ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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PREPRINT 88-168

НОВОСИБИРСК

QUANTUM RADIATION THEORY IN INHOMOGENEOUS EXTERNAL FIELDS

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A b s t r a c t

Some general expressions for radiation intensity (probability) in an inhomogeneous field, including the expansion in field gradients and end effects, have been derived on the basis of the quasi-classical operator approach. The results obtained are applied to the discussion on radiation from beam-beam collision in linear colliders.

1. Projects for linear electron and positron colliders are being actively developed /1-4/. What is of importance here is the radiation from beam-beam collision (the so-called 'beam-strahlung'). If at a particle energy of about 100 GeV, classical electrodynamics is still applicable, then for the so-called supercolliders ($\varepsilon \gtrsim 1$ TeV), quantum effects during the radiation are very significant. For relativistic particles in an external field (including the inhomogeneous one, which is required to treat the problem), the quantum radiation theory has been developed by two authors of Refs. /5/ to /7/. In recent years, relevant theoretical analyses have been made of the radiation from beam-beam collisions /8-18/ and of the radiation or pair-creation of high-energy particles or photons in aligned single crystals /19-21/. The present paper deals with a general theoretical analysis of radiation in an inhomogeneous field, and the results are applied to the radiation problem in linear colliders.

The picture of particle radiation in an external field depends on the parameter ($\hbar = c = 1$),

$$\chi = \frac{e}{m^3}\sqrt{|F^{\mu\nu}p_\nu|^2} = \frac{w\gamma^2}{m}, \qquad (1)$$

where $p^\nu(\varepsilon, \underline{p} = \varepsilon \underline{v})$ is the four-momentum of a particle, $F^{\mu\nu}$ is the external electromagnetic field, $\gamma = \varepsilon/m$, and $w$ is the acceleration. Introducing $\underline{E}_\perp = \underline{E} - \underline{v}(\underline{v}\underline{E})$, we have $\chi = \gamma|\underline{F}|/H_0$, where $\underline{F} = \underline{E}_\perp + (\underline{v}\times\underline{H})$, $\underline{E}$ and $\underline{H}$ are electric and magnetic fields in the lab. system, and $H_0 = 4.41\times10^{13}\,Oe$. For $\chi \gtrsim 1$, the radiation is essentially quantum. The fast growth of $\chi$ in supercolliders is due both to the growth of the energy itself and, to a greater extent, to the increase in field intensity of the colliding beam. The latter is associated with a considerable decrease in the beam dimensions and with an increase in its density, which are necessary in order to reach an acceptable luminosity.

The quasi-classical radiation theory, developed in Refs./5/ to /7/, describes the radiation of particles with any spin and is suitable for an arbitrary external field. In this theory, the radiation probability is of the form

$$dW = \frac{\alpha}{(2\pi)^2}\frac{d^3k}{\omega}|M|^2 , \tag{2}$$

$$M = \int dt\, R(t) \exp(-ik'x(t)),$$

where $\alpha = e^2 = 1/137$, $k' = (\varepsilon/\varepsilon')k$ , $k(\omega, \underline{k})$ is the four-momentum of a photon, $\varepsilon' = \varepsilon - \omega$ , $x(t) = (t, \underline{r}(t))$, $t$ is the time, and $\underline{r}(t)$ is the coordinate on the classical particle trajectory. For a spinor particle we have, with relativistic accuracy,

$$R = \varphi_f^+ (A + i\underline{\sigma}\underline{B})\varphi_i ,$$

$$A = \frac{1}{2}\left(1 + \frac{\varepsilon}{\varepsilon'}\right)\underline{e}\underline{v} , \quad \underline{B} = \frac{\omega}{2\varepsilon'}(\underline{e}\times\underline{b}) , \tag{3}$$

$$\underline{b} = \underline{n} - \underline{v} + \underline{n}/\gamma$$

where the velocity $\underline{v} = \underline{v}(t)$ is taken on the particle trajectory, $\varphi_{f(i)}$ are two-component spinors describing the electron polarization, $\underline{e}$ is the photon polarization vector, and $\underline{n} = \underline{k}/\omega$.

