in a random potential [8].

That (1) and (2) cannot have a very strong random character is seen from an equivalent representation of (1) (with $\tau>0$) in the form of a map on a 2-d torus:

$$\lambda_{n+1}=\{\lambda_n+\theta_n\},$$
$$\theta_{n+1}=\{\theta_n+\tau\}. \qquad (3)$$

It can be shown that even for irrational τ this map is not mixing and has therefore zero entropy.

We investigated two cases: (2) and (1) by taking different values for τ and θ_0. The typical results are illustrated by the cases represented in Figs 1—4. For comparison, we also used a standard multiplicative random sequence. Computations were performed in 10^{-14} precision.

As a first test, we investigated the distribution of spacing between neighbouring levels. A typical result for sequence (2) is shown in Fig. 1. At first glance the agreement with the Poisson distribution seems very good, except for the first interval. The χ^2-value for the whole considered interval is $\chi^2_{151}\approx 238$, for 151 subintervals, with a poor confidence level $<10^{-5}$. Also, looking just at the first

interval, we found $\chi^2_{10}\approx 125$ (with 10 subintervals), with a negligible confidence level.

Therefore, we conclude that, while the spacings are distributed according to the exponential law, their fluctuations do not appear to be in agreement with Poisson statistics. In contrast to this, a similar analysis for a different sequence with a nonzero «linear shift» $(\theta_0-\tau/2)$ gave a much better confidence; e. g. for the case of $\tau=1/\sqrt{3}$, $\theta_0=\sqrt{\tau}-\tau/2$ the χ^2-value is $\chi^2_{151}\approx 173.2$ with a good confidence level ≈ 0.2, also $\chi^2_{10}\approx 5.6$ which gives for the confidence level ≈ 0.8. These conclusions were confirmed by a detailed analysis of fluctuations (Fig. 2). To this end, we used the same approach as in Ref. [5]: we constructed a histogram for the normalized deviations of the observed number of spacings in 1000 intervals, chosen in such a way that the expected number of levels in eash interval according to the Poisson law is 100. We observe a strong deviation from the Gaussian distribution in Fig. 2,*a* (corresponding to data of Fig. 1) which indicates a strong correlation in fluctuations about the Poisson distribution, not restricted just to the first interval (0; 0.02).

Instead, the introduction of the linear shift greatly improves the agreement with the Gaussian distribution: see Fig. 2,*b*, in which χ^2 from Gaussian law is ≈ 29.7 with 24 subintervals. This can be compared with Fig. 2,*c* which was gotton by standard pseudorandom sequence, here $\chi^2\approx 32.6$. Thus the sequence with the linear shift proves as good as the pseudorandom one.

A different approach to the analysis of correlations of fluctuations is provided by the Δ_3-statistics, that characterizes the so-called «rigidity» of the spectrum [9]. For a given number L of quasi-energies λ_n we found the dependence $\bar{\Delta}_3(L)$ (as in Ref. [5]), by averaging $\Delta_3(\lambda_n, L)$ computed along a segment of L levels starting with λ_n, over a string $\lambda_1\leqslant\lambda_n\leqslant\lambda_N$. Typically, $N=10^4$. The results are shown in Fig. 3. We see that for $L\ll N$ we have a good agreement with the theoretical Poisson behaviour for all the investigated cases: a pseudorandom sequence, a sequence (1) with a nonzero linear shift, and a sequence (2). When L is increased, deviations appear in all cases. The main reason of these deviations is that when L is not small enough compared to N, a correlation in fluctuations appears, due to insufficient statistics. Indeed, a closer agreement with Poisson statistics was gotton by increasing N up to $5\cdot 10^4$. This conclusion is also supported by the fact that, by averaging Δ_3 in a different way, namely, over independent strings of length L, we could not observe so large systematic deviations (Fig. 4). It is interesting to note, that no substantial difference emerges in three cases investigated.

A somewhat similar behaviour of Δ_3 was obtained in [5]. However, in our opinion, the sequence investigated there differs from ours, in that its «local» rigidity is not homogeneous over the spectrum. This feature could hardly survive in our case, because of the additional mixing provided by taking fractional parts.

From our data we can draw two different types of conclusion. The first is concerned with the properties of the sequence (1) and (2); the interesting fact emerges, that introducing a linear shift greatly improves the statistical properties. Indeed, the agreement with Poisson statistics is quite bad if no linear shift is present, as it is particulary clear from the analysis of fluctuations about the exponential distribution of spacings. Nevertheless, supplementary checks of a different nature are required, in order to decide about the usefulness of (1) as an effective pseudorandom sequence.

