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F.M. Izrailev

INTERMEDIATE STATISTICS OF QUASI-ENERGY SPECTRUM AND QUANTUM LOCALIZATION OF CLASSICAL CHAOS

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Intermediate Statistics of
Quasi-Energy Spectrum and Quantum
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F.M. Izrailev

Institute of Nuclear Physics
630090, Novosibirsk 90, USSR

ABSTRACT

The statistical properties of quasi-energy spectrum in a simple quantum model are investigated for the case when the correspondent classical system is fully chaotic meanwhile quantum chaos is restricted by the localization effects. It is shown that the level spacing distribution depends effectively on some parameter which is the ratio of the dimension of eigenfunctions (mean localization length) to the total number of the quasi-energy levels. Numerical data in a wide range of parameters of the system are given.

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1. INTRODUCTION

The problem of the properties of quantum systems whose classical counterparts reveal the chaotic motion is still attractive for many scientists. One of the most important results in this field is close relation between the spectral properties of quantum chaos and those of random matrices of certain symmetry [1—5]. This relation is far from being trivial if only for one reason: The quantum systems under consideration have no random parameters. Nevertheless, numerical experiments have shown that random matrix theory (RMT) can be well applied to describe the statistical properties of energy [4] (or quasi-energy [5]) spectrum as well as chaotic structure of eigenfunctions [6]. Specifically, the spacing distribution $P(s)$ of nearest-neighbour levels for such systems is described with a high accuracy by simple Wigner — Dyson surmise [7—9]:

$$P(s) = As^{\beta} \exp(-Bs^2), \qquad (1)$$

where A and B are normalizing constants, and β is a parameter depending on the symmetry of the system and characterizing the repulsion between neighbour levels.

On the other hand, it was discovered the so-called quantum localization which can strongly suppress chaos in quantum system compared with classical one [10, 11]. Such a localization is analogous to the Anderson localization in solid state physics but, in principle, is different because of strongly deterministic nature of the system. As a result, it turns out that maximal quantum chaos appears under certain conditions when all eigenfunctions (EF) are random and fully extended (delocalized) in the restricted phase space of the system [5, 6]. It is clear that such situation corresponds to the case of strongly enough perturbation which covers all unperturbed states. Nevertheless, another case of so-called «intermediate» quantum chaos is possible which is characterized by localized chaotic states of the system [12].

In this paper we study the spacing distribution $P(s)$ of nearest-neighbour levels taking into account the finite length of localization of chaotic EF. To this time much attention has been paid to the properties of quantum chaos. Nevertheless, correlation between the rate of quantum localization and the statistical properties of spectrum (see also [13, 14] is not enough investigated.

It should be noted that Berry—Robnik approach [15—17] to describe spacing distribution $P(s)$ concerns completely different situation for which corresponding classical system is not fully chaotic and deviation $P(s)$ from Wigner—Dyson dependence (1) is caused by the existing of the stable regions in the phase space. It is known that in other limit case of completely integrable classical systems the level spacing distribution of quantum systems is very close to Poissonian $P(s)\sim\exp(-s)$ (for generic systems, see, [18—20]). For this reason the intermediate statistics in [15—17] is considered as a sum of two types of distribution (Poisson and Wigner—Dyson ones) depending on how phase space of classical systems is divided by the regions with stable and chaotic motion.

2. THE MODEL OF KICKED ROTATOR ON THE TORUS

Let us consider the well known kicked rotator (see e. g., [10, 11])

$$\hat{H}=-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2}+\varepsilon\cos\theta\,\delta_T(t);\qquad \delta_T(t)\equiv\sum_{m=-\infty}^{\infty}\delta(t-mT).\tag{2}$$

It is convenient to describe the motion of such system by the mapping for Ψ-function after one period T of perturbation

$$\Psi(\theta,t+T)=\exp\left(i\frac{T\hbar}{4I}\frac{\partial^2}{\partial\theta^2}\right)\exp\left(-i\frac{\varepsilon}{\hbar}\cos\theta\right)\exp\left(i\frac{T\hbar}{4I}\frac{\partial^2}{\partial\theta^2}\right)\Psi(\theta\,t).\tag{3}$$

It is written in a symmetric form where Ψ-function is determined just in the middle of free rotation, between two successive kicks. It is clear from (3) that the bevaviour of the system depends only on two parameters: $\tau\equiv\hbar T/I$ and $k\equiv\varepsilon/\hbar$. It is known [21] that the corresponding classical system (the so-called «standard mapping») has strongly chaotic motion under the condition $\mathcal{K}\equiv k\tau\gg1$.

According to numerical data (see e. g., [10, 11]) the quantum model (2), (3) imitates (under additional condition $k\gg1$ which means a large number of unperturbed levels covered by one kick) such a rough statistical property as diffusion of energy in time and relaxation of the distribution function in momentum space. But it occurs only for some time $t\leqslant t^*$ after which the quantum interference effects start to influence more and more. As a result, for $t\geqslant t^*$ classical diffusion is suppressing and after all stopping itself (for generic irrational values $\tau/4\pi$). It was established [11, 12] that this time t^* of correspondence to the classical diffusion ($E_{cl}=Dt/2$) is determined by the rate of diffusion: $t^*\approx D\sim k^2$. The mechanism of this interesting effect is caused by the localization of all eigenfunctions in unrestricted (infinite) momentum space of the system. The mean localization length of EF, as it was shown in [11, 12, 22], is related to the classical diffusion coefficient D:

$$l_D\approx\frac{D}{2}\approx\frac{k^2}{4}.\tag{4}$$

For model (2), (3) the level spacing distribution $P(s)$ must be Poissonian as far as localization length remains finite for any finite value of k and therefore, relative number of overlapped EF in infinite momentum space is vanishing. Nevertheless, if we are interested in level statistics of those EF which are overlapped if only by part then we can found some repulsion of nearest levels (see also [13]). It is naturally to expect the rate of repulsion to be dependent on the rate of overlapping of EF choosen from the total (infinite) number of states.

For our purpose to investigate the influence of localization on statistical properties of quasi-energy spectra it is convenient to pass to a model with finite number N of levels

$$\Psi_n(t+T)=\sum_{m=1}^{N}U_{nm}(k,\tau)\,\Psi_m(t);\qquad n,m=1,2,\dots,N.\tag{5}$$

Here the finite unitary matrix U_{nm} determines evolution of any

N-dimensional vector (Fourier transform of Ψ-function) of the system. It has symmetric form:

$$U_{nm}=G_{nn'}\,B_{n'm'}\,G_{m'm}\,, \qquad (6)$$

where diagonal matrix $G_{ll'}$ corresponds to free rotation during a half period $T/2$:

$$G_{ll'}=\exp\left(i\frac{T}{4}l^2\right)\delta_{ll'} \qquad (7)$$

and matrix $B_{n'm'}$ describes the result of one kick:

$$B_{n'm'}=\frac{1}{2N+1}\sum_{l=1}^{2N+1}\left[\cos(n'-m')\frac{2\pi l}{2N+1}-\cos(n'+m')\frac{2\pi l}{2N+1}\right]\times$$
$$\times\exp\left(-ik\cos\frac{2\pi l}{2N+1}\right). \qquad (8)$$

This model (5) — (8) with finite number of states can be regarded as quantum analog of classical standard mapping on the torus with closed momentum p and phase θ. The difference of (5) — (8) from those investigated in [5, 6] is that matrix U_{nm} describes only odd states of the system ($\Psi(\theta)=-\Psi(-\theta)$).

Such a model can be deduced from the model (2), (3) in a following way [5, 12]. Let us first consider (2), (3) for rational values of $\tau/4\pi=r/q$ (with r, q — integers). It corresponds to the so-called quantum resonance [23, 24] for which all EF in momentum representation are analogous to the Bloch states in a periodic crystal. Therefore, each EF is multiplied by phase factor $\exp(i\theta_0)$ under the shift in period q. By selection of only periodic EF with $\theta_0=0$ in the model (2), (3) we can construct the finite matrix of size q which describes evolution of periodic (in momentum space) states [5, 6]. The phase space of corresponding classical model is closed in momentum p with the size $2\pi m_0$ where $m_0=2r$ comes from the periodicity in p. Then, selecting only odd states it is easy to pass to the matrix U_{nm} with the reduced size $N=(q-1)/2$ (here q is odd number).

Our model (5), (6) can be, in principle, interpreted also as a model of some conservative system with finite number of levels on the closed energy surface. Therefore, statistical properties of quantum chaos investigated here are typical also for autonomous systems with chaotic counterpart in the classical limit. Similar models have been also considered in [14, 25].