In Ref./5/, the theory has been used to describe the radiation in a constant field, whilst in Ref./6/ it has been applied to the radiation problem in a Coulomb (i.e. considerably inhomogeneous) field. Then, in terms of this theory, the radiation has been considered during the quasi-periodic motion. In Ref./22/ the problem has been solved in dipole approximation, whilst in Ref./23/ the general problem, which includes in particular the radiation in the field of a plane wave /24/, has been solved. The type of field inhomogeneity has considerable influence on the characteristics of the radiation, including spectral, angular, and polarization properties. Recently, the quasi-classical theory of radiation and pair creation has served as the foundation for the creation of a specific crystal electrodynamics /19-21/. It has turned out that the action mechanism of these processes depends on the external parameter, the angle of incidence $\vartheta_0$ with respect to the axes or the planes of a crystal. At very small angles of incidence, the constant-field approximation may be applied, whilst at relatively large angles the derived general expressions reduce to formulae of the theory of coherent radiation and pair-creation.

The quasi-classical nature of particle motion in colliders is very reliably provided. For example, for round beams the estimation of the transverse-motion phase yields.

$$\varphi_\perp \sim p_\perp \sigma_\perp \sim \frac{\alpha N}{\sqrt{1+D}} , \tag{4}$$

where $N$ is the number of particles in the beam, $D$ is the disruption parameter characterizing a change in the beam shape, and $\sigma_\perp$ ($\sigma_z$) is the transverse (longitudinal) beam size

$$D = \frac{N r_e \sigma_z}{\gamma \sigma_\perp^2} , \tag{5}$$

where $r_e = \alpha/m$ is the classical electron radius.

2. Expressions (2) and (3) may be represented in a form that is convenient for the calculations, where all cancellations of the dominant terms have already been made. For unpolarized initial particles, the spectral density of the radiation probability, summed over the polarization of the final particles and integrated over the angles of photon emission, is of the form [see formulae (2.3), (2.4), and (4.2) in Ref./23/ *)]

$$\frac{dw}{d\omega} = \frac{i\alpha}{8\pi} \iint \frac{dt\,d\tau}{\tau - i0} \left[ \frac{4}{\gamma^2} + \beta \left( \mathbf{v}(t_1) - \mathbf{v}(t_2) \right)^2 \right] \times$$

$$\times \exp\left\{ -\frac{i\omega\varepsilon\tau}{2\varepsilon'} \left[ \frac{1}{\gamma^2} + \frac{1}{\tau} \int_{t_1}^{t_2} \mathbf{\Delta}^2(t')\,dt' \right] \right\},$$

$$\mathbf{\Delta}(t') = \mathbf{v}(t') - \frac{1}{\tau} \int_{t_1}^{t_2} \mathbf{v}(t')\,dt', \quad \beta = \frac{\varepsilon}{\varepsilon'} + \frac{\varepsilon'}{\varepsilon}, \qquad (6)$$

where $t_{2,1} = t \pm \tau/2$. It is evident that expression (6) does not change after the substitution $\mathbf{v}(t) \to \mathbf{v}(t) - \mathbf{v}_0$, where $\mathbf{v}_0$ is the time-independent vector. Putting into formula (6) the velocity of a particle moving, generally speaking, in an arbitrary external field, and calculating the corresponding quadratures, we find the desired probability of the process.

If the field varies slightly along the length of the photon formation, the vector $\mathbf{\Delta}(t')$ can be expanded in powers of $t' - t$.

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*) In these formulae an expression is also given for the probability with the polarization characteristics of radiation taken into account.



with the required number of the expansion terms. The basic terms of this expansion, which incorporates the particle acceleration, give the constant-field limit, and the remaining terms are the corrections to this approximation. When calculating the corrections, we also have to expand the exponential factor that contains terms with the derivatives of acceleration. The above procedure has been used in Ref./19/ [see formula (6)] to obtain the expression for the spectral intensity of radiation, which incorporates the field inhomogeneity along the particle formation length*):

$$\frac{dI}{d\omega} = \frac{\alpha m^2 \omega}{\sqrt{3}\pi\varepsilon^2} \left\{ \beta K_{2/3}(\lambda) - \int_\lambda^\infty dy\, K_{1/3}(y) - \right.$$