The second concerns the effectiveness of the statistical tests. In this respect, Δ_3-statistics appears to be not so good as the analysis of the distribution of fluctuations Fig. (2); indeed, the latter exposes a much clearer difference between (1) and (2) than was possible guess just by Figs 3, 4.

A more general indication can also be drawn, since, as already quoted, the statistical properties of (2) determine the most important feature of the dynamics of a lattice model related to the quantum kicked rotator, namely, they determine whether localization occurs. This would be certainly the case if λ_n were a completely random sequence. On the other hand, it is known that for «good» irrational numbers τ, the sequence (2) always yields localization [7, 8]. It therefore appears, that localization does not require very strong statistical properties.

The authors would like to express their gratitude to Prof. B.V. Chirikov for fruitful discussions and valuable comments.

REFERENCES

1. *O. Bohigas, M.-J. Giannoni, C. Schmit.* Phys. Rev. Lett., 52 (1984) 1.
2. *F.M. Izrailev.* Phys. Rev. Lett., 56 (1986) 541.
3. *J.V. Jose, R. Cordery.* Phys. Rev. Lett., 56 (1986) 290.
4. *M.V. Berry, M. Tabor.* Proc. Roy. Soc., A356 (1977) 375.
5. *G. Casati, B.V. Chirikov, I. Guarneri.* Phys. Rev. Lett., 54 (1985) 1350.
6. *M.V. Berry,* Proc. Roy. Soc., A400 (1985) 229.
7. *B.V. Chirikov, F.M. Izrailev, D.L. Shepelyansky.* Sov. Sci. Rev. 2C (1981) 209.
8. *S. Fishman, D.R. Grempel, R.E. Prange.* Phys. Rev. 29A (1984) 1639.
9. *M.L. Mehta.* Random Matrices. Academic, New York, 1967.