3. DIMENSION OF CHAOTIC EIGENSTATES: DEFINITION

Recently it was shown [5, 6] that under the conditions $\mathscr{K}\gg 1$ (strong classical chaos) and $\Lambda\equiv l_D/N\gg 1$ (delocalization of all EF of the system) the quantum chaos in model (5) — (8) is maximal. It means that statistical properties of quasi-energy spectrum and chaotic structure of EF are maximal. Specifically, the level spacing distribution $P(s)$ for quasi-energies ω of U_{nm} is in excellent agreement with the dependence (1) for $\beta=1$. Moreover, distribution of the components of EF in unperturbed basis with a high accuracy corresponds to microcanonical distribution of eigenvector components of finite random matrices [9]:

$$W_N(\varphi_n)=\frac{\Gamma\left(\frac{N}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{N-1}{2}\right)}\,(1-\varphi_n^2)^{\frac{N-3}{2}}\,;\quad \sum_{n=1}^{N}\varphi_n^2=1\,. \qquad (9)$$

As long as matrix U_{nm} is unitary and symmetric the real and imaginary parts of its EF are equal to each other and equal to EF of the real and imaginary part of U_{nm}. Therefore, the quantity φ_n in (9) is either real or imaginary part of EF of matrix U_{nm}. Let us note that for $N\to\infty$ microcanonical distribution (9) goes to Gaussian one. It means that in the semiclassical region all EF of the system with a maximal quantum chaos are Gaussian random functions. At the same time, distribution $P(s)$ for $\Lambda<1$ turns out to be intermediate between Wigner — Dyson (1) and Poissonian ones [5, 12] and eigenvectors of U_{nm} are random only on some localization scale in the momentum space.

In what follows we shall introduce a new definition of localization length of EF as far as relation (4) has sense only for the model (2), (3) with infinite momentum space (or, just the same, for the model (5) — (8) with $l_D\ll N$). Unlike the traditional definition of localization length as inverse rate of amplitude decreasing of EF for $n\to\pm\infty$ (n is the number of unperturbed state) we determine l through the «entropy» $\mathscr{H}$ of EF (not confuse with thermodynamical entropy):

$$\mathscr{H}_N^{(m)}=-\sum_{n=1}^{N}w_{nm}\ln w_{nm}\,;\qquad w_{nm}\equiv\varphi_{nm}^2\,. \qquad (10)$$

Here m stands for the individual eigenvector of matrix U_{nm} ($m=1,\ ...,\ N$).

In the limit case of microcanonical distribution of φ_n (see (9)) the entropy $\mathscr{H}_N^{(m)}$ can be easily found from (10):

$$\mathscr{H}_N^{(m)} \approx \ln\left(\frac{N}{2}\alpha\right) + \frac{1}{N}; \quad N \gg 1, \tag{11}$$

where α — some constant:

$$\alpha = \frac{4}{\exp(2-\gamma)} \approx 0.96 \tag{12}$$

with γ being the Euler constant ($\gamma \approx 0.577$). Now it can be seen that the quantity L_m,

$$L_m = \exp(\mathscr{H}_N^{(m)}) \tag{13}$$

has the meaning of effective number of components φ_{nm} with not too small values. As an example let us take steady-state distribution $w_{nm} = 1/N$. Then the number L_m is equal to the maximal dimension of EF ($L_m = N$). In comparison, for microcanonical distribution (9) we can get from (10), (11):

$$L_m \approx \alpha \frac{N}{2}. \tag{14}$$

It means that in spite of ergodicity of EF ($\langle \varphi_{nm}^2 \rangle = 1/N$) the fluctuations are resulting in very small values $w_{nm} \approx 0$ approximately for a half of components of φ_{nm}. This fact is related to the particular form of distribution of w_{nm}, which is χ^2-distribution with the divergence for $w_{nm} \to 0$. As a result, the probability density of EF turns out to be full of «holes» both in momentum p and in «coordinate» θ space.

Numerical data show that for $\mathscr{K} = \text{const} \gg 1$ the scale on which EF can be considered as random is less than maximal dimension N and is decreasing with the decrease of quantum parameter k. Therefore, in accordance with (9) — (13) the mean localization length can be associated with the average dimension d of EF and determined by the «entropy»

$$l_{\mathscr{H}} \equiv d = \langle d_m \rangle = \frac{2}{\alpha} \langle L_m \rangle \approx 2 \langle \exp(\mathscr{H}_N^{(m)}) \rangle; \quad d \gg 1, \tag{15}$$

where d is averaged over all eigenvectors of matrix U_{nm}.

In essence, relation (15) is a definition both the mean localization length and the dimension of chaotic EF. In the limit of maximal quantum chaos it gives $d = N$ but for $d \ll N$ numerical data show a good agreement with the usual definition of localization length using the decay of EF on the «tailes» (see further).

It should be pointed out that for $d = N$ our matrix, in principle, is not random one (it depends only on two dynamical parameters, $\mathscr{K}$ and k). Nevertheless, in this limit case all statistical properties are very well described by random matrix theory (RMT). For $d < N$ the situation is much more difficult because RMT is not already applicable. It seems that eigenvectors of U_{nm} with chaotic localized structure are isotropic only in some part of N-dimensional Hilbert space. It is interesting whether it is possible to develop mathematical theory for such type of matrices.

4. THE MAIN PROPERTIES OF LOCALIZED CHAOTIC STATES: NUMERICAL DATA

Now we investigate the dependence of dimension d on quantum parameter k in our model (5) — (8), when classical parameter $\mathscr{K} \approx 5$ is large enough to provide strong classical chaos [21]. All semiclassical conditions are supposed to be fulfilled: $N \gg 1$; $k \gg 1$, $\tau = 4\pi r/q \ll 1$; $(q = 2N+1)$. The result for $N = 398$ appears in Fig. 1 where dimension d have been computed according to (10) — (15) with φ_n being the real part of all EF of matrix U_{nm}. So far there is no analytical description of dependence $d(k)$. Nevertheless, it is reasonable to compare $d(k)$ with the known relation [11, 12, 22] between localization length l (see (4)) and k for small values $d \ll N$ (or $k^2 \ll N$). As it was mentioned above, numerical data for free rotator model (2), (3) show [12, 26] that localization length measured by the decay rate of EF is equal $l \approx k^2/4$ (for $\mathscr{K} \approx 5$). As a rough estimate, let us suppose EF to be of the form $\varphi_n = l^{-1/2} \exp(-|n - n_0|/l)$ without taking into account fluctuations of its amplitude. Here n_0 is a center of «gravity» of EF. Substituting this expression into (10), (15) we have, for $|n_0| \ll N$:

$$d \approx 2el \approx 5.4\, l \approx 1.25\, k^2 \tag{16}$$

while the fitting line in Fig. 1 (see, insertion) corresponds to $d \approx 0.87 k^2$. It is quite good correspondence of dimension $d \ll N$ to the common definition of localization length. Nevertheless, further numerical experiments should be carried out not only in the region

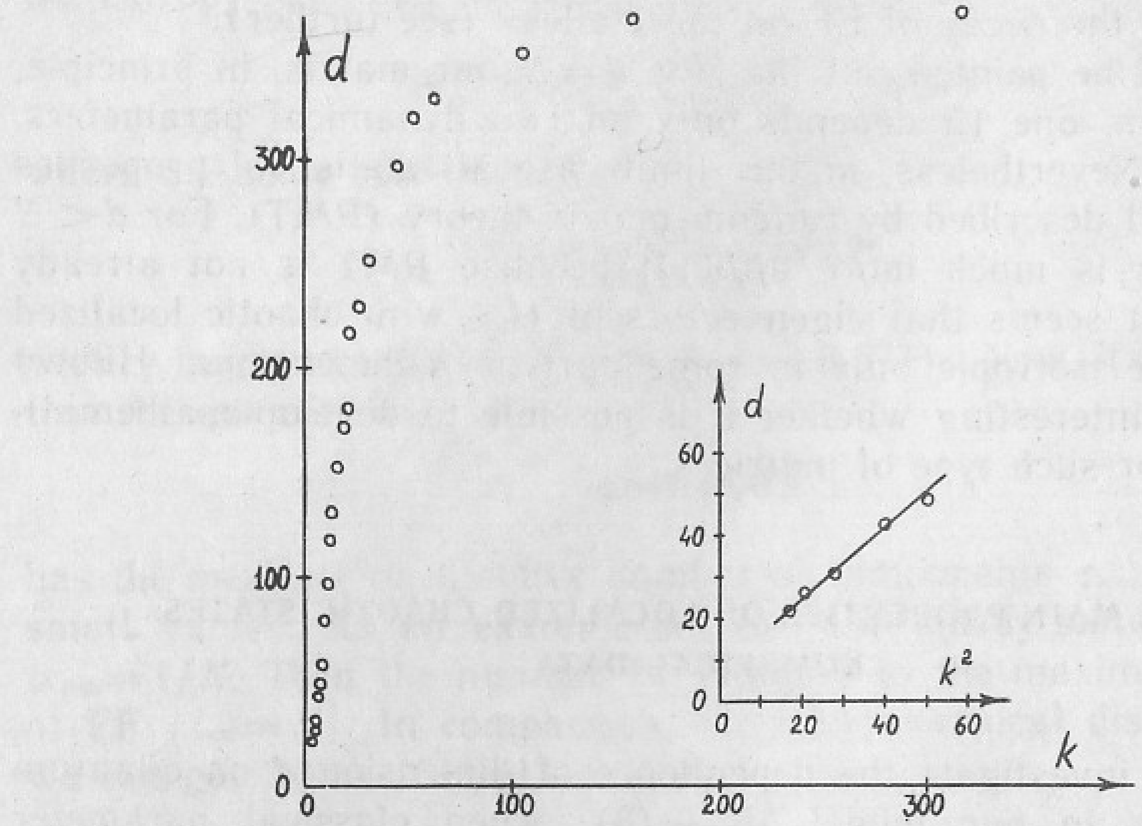

Fig. 1. The mean localization length (dimension d) versus quantum parameter k for the fixed value of classical parameter $\mathscr{K}=5$.