$$- \frac{1}{3\mathbf{b}^4} \left[ (\mathbf{b}, (\mathbf{V}\nabla)^2 \mathbf{b}) \beta \left[ K_{2/3}(\lambda) - \frac{2}{3\lambda} K_{1/3}(\lambda) \right] - \right.$$

$$- \frac{1}{10} \left[ ((\mathbf{V}\nabla)\mathbf{b})^2 + 3(\mathbf{b}, (\mathbf{V}\nabla)^2 \mathbf{b}) \right] \left[ \lambda K_{1/3}(\lambda) - \right.$$

$$- \frac{4}{3} K_{2/3}(\lambda) + \beta \left( 4K_{2/3}(\lambda) - (\lambda + \qquad (7)$$

$$\left. \left. \left. + 16/9\lambda) K_{1/3}(\lambda) \right) \right] \right] \right\},$$

where $\mathbf{V}$ is a difference between the velocities of the particle and the opposite beam,

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*) It is worth mentioning that as far back as 1981 (see Ref./23/), similar corrections were calculated for the case of harmonic transverse motion.

$$\lambda = 2m^2\omega/(3\varepsilon\varepsilon'|\underline{b}|)\,, \quad \underline{b} = e\underline{F}/m\,,$$

and $K_\nu$ are the MacDonald functions. The first two terms in formula (7) present the magnetic bremsstrahlung limit $dI_0/d\omega$. Taking advantage of the asymptotics of $K_\nu$ functions, we obtain the following estimate of the relative contribution from the correction terms to formula (7):

$$\frac{dI - dI_0}{dI_0} \sim c_1 \frac{\ell_c^2}{\sigma_z^2} + c_2 \frac{\ell_c^2 \mathcal{V}_\perp^2}{\sigma_\perp^2}\,, \tag{8}$$

$$\ell_c = \frac{1}{|\underline{b}|}(1 + \mathcal{X}/u)^{1/3} = \frac{\ell_0}{\mathcal{X}}(1 + \mathcal{X}/u)^{1/3}\,, \quad u = \omega/\varepsilon'\,,$$

where $\ell_0 = \gamma \lambda_c$ ; $\lambda_c = 1/m$ is the Compton wavelength of the electron. Here $\ell_c$ is the length of the formation of a photon in a constant external field (see, for example, Ref./25/) and $\ell_0$ is the characteristic length of the bremsstrahlung. The estimation (8) has a simple physical meaning if it is borne in mind that formula (7) was derived assuming that the field varies slightly along the photon formation time. Estimation of the relative contribution of the terms with transverse inhomogeneity in formulae (7) gives

$$\frac{\mathcal{V}_\perp^2 \sigma_z^2}{\sigma_\perp^2} \sim \frac{\varphi_\perp^2 \sigma_z^2}{\varepsilon^2 \sigma_\perp^4} \sim \frac{\mathcal{D}^2}{1+\mathcal{D}}\,, \tag{9}$$

where we have made use of formulae (4) and (5). If the disruption parameter $\mathcal{D}$ is small $(\mathcal{D} \ll 1)$, the transverse inhomogeneity of the field can be neglected. If the inhomogeneity should be taken into account, we have also to average over all the directions of $\underline{V}_\perp$ , with a proper particle distribution function with respect to $\underline{V}_\perp$ .



Corrections to the constant-field limit that contain field gradients have been studied with respect to the problem of radiation of ultrarelativistic particles in crystals /19,20/. The situation considered there corresponds, in the terms used here, to the case $\mathcal{D} \gg 1$. In crystals, these corrections determine a behaviour of the orientation dependence curve, but their region of applicability appears to be narrow enough, owing to a rapid increase of the radiation formation length in a region of decreasing field. Results obtained in Refs./19/ and /20/ are very instructive for the beamstrahlung problem.