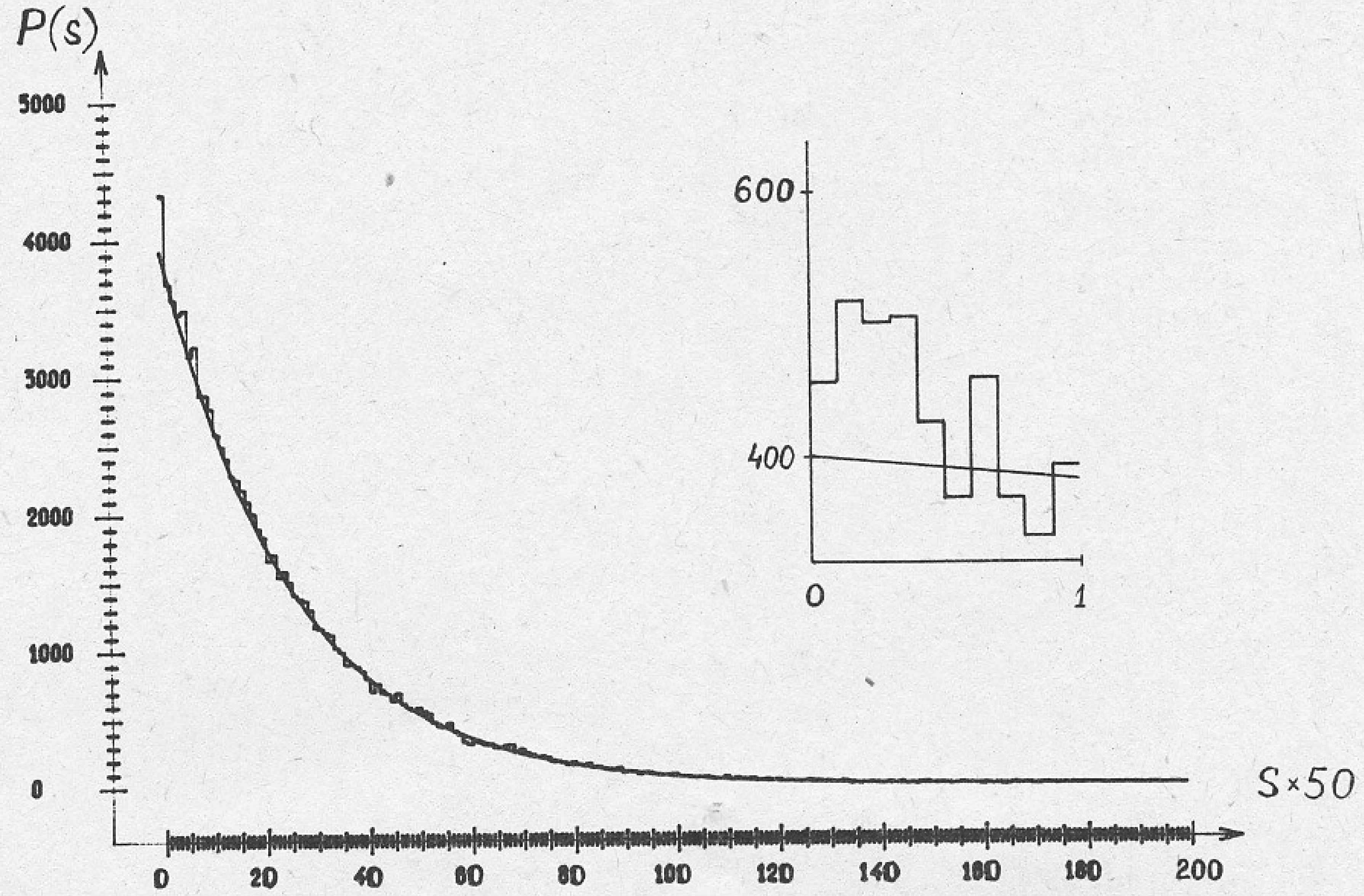

Fig. 1. Distribution $P(s)$ of nearest level spacings, computed over 10^5 values of (1) with $\tau=1/\sqrt{3}$, $\theta_0=\tau/2$; here s is in units of the average spacing. Smooth curve: the Poisson distribution. The detail shows the distribution inside first interval (0; 0.02). Straight line is the Poisson distribution.

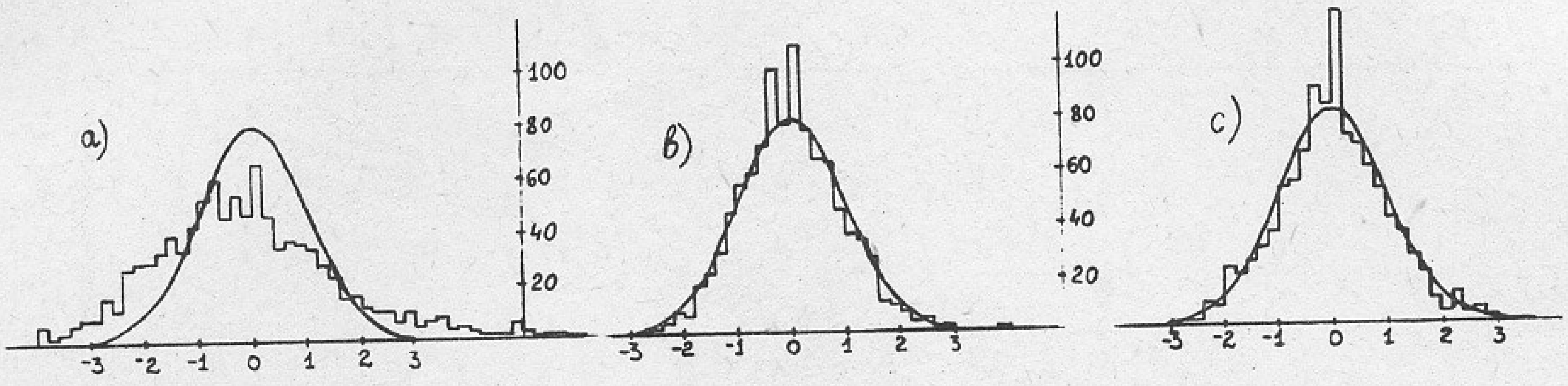

Fig. 2. Hystograms of the distributions of deviations $m_i=(n_i^0-n_{ex})/\sqrt{n_{ex}}$ of the observed number of spacings n_i^0 from the expected n_{ex} in the i-th interval. Full lines show the Gaussian distribution with $\sigma=1$.

a — a sequence (2) with $\tau=1/\sqrt{3}$, $\theta_0=\tau/2$; χ^2_{1000} for deviation from Poisson distribution was here 3184; b — a sequence (1) with $\tau=1/\sqrt{3}$, $\theta_0=\tau/2-\sqrt{\tau}$. Here $\chi^2_{1000}\approx 934$, c — pseudorandom sequence; $\chi^2_{1000}\approx 1068$. All data are obtained from 10^5 sequence values. Confidence levels for Gaussian distributions: 0.05 (b), 0.03 (c).