$k\gg 1$, $1\ll d\ll N$ (see also [26]) but also for $k\gtrsim 1$ $(d\gtrsim 1)$, which is slightly above the quantum stability border.

Remarkable property of localized chaotic states is large fluctuations of localization length d_m of the individual EF. As an example, Fig. 2 represents localization length distribution for three values of $k\approx 3.3$; 21.1; 317 (respectively, $r=95$; 15; 1 for $\tau=4\pi r/(2N+1)$), with the horizontal scale being the ratio of dimension d_m (localization length $l_{\mathscr{K}}$) to the total number of levels N. It is seen that the most large fluctuations correspond to the value $d/N\approx 0.5$. In this case there are both strong localized states ($d_m\ll N$) and completely extended states ($d_m\approx N$). Nevertheless, inspite of these fluctuations, the average dimension d can be described by «good» smooth dependence $d(k)$ (see Fig. 1).

Our approach to determine localization length using «entropy» of EF is well associated with the simple idea of localization length as an effective size on which the main probability of EF is concentrated. This is confirmed by the data on Fig. 3 where «entropy» localization length d versus «probability» localization length l_w is shown. The latter have been computed as a number of unperturbed

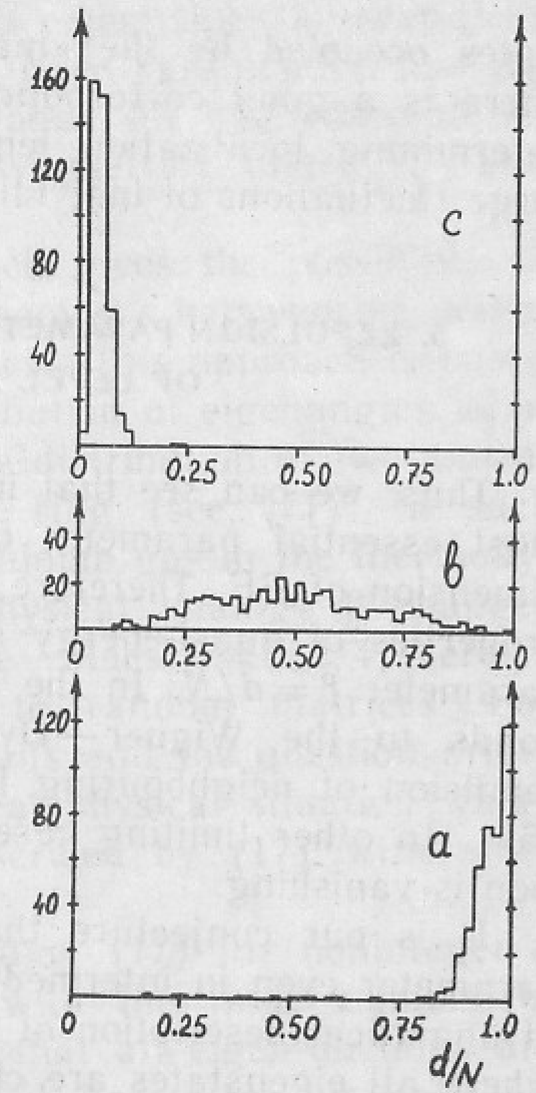

Fig. 2. Three examples of the distribution of localization length d_m for individual EF with the different values of k and fixed ($\mathscr{K}=5$):

a) $r=1$; $k\approx 317$; $\beta\approx 0.95$;

b) $r=15$; $k\approx 21.1$; $\beta\approx 0.50$;

c) $r=95$; $k\approx 3.3$; $\beta\approx 0.05$.

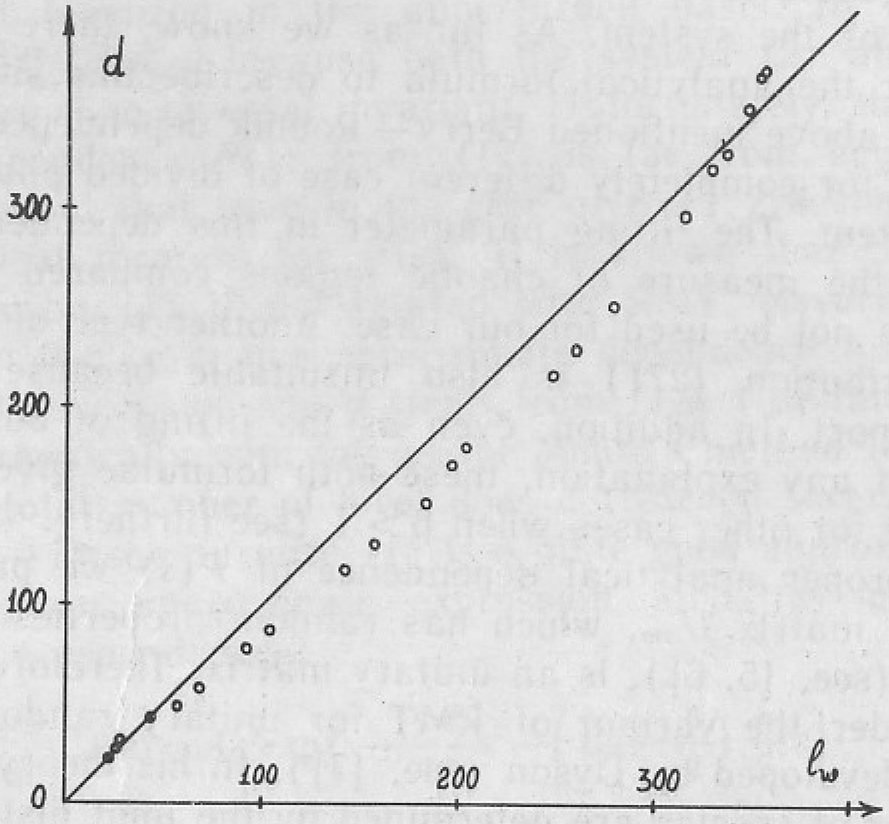

Fig. 3. Relation between «entropy» localization length (dimension d) and «probability» localization length l_w.

states occupied by the «main» part (95% in probability) of EF. There is a good correspondence between these two approaches in determining localization length, especially, if taking into account large fluctuations of individual EF.

5. REPULSION PARAMETER β AND ANALYTICAL DESCRIPTION OF LEVEL SPACING DISTRIBUTION

Thus, we can see that in the case of strong classical chaos the most essential parameter of quantum localization is the average dimension of EF. Therefore, it is naturally to expect that statistical properties of quasi-energy spectrum also effectively depend on the parameter $\beta = d/N$. In the limit $d \to N$ we have $\beta \to 1$ which corresponds to the Wigner—Dyson distribution for $P(s)$ with linear repulsion of neighbouring levels (it was confirmed numerically in [5]). In other limiting case of integrable systems, for $\beta \to 0$, repulsion is vanishing.

It is our conjecture that this parameter $\beta = d/N$ is repulsion parameter even in intermediate case of $0 < \beta < 1$. Then the problem of analytical description of distribution $P(s)$ arises for the situation where all eigenstates are chaotic but not full extended in available phase space of the system. As far as we know, there is no good candidate for the analytical formula to describe this situation. For example, the above mentioned Berry—Robnik dependence [15] have been derived for completely different case of divided phase space of classical system. The fitting parameter in this dependence has the meaning of the measure of chaotic regions compared with stable ones and can not be used for our case. Another type of distribution (Brody distribution [27]) is also unsuitable because it has no physical support. In addition, even as the fitting of our numerical data, without any explanation, these both formulae give completely wrong result for other cases when $\beta > 1$ (see further).