For a total radiation intensity integrated over all the frequencies, at $\mathcal{X} \ll 1$ there appears an extra factor $\mathcal{X}$ in the right-hand side of formula (8). This is connected with the fact that in the classical region $(\mathcal{X} \ll 1)$ the radiation intensity in an arbitrary external field has a local form $[I = (2/3)\alpha m^2 \mathcal{X}^2]$,which is the same as that of $I_0$ (intensity in magnetic bremsstrahlung limit) in formula (7). For this reason, the correction terms make an insignificant contribution to the total radiation intensity at $\mathcal{X} \ll 1$ , and this may be ignored. Bearing in mind that $I_0 \propto \mathcal{X}^{2/3}$ for $\mathcal{X} \gg 1$ , we see from formula (8) that the absolute value of the corrections in formula (7) is largest at local $\mathcal{X}(t) \sim 1$. Note that for formula (7) to be aplicable, the condition $\ell_0 \ll \sigma_z$ should be satisfied. In the opposite case, $\sigma_z \lesssim \ell_0$ , the constant-field approximation may be used only under the condition $\mathcal{X}(t) \gg (\ell_0/\sigma_z)^{3/2} \gtrsim 1$, and to calculate the corrections we ought to return to formula (6).

3. The boundary (end) effects of photon emission (assuming that the length of field inhomogeneity $\sigma$ at this boundary is considerably smaller than the characteristic bremsstrahlung length $\ell_0$) are calculated in Appendix A. In this case ($\sigma \ll \gamma \lambda\!\!\!^{-}_c$), the spectral distribution of the energy losses is of the form [see formulae (A.17) and (A.18)]

$$\frac{1}{\varepsilon}\frac{d\varepsilon_L}{dx} = \frac{1}{\varepsilon}\frac{d\varepsilon_F}{dx} + \frac{1}{\varepsilon}\frac{d\varepsilon_b}{dx}, \quad x = \omega/\varepsilon,$$

where

$$\frac{1}{\varepsilon}\frac{d\varepsilon_F}{dx} = \int_{-\infty}^{\infty}\frac{1}{m^2}\frac{dI_0}{dx}\frac{dt}{\ell_0} \qquad (10)$$

and

$$\frac{1}{\varepsilon}\frac{d\varepsilon_b}{dx} = \frac{\alpha}{\pi}\Big\{-2(1-x)+\left(1+(1-x)^2\right)\Big[-C + \int d\eta(t)\sin\eta(t)\ln\Gamma(t) + \int\frac{dt}{\ell_\omega}\int_0^\infty\frac{d\tau}{\tau}\times \left(R(t,\tau)-R_i(t,\tau)-R_0(t,\tau)\right)\Big]\Big\}. \qquad (11)$$

Here

$$\eta(t) = \frac{1}{\ell_\omega}\int_{-\infty}^{t} \underline{\psi}^2(t')dt', \quad \underline{\psi}(t) = \frac{e}{m}\int_{-\infty}^{t} \underline{F}(t')dt',$$



$$\underline{\Phi}(t) = \int_{-\infty}^{t} \underline{\psi}(t')dt', \quad \ell_\omega = 2\varepsilon'\ell_0/\omega = 2\ell_0(1-x)/x,$$

$$\Gamma(t) = \ell_\omega^2\,\eta(t)/\underline{\Phi}^2(t), \quad C = 0.577\,216..$$

The intensity $dI_0$ is determined by the first two terms in formula (7), describing the constant field limit,

$$R(t,\tau) = \left[\underline{\psi}(t+\tau/2)-\underline{\psi}(t-\tau/2)\right]^2 \sin\Big\{\eta(t+\tau/2)-\eta(t-\tau/2) - \frac{1}{\tau\ell_\omega}\left[\underline{\Phi}(t+\tau/2)-\underline{\Phi}(t-\tau/2)\right]^2\Big\},$$

$$R_i(t,\tau) = \psi^2(t+\tau/2)\sin\left[\eta(t+\tau/2) - \frac{1}{\tau\ell_\omega}\underline{\Phi}^2(t+\tau/2)\right], \qquad (12)$$

$$R_0(t,\tau) = \dot{\psi}^2(t)\tau^2\sin\left[\dot{\psi}^2(t)\tau^3/12\ell_\omega\right].$$