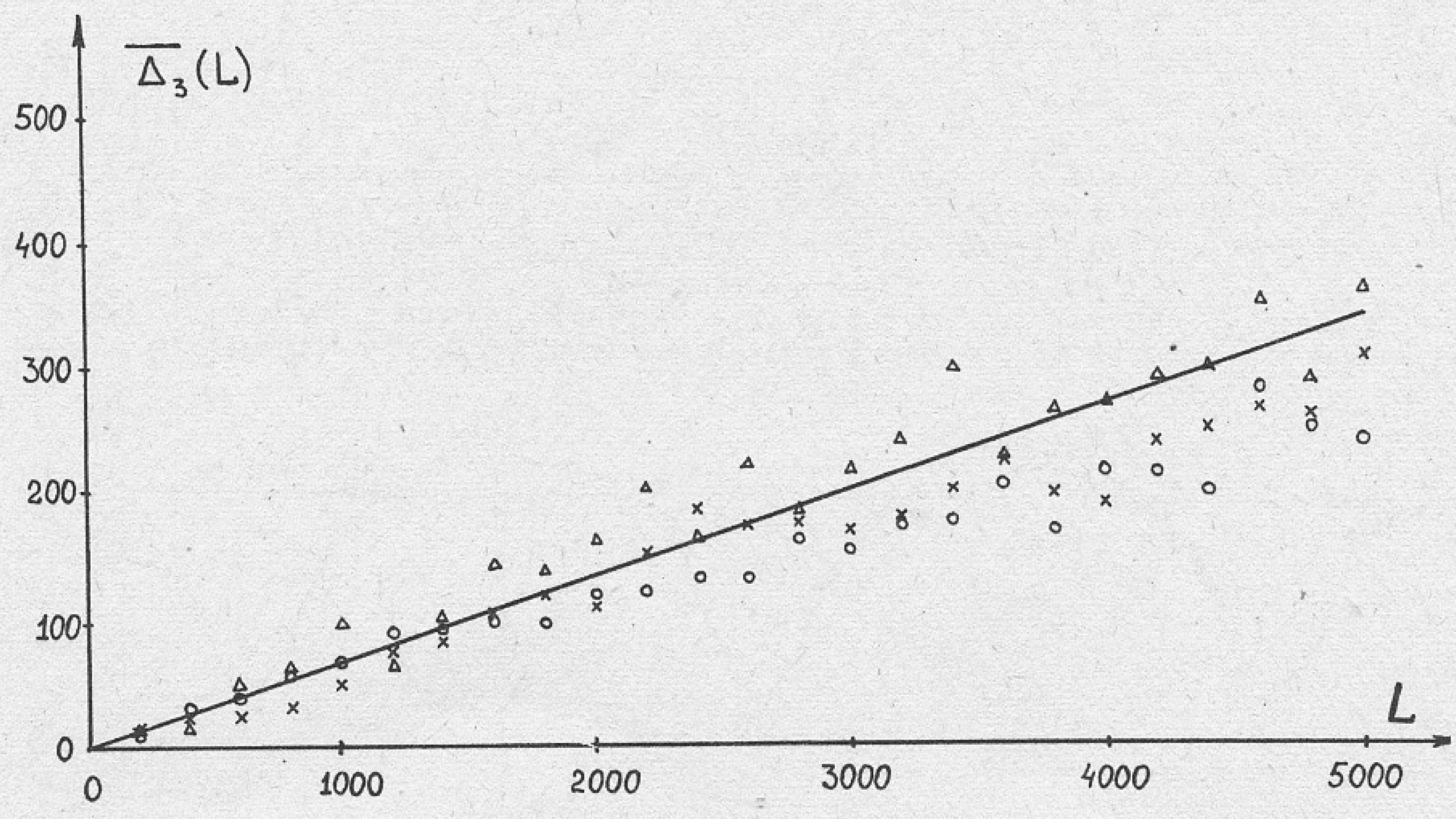

Fig. 3. Statistics $\bar{\Delta}_3(L)$ in three cases. Triangles: $\tau=1/\sqrt{3}$, $\theta_0=\tau/2$. Circles: $\tau=1/\sqrt{2}$, $\theta_0=0.1$. Crosses: pseudorandom sequence, $N=10^4$. The effect of increasing N to 5×10^4 is shown for $L=1000$, 1500.