To get proper analytical dependence of $P(s)$ we pass to RMT [7—9]. Our matrix U_{nm}, which has random properties in the limit case $l_D \gg N$ (see, [5, 6]), is an unitary matrix. Therefore, it is natural to consider the variant of RMT for unitary random matrices, thoroughly developed by Dyson (see, [7]). In his theory all statistical properties of spectra are determined by the joint distribution

$$Q(\omega_1, \dots, \omega_N) = Q_0 \prod_{n \neq m} |e^{i\omega_n} - e^{i\omega_m}|^{\beta} d\omega_1 \dots d\omega_N \qquad (17)$$

of eigenangles ω_j which are related with eigenvalues $\lambda_j = \exp(i\omega_j)$ of random unitary matrix of size $N \gg 1$. Here parameter β has the meaning only for three cases: $\beta = 1$ stands for the ensemble of symmetric matrices, $\beta = 2$ — for nonsymmetric matrices, and $\beta = 4$ — for symplectic matrices.

Starting from (17), Dyson's approach gives the possibility to derive, in principle, distribution for the spacing s between the neighbouring values ω_j located on the unit circle. This approach is based on the correspondence between the distribution of eigenangles ω_j of random unitary matrices and steady-state distribution of two-dimensional Coulomb particles located on a ring (see, [7]). In such model β is an inverse temperature of Coulomb gas in the thermodynamic equilibrium. Therefore, in this physical analogy β changes from zero to infinity, but only for three values $\beta = 1; 2; 4$ there is rigorous mathematical correspondence to random matrices. For other values of β, this correspondence fails and the question arises whether it is possible to find out the real physical situation where statistical properties of spectra are described by (17) with other (noninteger) values of β.

Our main conjecture is that distribution (17) for noninteger β corresponds to the quantum systems with the finite number of quasi-energy states under the condition that all eigenfunctions are chaotic and localized in the unperturbed basis. In our case we expect that $0 \leqslant \beta \leqslant 1$ because both the system (2) and the model (5) — (8) are time-reversal invariant. Unfortunately, the question of deriving dependence $P(s)$ from (17) is far from being trivial. It should be noted that even in the case of $\beta = 1; 2; 4$ there is no correct analytical formula for $P(s)$. It is known that the commonly used expression (1) is not related with RMT. Nevertheless in the main region $0 < s \leqslant 2$, this approximate dependence turns out to be very close to exact one which stems from (17) (the latter have been obtained numerically with the use of Mehta's method (see, [7, 8])). So far the total number of level does not exceed several thousands, the Wigner—Dyson surmise (1) is a quite good approximation.

Here as an approximate expression for $P(s)$ in the region $0 \leqslant \beta \leqslant 2$ the dependence

$$P(s) = A s^{\beta} \exp\left\{ -\frac{\beta\pi^2}{16} s^2 - \left(C_0 - \frac{\beta}{2} \right) \frac{\pi}{2} s \right\} \qquad (18)$$

is suggested. Two normalized parameters A and C_0, in (18) are determined by usual relations:

$$\int_0^\infty P(s)\,ds=1\,; \quad \int_0^\infty sP(s)\,ds=1\,,$$

where $s=1$ is the mean distance between neighbouring levels. The dependence (18), written in the form which approximately takes into account the asymptotic expression of $P(s)$ for $s\to\infty$, have been obtained by Dyson [7]. On the other hand, it is quite close to (1) when $\beta=1;2$. In addition, for $\beta=0$ the dependence (18) is Poissonian one with the correct values of A and C_0. In Fig. 4 the expressions (18) and (1) together with the numerical data of RMT [7, 8], are shown. It is seen that the deviation of (18) does not exceed 5% for the most essential (from practical point of view) region $s\approx 1\div 2$. It means that the dependence (18) can be regarded as a good approximation of (1) if the total number N of levels does not exceed $N\approx 10^4$. Much better correspondence occurs for $\beta=2$ (see, Fig. 5). Thus, our formula (18) is expected to be close to exact (but unknown!) one, which stems from (17) with arbitrary values $0\leqslant\beta\leqslant 2$.

6. NUMERICAL DATA FOR INTERMEDIATE STATISTICS ($0<\beta<1$)

Now we come back to our question of intermediate statistics $P(s)$ for the model (5) – (8) in dependence on the localization of quantum chaos. Let us compare numerical data for $P(s)$ with the expression (18) where parameter β is determined by the localization length of chaotic localized EF through the expressions (10), (15). For this, the dimension d of EF of matrix U_{nm} and spacing distribution $P(s)$ for quasi-energies ω have been computed independently in a wide range of quantum parameter k. In all cases the classical parameter $\mathscr{K}$ was fixed ($\mathscr{K}=5$). To improve the statistics, the summing of $P(s)$ for four matrices U_{nm} of size $N=398$ have been performed, with slightly different values of k ($\Delta k\ll k$). Quasi-energies ω_j have been found from the eigenvalues $\lambda_j=\exp(i\omega_j)$ of matrix U_{nm}. To compute dimension d we use one of four matrices U_{nm} with the averaging over all its EF.

The typical examples of $P(s)$ for three values $k\approx 39{,}8$; 21.1; 9.1 (respectively for $r=8$; 15; 35) are given in Fig. 6. We can see good

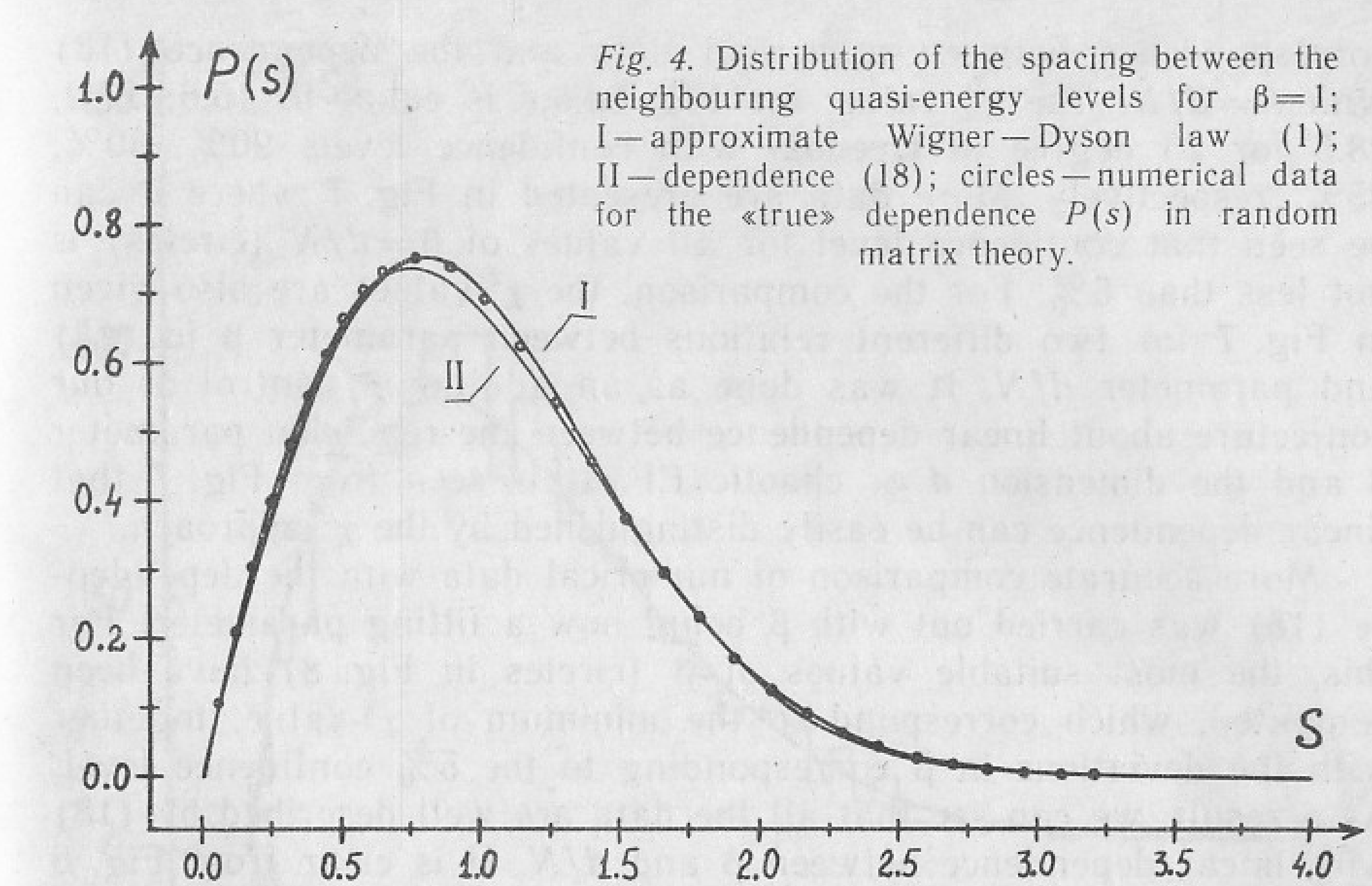

Fig. 4. Distribution of the spacing between the neighbouring quasi-energy levels for $\beta=1$; I — approximate Wigner — Dyson law (1); II — dependence (18); circles — numerical data for the «true» dependence $P(s)$ in random matrix theory.