If we make allowance for the fact that the main contribution to the integral (11) comes from the region $\eta(t)\approx 1$, and that here

$$\Gamma(t) = \frac{\ell_\omega\int_{-\infty}^{t}\psi^2(t')dt'}{\left(\int_{-\infty}^{t}\psi(t')dt'\right)^2} \simeq \frac{\ell_\omega\int_{t_0}^{t_0+t_b}\psi^2(t')dt'}{\left(\int_{t_0}^{t_0+t_b}\psi(t')dt'\right)^2} \simeq \frac{\ell_\omega}{t_b} \qquad (13)$$

where $t_0$ and $t_b$ are defined from the relations

$$\psi(t_0) \simeq 1 \,, \quad \tau(t_0 + t_b) \simeq 1 \,, \tag{14}$$

we obtain, with logarithmic accuracy, the following expression for the probability of end-photon emission:

$$dW_b = \frac{\alpha}{\pi}\frac{d\omega}{\omega}\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\ln\frac{l_\omega}{t_b} \,. \tag{15}$$

The probability (15) can be readily derived if we take into account that the angular distribution of collinear photons*) at large $\vartheta \gg 1/\gamma$ is as follows (see, for example, Ref./7/):

$$dW_b = \frac{\alpha}{\pi}\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\frac{d\omega}{\omega}\frac{d\vartheta^2}{\vartheta^2} \,. \tag{16}$$

It is worth noting that the upper boundary of $\vartheta^2$ to which formula (6) can be applied is established from a relation between the time of collinear photon formation inside the region occupied by the field, and its emission angle (see, for example, Ref./7/):

$$t_b \simeq \frac{l_\omega}{1+\gamma^2\vartheta_m^2} \,, \quad \ln(\gamma^2\vartheta_m^2) \simeq \ln\frac{l_\omega}{t_b} \tag{17}$$

Using formulae (14) and (15), the contribution of collinear photons to the energy losses can be estimated for any particular potential of the field. So, for example, for the case of a step-like constant field, we have $t_b \simeq l_c = l_\omega (u/\varkappa)^{2/3}$, and in the

*) The direction of the photon emission coincides, on the average, with that of the initial or final velocity. In quantum electrodynamics, the emission from the ends of the lines of fast charged particles is well known. It is described, with logarithmic accuracy, in a quasi-real electron approximation (see Ref./26/, p.375).



case of an exponential field reduction such as $\exp(-|t|/\sigma)$, we obtain $t_b \simeq \sigma$.

Despite the seeming awkwardness of formulae (11) and (12), which describe the radiation of collinear photons with power accuracy, the calculations using these formulae are fairly simple. This is because the characteristic scale of the inhomogeneity really enters only into $\Gamma(t)$, whilst the remaining terms are scale-invariant and depend only on the asymptotic form of the potential at its boundary. Appendix B illustrates the calculation according to formulae (11) and (12) with the field intensity of the form $F = F_0 (t/\sigma)^n, (n \geq 0)$.

For this case, the contribution of the initial collinear photon is as follows:

$$\frac{d\varepsilon_i}{d\omega} = \frac{\alpha}{\pi}\left\{\left[\frac{1}{2}\mathcal{D}_n + \ln(n+2) + \frac{1}{2n+3}\left[\ln X + n\ln\frac{l_0}{\sigma} + (n+1)\left(\ln\frac{2\varepsilon'}{\omega} - C\right) - \frac{1}{2}\ln(2n+3) - 2(n+2)\ln(n+1) - 3\right]\right]\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right) - \frac{\varepsilon'}{\varepsilon}\right\} \tag{18}$$

where

$$\mathcal{D}_n = \int_0^\infty \frac{d\eta}{\eta}\left[\frac{(z^{n+1}-1)^2}{z^{2n+3}-1-\frac{a_n}{\eta}(z^{n+2}-1)^2} - \frac{b_n}{\eta}\right], \tag{19}$$

and

$z=1+\eta$, $a_n=(2n+3)/(n+2)^2$ and $b_n=12/(2n+3)$.

If the potential is symmetric, the same contribution will come from the final collinear photon, and the result obtained should be multiplied by a factor of 2. There is no difficulty in calculating the contribution to the total energy losses; to do this, it suffices to take the integrals

$$\int_0^{\varepsilon} d\omega\,\frac{\varepsilon'}{\varepsilon}=\frac{1}{2}\varepsilon,\quad \int_0^{\varepsilon} d\omega\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)=\frac{4}{3}\varepsilon,\quad \int_0^{\varepsilon} d\omega\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\ln\frac{\varepsilon'}{\omega}=\frac{1}{2}\varepsilon .$$