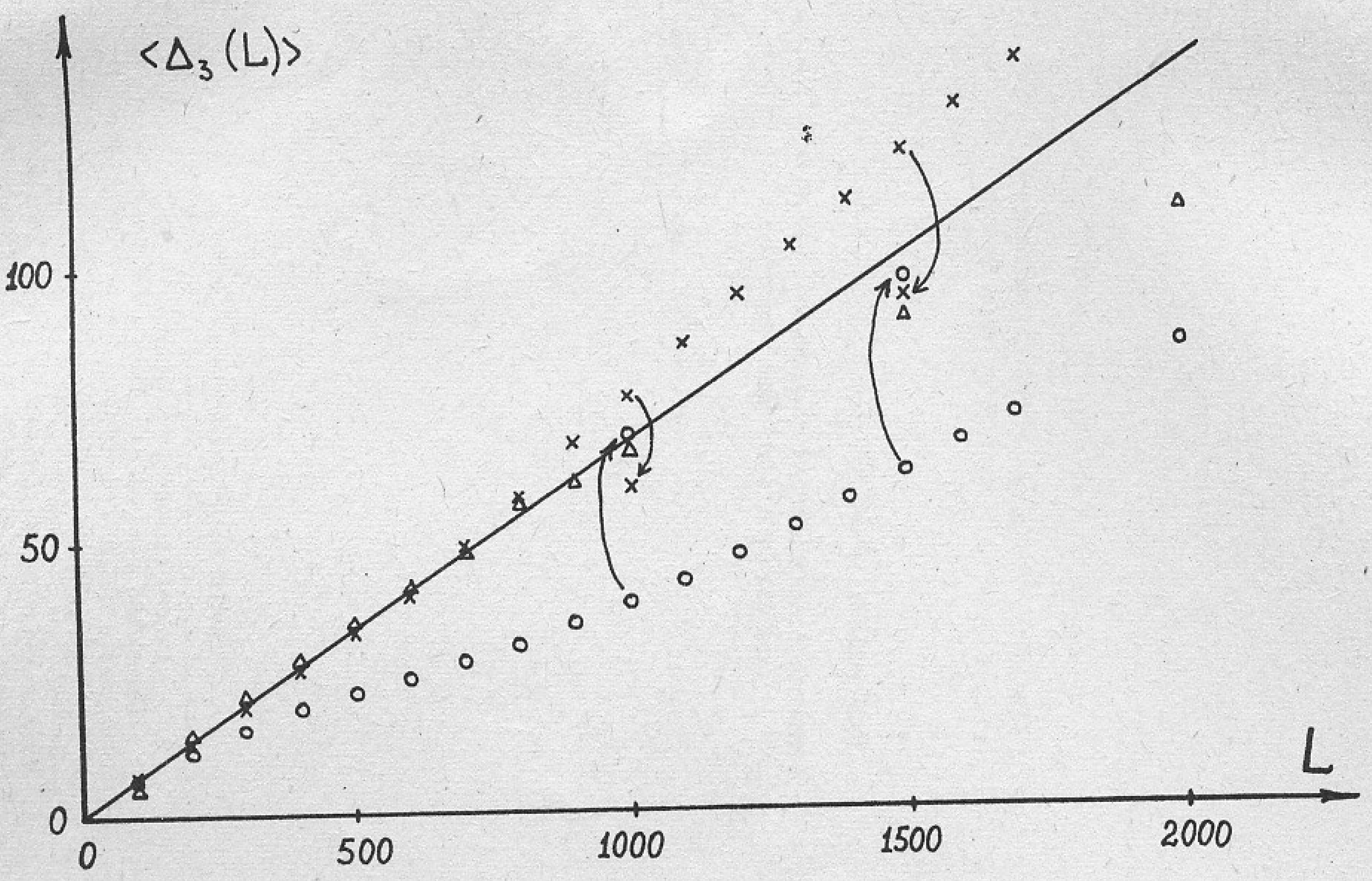

Fig. 4. The same as in Fig. 3, by averaging $\Delta_3(L)$ over independent intervals, $N=5\cdot10^4$.

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Статистические свойства спектра квазиэнергий простой интегрируемой системы

Ответственный за выпуск С.Г.Попов

Работа поступила 11 марта 1987 г.
Подписано в печать 25.03.1987 г. МН 08659
Формат бумаги 60×90 1/16 Объем 0,9 печ.л., 0,8 уч.-изд.л.
Тираж 290 экз. Бесплатно. Заказ № 31

Набрано в автоматизированной системе на базе фотонаборного автомата ФА1000 и ЭВМ «Электроника» и отпечатано на ротапринте Института ядерной физики СО АН СССР,
Новосибирск, 630090, пр. академика Лаврентьева, 11.