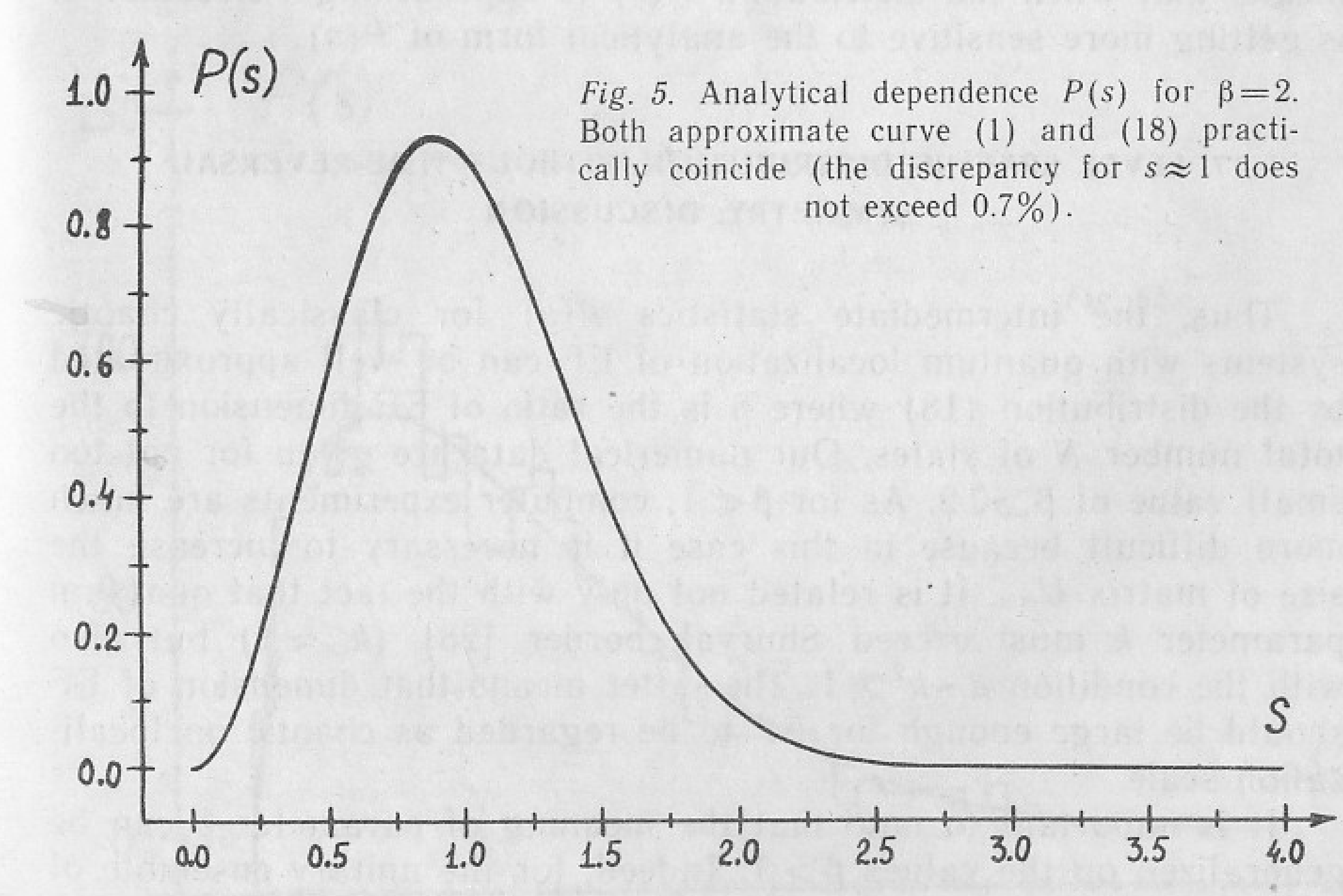

Fig. 5. Analytical dependence $P(s)$ for $\beta=2$. Both approximate curve (1) and (18) practically coincide (the discrepancy for $s\approx 1$ does not exceed 0.7%).

correspondence between numerical data and the dependence (18) with $\beta=d/N$. The χ^2_{23}-value, for Fig. 6*a*,*b*,*c* is equal to 15.6; 27.2; 28.5 for 23 degree of freedom with confidence levels 90%, 30%, 35%, respectively. More data are presented in Fig. 7 where it can be seen that confidence level for all values of $\beta=d/N$ (circles) is not less than 5%. For the comparison, the χ^2-values are also given in Fig. 7 for two different relations between parameter β in (18) and parameter d/N. It was done as an additional control of our conjecture about linear dependence between the repulsion parameter β and the dimension d of chaotic EF. It is seen from Fig. 7 that linear dependence can be easily distinguished by the χ^2-approach.

More accurate comparison of numerical data with the dependence (18) was carried out with β being now a fitting parameter. For this, the most suitable values of β (circles in Fig. 8) have been computed, which correspond to the minimum of χ^2-value, together with the deviations in β corresponding to the 5% confidence level. As a result, we can see that all the data are well described by (18) with linear dependence between β and d/N. It is clear from Fig. 8 that the spread in β is decreasing with the decrease of value β. It means that when the distribution $P(s)$ is approaching Poissonian it is getting more sensitive to the analytical form of $P(s)$.

7. LEVEL SPACING DISTRIBUTION WITHOUT TIME-REVERSAL SYMMETRY: DISCUSSION

Thus, the intermediate statistics $P(s)$ for classically chaotic systems with quantum localization of EF can be well approximated by the distribution (18) where β is the ratio of EF-dimension to the total number N of states. Our numerical data are given for not too small value of $\beta \geqslant 0.2$. As for $\beta \ll 1$, computer experiments are much more difficult because in this case it is necessary to increase the size of matrix U_{nm}. It is related not only with the fact that quantum parameter k must exceed Shuryak border [28] ($k_{cr} \approx 1$) but also with the condition $d \sim k^2 \gg 1$. The latter means that dimension of EF should be large enough for EF to be regarded as chaotic on localization scale.

It is important to note that the meaning of parameter β can be generalized on the values $\beta > 1$. Indeed, for the unitary ensemble of random matrices ($\beta=2$ in (1)) the maximal dimension of chaotic states is equal to $2N$. It is related to the fact that each EF has now