For $n=0$, we get for the contribution of the initial photon $(\mathcal{D}_0=0)$:

$$\frac{d\varepsilon_i}{d\omega}=\frac{\alpha}{2\pi}\left\{\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\left(\ln\chi^{2/3}+\frac{8}{3}\ln 2-\frac{2}{3}C-\frac{1}{3}\ln 3-2+\frac{2}{3}\ln\frac{\varepsilon'}{\omega}\right)-2\frac{\varepsilon'}{\varepsilon}\right\}. \qquad (20)$$

For total energy losses, in the symmetric case we have the following contribution of collinear photons $(n=0)$:

$$\frac{\varepsilon_b}{\varepsilon}=\frac{8\alpha}{9\pi}\left(\ln\chi-2.104\right) \qquad (21)$$

The radiation spectrum (20) and energy losses (21) are just the same as those in Ref./9/, the authors of which have used formula (6) for this case ( losses in a step-like field). This result was first obtained in Ref./16/, with logarithmic accuracy.



For a field decreasing at infinity as $F_0(\sigma/t)^n\ (n\geq 2)$, we have correspondingly (see Appendix C)*)

$$\frac{d\varepsilon_i}{d\omega}=\frac{\alpha}{\pi}\left\{\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\left[\frac{1}{2}\mathcal{E}_n+\ln(n-2)+\frac{1}{2n-3}\left[n\ln\frac{l_0}{\sigma}-\ln\chi+(n-1)\left(\ln\frac{2\varepsilon'}{\omega}-C\right)-(n-2)\ln(2n-3)+\ln(n-1)\right]\right]-\frac{\varepsilon'}{\varepsilon}\right\}, \qquad (22)$$

where

$$\mathcal{E}_n=\int_0^{\infty}\frac{d\eta}{\eta}\left[\frac{(z^{n-1}-1)^2}{\left(z^{2n-2}-z-\frac{c_n}{2}(z^{n-1}-z)\right)^2}-\frac{1}{1+c_n/2}-\frac{d_n}{\eta}\right];$$

and

$z=1+\eta$, $c_n=(2n-3)/(n-2)^2$ and $d_n=12/(2n-3)$.

For an exponentially decreasing field of the form $F_0\exp(-|t|/\sigma)$, the contribution of collinear photons to the radiation spectrum is as follows:

$$\frac{d\varepsilon_b}{d\omega}=\frac{d\varepsilon_i}{d\omega}+\frac{d\varepsilon_f}{d\omega}=2\frac{d\varepsilon_i}{d\omega}=\frac{\alpha}{\pi}\left\{\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\left(\mathcal{E}_{exp}-C+\ln\frac{l_\omega}{2\sigma}\right)-\frac{2\varepsilon'}{\varepsilon}\right\}, \qquad (23)$$

where

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*) In this Appendix the limiting transition $n\rightarrow 2$ is considered as well.

$$
\mathcal{E}_{exp} = \int_0^\infty d\eta \left( \frac{1}{\eta \,\mathrm{cth}\,\eta - 1} - \frac{1}{1+\eta} - \frac{3}{\eta^2} \right) = -1.137..
$$

Note that the result (23) can be obtained by means of the limiting transition $n \to \infty$ from the preceding formula (22) after the substitution $\sigma \to n\sigma$. Then $\lim\limits_{n\to\infty} \mathcal{E}_n = \mathcal{E}_{exp}$. The same may be applied to formula (18), but in this case $\mathcal{E}_{exp} = \lim\limits_{n\to\infty} (\mathcal{D}_n - \ln \frac{n}{2})$.

Lastly, let us consider an important case for colliders, when $F = F_0 \exp(-z^2/2\sigma_z^2) = F_0 \exp(-2t^2/\sigma_z^2)$. Taking into account that $\zeta(t) \approx 1$ at $t/\sigma_z \approx \sqrt{\ln au/2}$, where $au \sim (\sigma_z/\ell_c)^3$, and that for such $t$ the asymptotic form of the field is determined by the expression $F \approx \tilde{F}_0 \exp(-2(\ln ua)^{1/2} \tilde{t}/\sigma_z)$ within the terms $\sim 1/\ln au$, it is possible to use formula (23) if we put $\sigma = \sigma_z/(2(\ln au)^{1/2})$. The direct calculation by formula (11), through integration by the Laplace method, yields the same result:

$$
\frac{d\varepsilon_b}{d\omega} = \frac{\alpha}{\pi} \left\{ \left(1 + \frac{\varepsilon'^2}{\varepsilon^2}\right) \left( \ln \frac{2\varepsilon' (\ln(au))^{1/2} \ell_0}{\omega \sigma_z} - 1.714 \right) - 2\frac{\varepsilon'}{\varepsilon} \right\}, \tag{24}
$$

where

$$
a = \frac{\chi^2 \sigma_z^3}{32 \ell_0^3} = \frac{1}{4} \left( \frac{\chi^{2/3} \sigma_z}{2 \lambda_c \gamma} \right)^3 . \tag{25}
$$



Having integrated over $\omega$, in (24) we obtain for the contribution of collinear photons to the energy losses in beam-beam collisions,

$$
\frac{\varepsilon_b}{\varepsilon} = \frac{4\alpha}{3\pi} \left[ \ln \frac{2\ell_0 \sqrt{\ln a}}{\sigma_z} - 2.089 + O\left(\frac{1}{\ln a}\right) \right]. \tag{26}
$$

This equation may be obtained also from energy losses in the exponentially decreasing field

$$
\frac{\varepsilon_b}{\varepsilon} = \frac{4\alpha}{3\pi} \left[ \ln \frac{\ell_0}{\sigma} - 2.089 \right], \tag{27}
$$

using the substitution mentioned above. If for our estimation we take the parameters of a supercollider ( $\sigma_z = 4\times10^{-5}$ cm, $\varepsilon = 5$ TeV, and $\chi_0 = \chi_{max} = 5\times10^3$), we get $\varepsilon_b(\text{super})/\varepsilon \simeq 6\times10^{-3}$. This result is about half of that obtained in Ref./16/ on the basis of a logarithmic approximation in the step-like field model. It is necessary also to mention that in Ref./15/ when calculating the corrections to the constant-field limit, formula (7) has been used for the above parameters of the supercollider when $\sigma_z \ll \ell_0$. As is shown in the foregoing text, formula (7) becomes inapplicable under the condition indicated: its use in Ref./15/ has given rise to a considerable overestimation of the contribution from the correction terms (by a factor of 6 for these parameters).

We have considered radiation of the particle in the field of an incident bunch. Besides this process, the particle radiates when scattering on some particles from the opposite bunch. Its contribution to energy losses is essentially smaller. This bremsstrahlung process is suppressed especially in the strong field of the bunch /27/.

Let us apply the results to the known collider projects. Maximum values of the $\chi$ parameter are $\chi \approx$ 0.5–1.2 in the energy range $\varepsilon$ = 0.5–1 TeV, and the ratio $\ell_o/\sigma_z$ is $\ell_o/\sigma_z \approx$ $\approx 10^{-3}$–$10^{-2}$. Then the energy losses are completely described by Eq. (7), and the estimation (8) shows that the relative contribution of the correction terms, including also end effects (compared with the constant field approximation contribution), is of the order of $< 10^{-4}$. Let us stress that even for $\chi \gtrsim$ 0.1, one has to use quantum formulae for $I_o$ (the difference between quantum and classical energy loss calculations is of the order of 50% (at $\chi$ = 0.1). For a supercollider with parameters $\varepsilon$ = 5 TeV, $\chi_{max} = 5\ 10^3$, and $\ell_o/\sigma_z \approx$ 10, the gradient correction terms in Eq. (7) are invalid in the region of their main contribution. In this situation, the main corrections to the energy losses calculated in the constant field approximation are due to collinear photon radiation. Their relative contribution is $\approx$ 5% of the constant-field losses, the latter being $\approx$ 13% of the initial energy in this case.