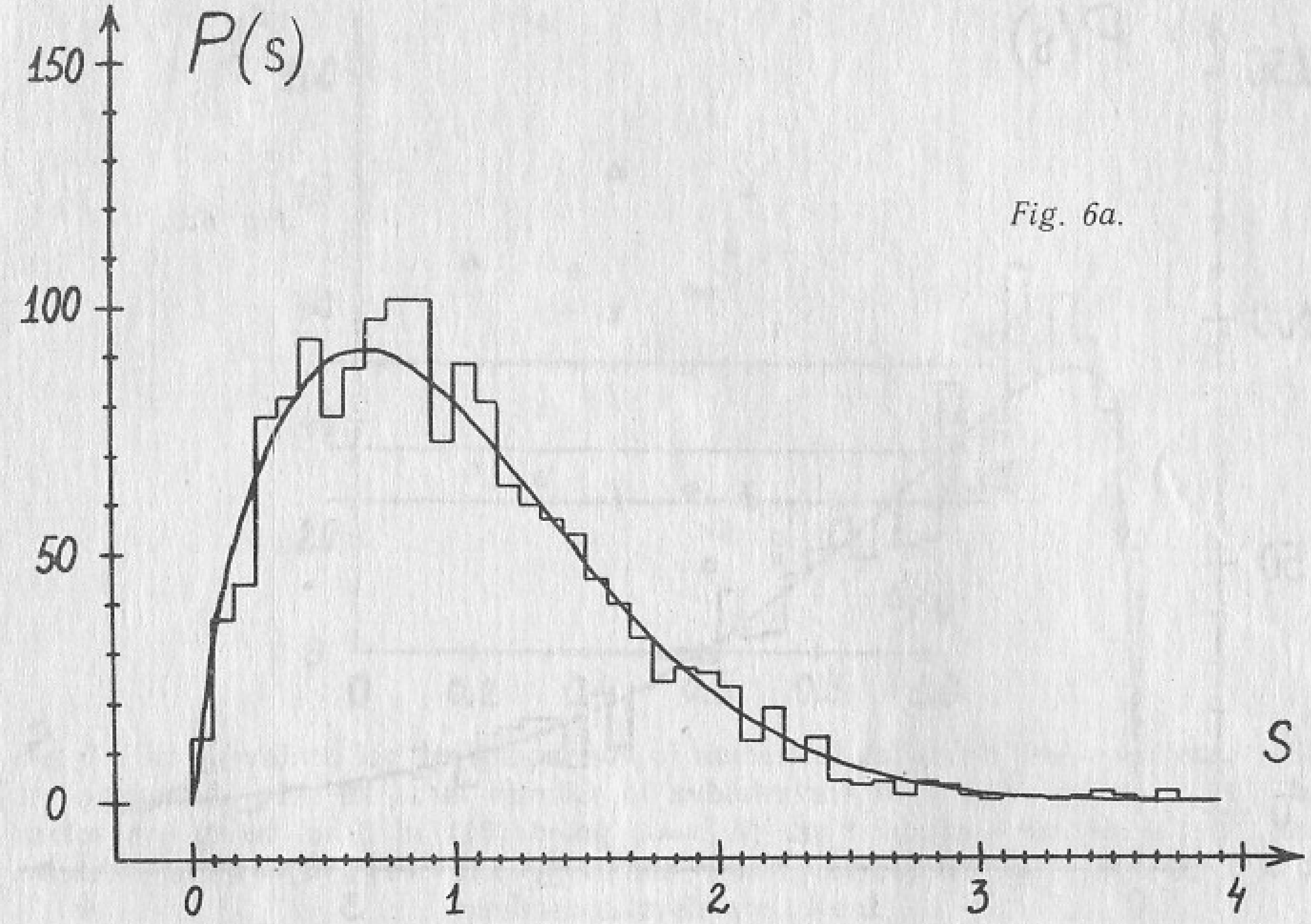

Fig. 6a.

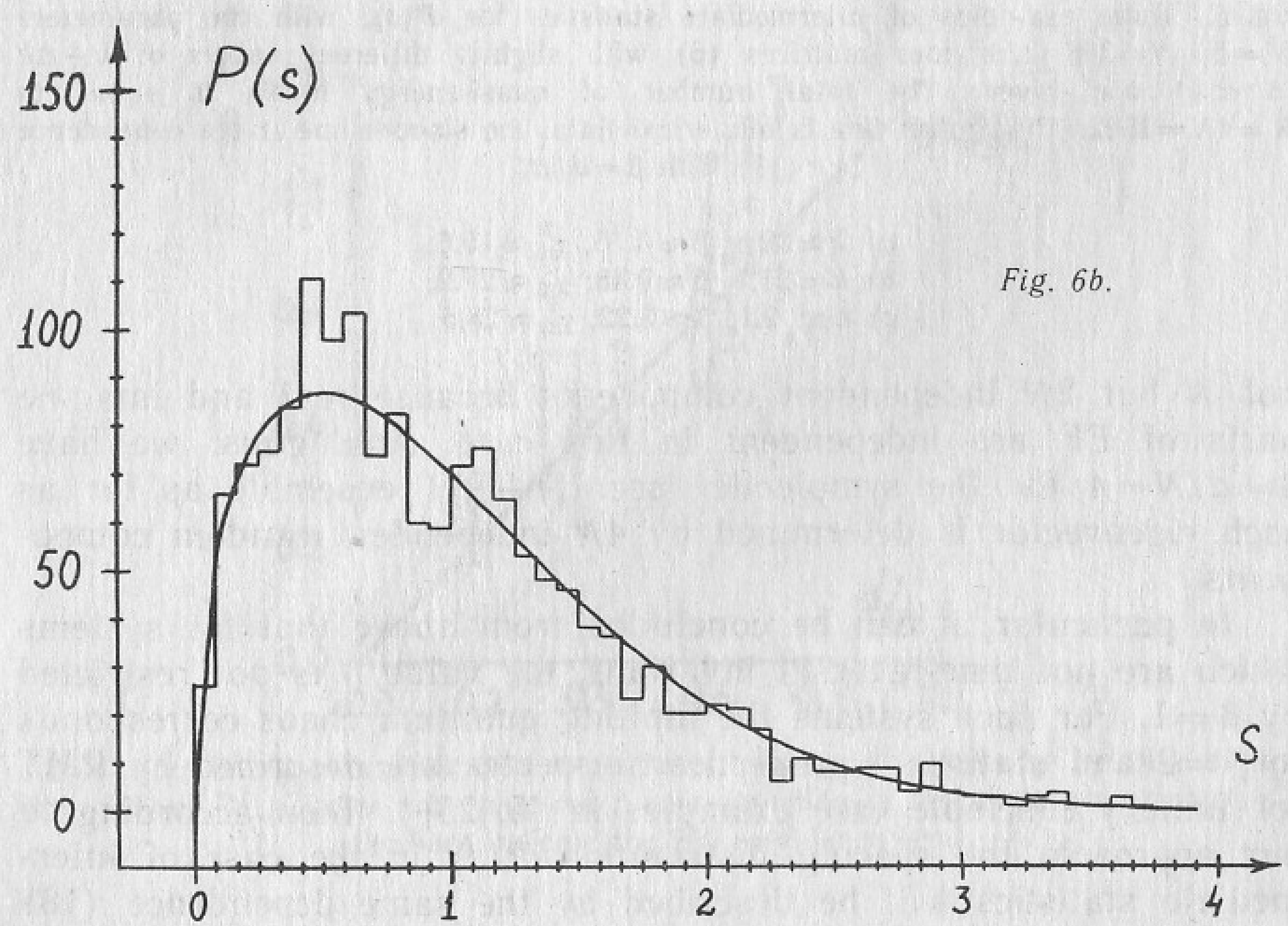

Fig. 6b.

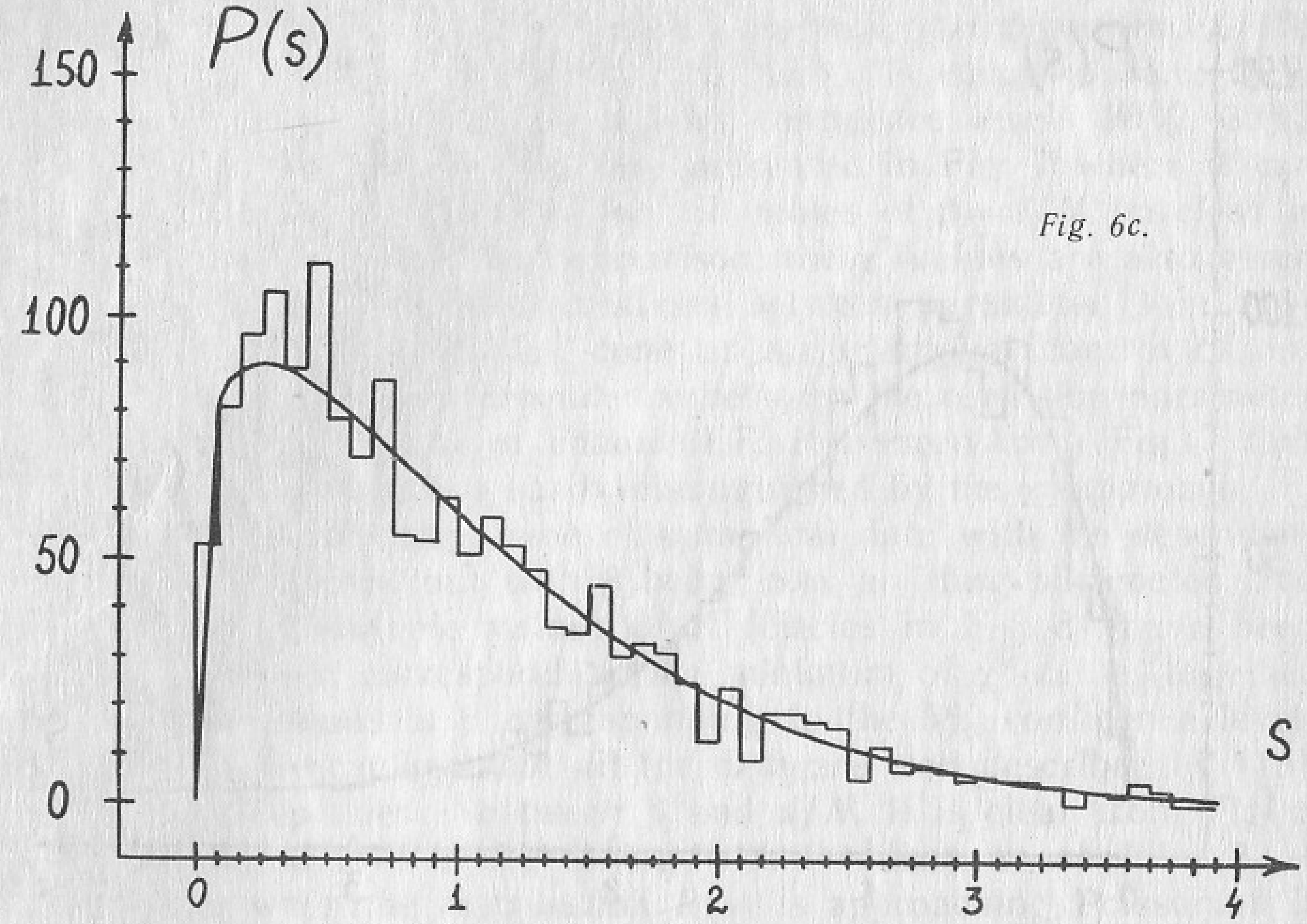

Fig. 6. Three examples of intermediate statistics for $P(s)$, with the parameters $\mathcal{K}=5$, $N=398$, and four matrices (6) with slightly different values of $k+\Delta k$ ($\Delta k \ll k$) are given. The total number of quasi-energy levels is equal to $M=4N=1592$. The broken line is numerical data, the smooth line is the dependence (18) with $\beta=d/N$.

a) $k \approx 39.8$; $\beta \approx 0.76$; $\chi^2_{23} \approx 15.6$;
b) $k \approx 21.1$; $\beta \approx 0.48$; $\chi^2_{23} \approx 27.2$;
c) $k \approx$ 9.1; $\beta \approx 0.22$; $\chi^2_{23} \approx 28.5$.

not N but $2N$ independent components because real and imagine parts of EF are independent in this case. Analogous, we have $\beta=d/N=4$ for the symplectic (see. [7—9]) ensemble as far as each eigenvector is determined by $4N$ independent random components.

In particular, it can be concluded from above that for systems which are not time-reversal invariant, the value β is not restricted by $\beta=1$. For such systems the limiting quantum chaos corresponds to $\beta=2$ and statistical properties of spectra are described by RMT for unitary ensemble (see examples in [5, 25]). Then according to our approach, the spacing distribution $P(s)$ in the case of intermediate statistics will be described by the same dependence (18) with $\beta=d/N$.

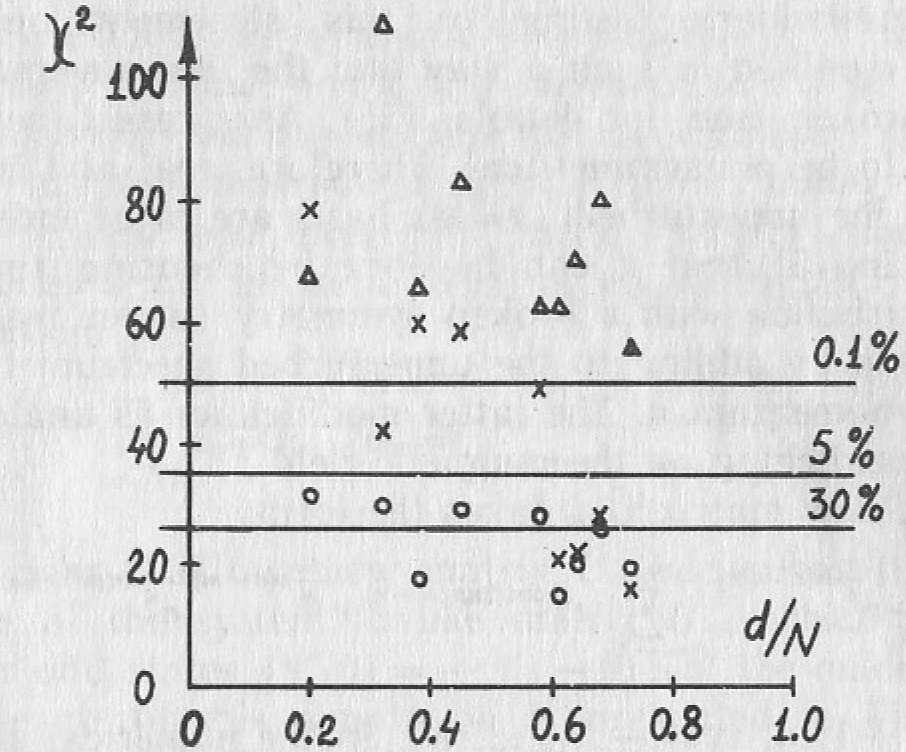

Fig. 7. The χ^2_{23} values for the comparison of numerical data with the dependence (18) are presented, with the total number of subintervals in s to be equal to 24. The circles are given for β in (18) being $\beta=d/N$; the triangles—for $\beta=(d/N)^2$; the crosses—for $\beta=(d/N)^{1/2}$. The χ^2_{23}-values which correspond to 0.1%, 5%, 30% confidence levels are given.

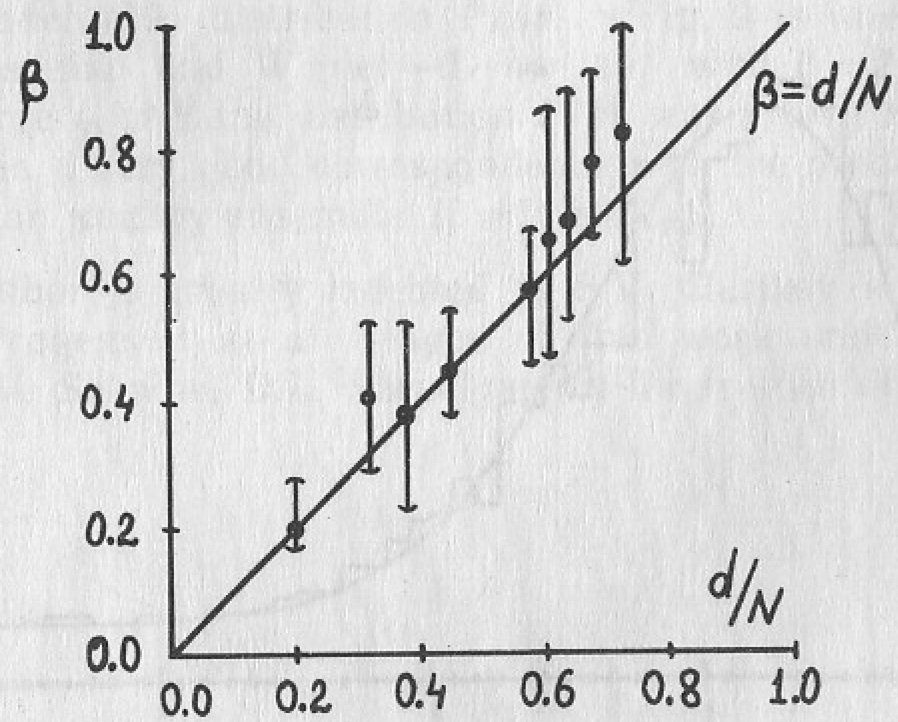

Fig. 8. The fitting parameter β in dependence (18) as a function of relative dimension d/N of EF. Circles are values of β corresponding to the minimum value of χ^2_{23}; the bars indicate the 5% confidence level.

For the preliminary testing of this statement our model (5) — (8) was modified in such a way that the time-reversal invariance is to be broken (see, for details, [5]). As a result, new matrix $\tilde{U}_{nm}$ turns out to be nonsymmetrical. Therefore, real and imaginary parts of EF in the unperturbed ($k=0$) basis are to be independent. It was shown in [5] that it can be done by choosing (instead of $\cos\theta$) the perturbation with a broken symmetry (under transformation $\theta\to-\theta$) and by adding to the unperturbed spectrum the linear dependence in momentum n. The latter modification is analogous, in essence, to the switching on the magnetic field.

As a result, new matrix $\tilde{U}_{nm}$ takes the form:

$$\tilde{U}_{nm}=\frac{1}{N}e^{\frac{1}{4}i\tau(n^2+\xi n)}\sum_{p=-N_1}^{N_1}e^{ik\cos(2\pi p/N+\eta)}e^{i\frac{2\pi}{N}p(n-m)}e^{\frac{1}{4}i\tau(m^2+\xi m)}, \qquad (19)$$

where $N=2N_1+1$; $n, m=-N_1, \dots, N_1$. In the numerical simulation the values of parameters are equal to $\tau=4\pi\cdot 16/N$; $N=199$;

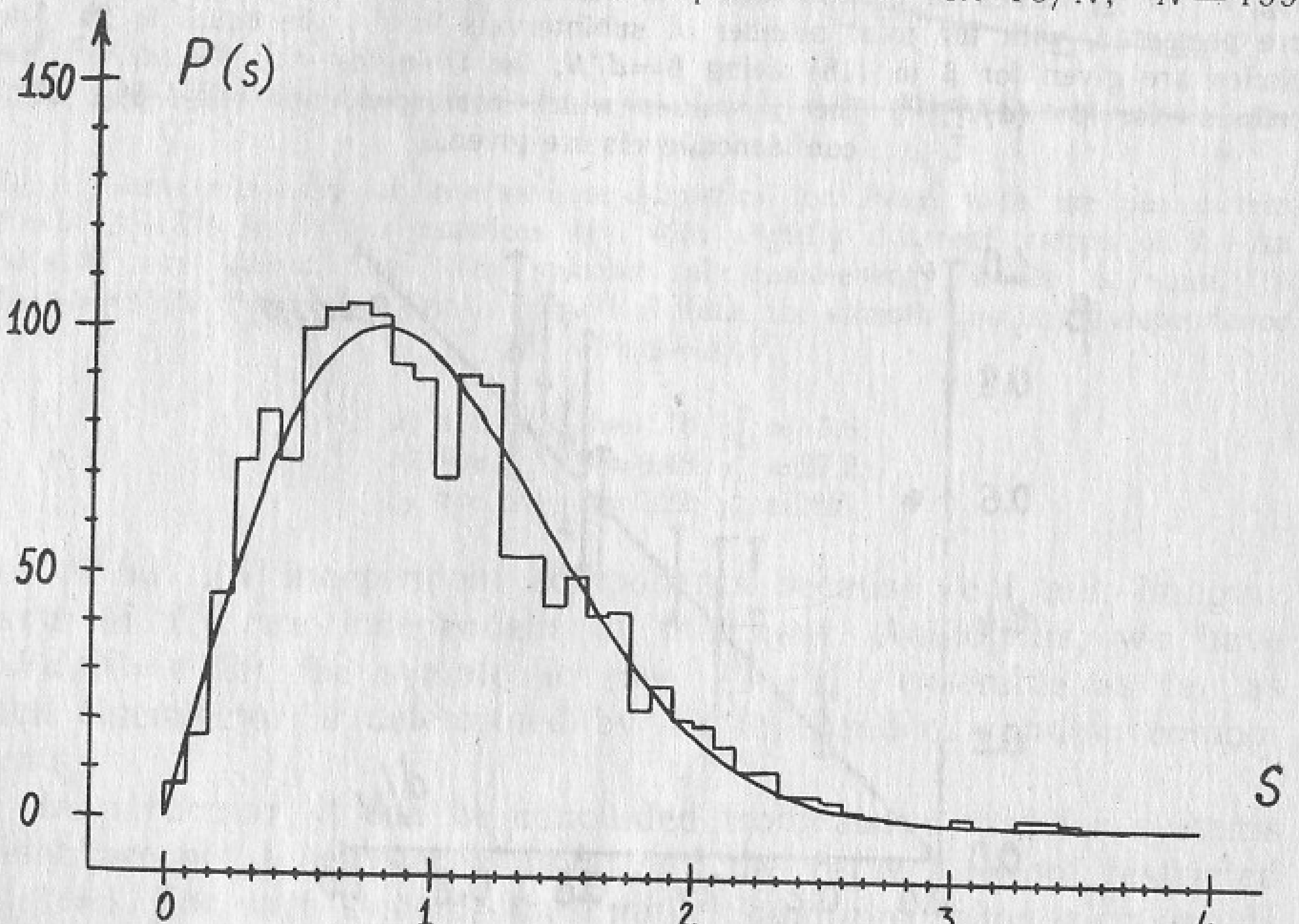

Рис. 9. The intermediate statistics of $P(s)$ for the model (19) which is not time-reverse invariant. The size of matrix $\tilde{U}_{nm}$ is equal to $N=199$, quantum parameter k for 8 matrices $\tilde{U}_{nm}$ changes in the interval $k=13.0\div13.9$ which corresponds, approximately to $d\approx N$ for the average dimension of EF. The smooth curve is analytical dependence (18) with $\beta=1$. χ^2-approach gives $\chi^2_{23}\approx31$ with 10% confidence level.

$\xi\approx1.88$; $\eta\approx0.81$. The strength of perturbation k was choosen in such a way that dimension d is to be equal to $d\approx N$. The quantity d was numerically found according to the formulae (10), (15) with the only exception that summing in (10) is running both over real parts of EF and over imaginary parts. Therefore, the total number of components in the sum (10) is equal to $2N$ with the usual normalization:

$$\sum_{n=1}^{2N}w_n=1; \qquad w_n=\begin{cases}(\mathrm{Re}\,\Phi_n)^2 & n=1,\dots,N\\ (\mathrm{Im}\,\Phi_n)^2 & n=N+1,\dots,2N\end{cases} \qquad (20)$$

It is clear that our new matrix $\tilde{U}_{nm}$ describes the evolution of any state of the system, unlike matrix U_{nm} which have been obtained for odd states ($\Psi(\theta)=-\Psi(-\theta)$) of the model (5) — (8).

The result of this simulation is presented in Fig. 9. Here the numerical data are the sum over the distributions $P(s)$ for 8 matrices $\tilde{U}_{nm}$ with slightly different values of k in the interval $13.0\leqslant k\leqslant13.9$. The matrix size is equal to $N=199$, therefore, the total number of quasi-energy levels is equal to $M=8\cdot N=1592$. It is seen from Fig. 9 that the correspondence between the numerical data and the dependence (18) is good. It should be pointed out that for the model (19) distribution $P(s)$ in Fig. 9 is intermediate between Poissonian and Wigner — Dyson (1) with $\beta=2$. In the limit case of large $l_D\gg N$ the distribution $P(s)$ was shown numerically in [5] to be in a very good correspondence with the prediction of RMT for Gaussian unitary ensemble ($\beta=2$ in (1)).

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REFERENCES

1. *Zaslavsky G.M., Filonenko N.N.* Zh. Eksp. Teor. Fis. 65 (1973) 643:
2. *Casati G., Valz-Gris F., Guarneri I.* Lett. Nuovo Cim., 28 (1980) 279.
3. *Berry M.V.* Ann. Phys. 131 (1981) 163.
4. *Bohigas O, Giannoni M.-J.* Lect. Notes in Physics, 209 (1984) 1.
5. *Izrailev F.M.* Phys. Rev. Lett., 56 (1986) 541; Preprint INP 84-63, Novosibirsk, 1984.
6. *Izrailev F.M.* Phys. Lett. A., 125 (1987) 250.
7. Statistical Theory of Spectra: Fluctuations, ed. C.E.Porter, Academic Press, 1965.
8. *Mehta M.L.* Random Matrices, Academic, 1967.
9. *Brody T.A., Flores J., French J.B., Mello P.A., Pandey A., Wong S.S.M.* Rev. Mod. Phys., 53 (1981) 385.
10. *Casati G., Chirikov B.V., Ford J., Izrailev F.M.* Lect. Notes in Physics, 93 (1979) 334.
11. *Chirikov B.V., Izrailev F.M., Shepelyansky D.L.* Sov. Sci. Rev., 2C (1981) 209.
12. *Izrailev F.M., Chirikov B.V., Shepelyansky D.L.* Preprint INP 86-166. Novosibirsk, 1986 (in Russian); to be published in Physica D.
13. *Feingold M., Fishman S., Grempel D.R., Prange R.E.* Phys. Rev. B31 (1985) 6852;
    Feingold M., Fishman S. Physica D25 (1987) 181.
14. *Frahm H., Mikeska H.J.* Z. Phys., B65 (1986) 249;
    Frahm H., Mikeska H.J. Phys. Rev. Lett., 60 (1988) 3.
15. *Robnik M., Berry M.V.* J. Phys., A17 (1984) 2413.
16. *Robnik M.* J. Phys., A20 (1987) L495.
17. *Bogomolny E.B.* Zh. Eksp. Teor. Fiz. (Pisma) 41 (1985) 55.
18. *Berry M.V., Tabor M.* Proc. Roy. Soc., A356 (1977) 375.
19. *Casati G., Chirikov B.V., Guarneri I.* Phys. Rev. Lett., 54 (1985) 1350.
20. *Casati G., Guarneri I., Izrailev F.M.* Phys. Lett. A124 (1987) 263.
21. *Chirikov B.V.* Phys. Rep., 52 (1979) 263.
22. *Shepelyansky D.L.* Phys. Rev. Lett., 56 (1986) 677.
23. *Izrailev F.M., Shepelyansky D.L.* Teor. Mat. Fiz., 43 (1980) 417.
24. *Chang S.-J., Shi K.-J.* Phys. Rev. Lett., 55 (1985) 269.
25. *Haake F., Kus M., Scharf R.* Z. Phys. B65 (1986) 381, Lect. Notes in Phys., 282 (1987) 3.
26. *Chirikov B.V., Shepelyansky D.L.* Radiofizika, 29 (1986) 1041.
27. *Brody T.A.* Lett. Nuovo Cim., 7 (1973) 482.
28. *Shuryak E.V.* Zh. Eksp. Teor. Fiz., 71 (1976) 2039.

F.M. Izrailev

Intermediate Statistics of Quasi-Energy Spectrum and Quantum Localization of Classical Chaos

Ф.М. Израйлев

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