## DERIVATION OF EQUATION (18)

In this Appendix we present the general analysis of end-effects in radiation for the case when the region $\sigma$, where the field is inhomogeneous, is much shorter than the bremsstrahlung formation length $\ell_o$ (or, more exactly, $\sigma \ll \ell_\omega$ ). The radiation of the initial electron is taken as an example, i.e. the radiation upon incidence of the particle into the region with the field. Let us choose the times $T_1$ and $T_2$ such that

$$\psi(T_1) \ll 1\ ,\quad \zeta(T_2) \gg 1\ ,\quad T_2 - T_1 \ll \ell_\omega \qquad (A.1)$$

where

$$\underline{\psi}(t) = \int_{-\infty}^{t} \frac{e\underline{F}(t')dt'}{m}\ ,\quad \psi(t) = |\underline{\psi}(t)|\ ,\quad \zeta(t) = \frac{1}{\ell_\omega}\int_{-\infty}^{t} \psi^2(t')dt'. \qquad (A.2)$$

The first equation in (A.1) means that at $t < T_1$ a momentum transfer from the field is much smaller than the particle's mass, and for such a time it may be neglected $[\psi(t<T_1) \simeq 0]$. The second equation may be fulfilled owing to the existence of a constant-field limit inside the region with the field, which is connected with a large value of the phase $\zeta(t)$. In the region where $\zeta(t) \gg 1$, one can use magnetic bremsstrahlung formulae, so what is of great interest for us is the region $t < T_2$. The third equation in (A.1) is valid when $\sigma \ll \ell_\omega$. We will divide the range of integration in Eq.(6) into three parts, in accordance with Eq.(A.1), after the substitution $t \to t - \tau/2$ $(t_2 = t,\ t_1 = t - \tau)$. In the first range

$t_1, t_2 \leqslant T_1$; in the second one, $t_1 < T_1$, $T_1 < t_2 \leq T_2$; and in the third, $T_1 \leq t_1, t_2 \leq T_2$. In the first range, the influence of the field on the particle motion may be neglected. Then the integration over $t$ and $\tau$ becomes elementary,

$$\int_{-\infty}^{T_1} dt \int_{t-T_1}^{\infty} \frac{d\tau}{\tau - i0} e^{-i\frac{\tau}{\ell_\omega}} = -\ell_\omega \int_{-\infty}^{0} dx \int_{-\infty}^{x} \frac{dy}{y} e^{-iy} = i\ell_\omega \tag{A.3}$$

and the radiation spectrum in this range has the form

$$d\varepsilon^{(1)} = -\frac{\alpha}{\pi} \frac{\varepsilon'}{\varepsilon} d\omega. \tag{A.4}$$

The main contribution to the second range comes from $t$, for which $\psi(t) \gg 1$, so the terms in Eq.(6) that contain no $\psi^2$ can be neglected. According to (A.1), one can put $\psi(t < T_1) = 0$. After neglecting these terms, we obtain for the energy losses in the second domain

$$d\varepsilon^{(2)} = \frac{\alpha\omega d\omega}{4\pi\gamma^2}\left(\frac{\varepsilon}{\varepsilon'} + \frac{\varepsilon'}{\varepsilon}\right) \int_{T_1}^{T_2} dt \int_{t-T_1}^{\infty} \frac{d\tau}{\tau} \sin\left[\frac{\tau}{\ell_\omega} + \xi(t) - \frac{\Phi^2(t)}{\tau \ell_\omega}\right] \psi^2(t), \tag{A.5}$$

where

$$\Phi(t) = \int_{-\infty}^{t} \psi(t')dt'$$

We now present the integral over $\tau$ in Eq.(A.5) in the form

$$\int_{t-T_1}^{\infty} d\tau [\dots] = \int_0^{\infty} d\tau [\dots] - \int_0^{t-T_1} d\tau [\dots] \tag{A.6}$$

The first of these integrals can be expressed by the MacDonald function



$$\int_0^{\infty} \frac{d\tau}{\tau} \sin\left(\frac{\tau}{\ell_\omega} - \frac{\Phi^2(t)}{\tau \ell_\omega} + \xi(t)\right) = 2\sin\xi(t) K_0\left(\frac{2\Phi(t)}{\ell_\omega}\right). \tag{A.7}$$

The main contribution to the integral over $t$ is given by a region $\xi(t) \sim 1$ (under the assumptions used), so the argument of the $K_0(z)$ function in Eq.(A.7) is small, and

$$K_0(z) \simeq \ln\frac{2}{z} - C, \; z \ll 1, \tag{A.8}$$

where $C$ = 0,577216... is the Euler constant. In the second integral in (A.6), one can neglect terms that are linear over $\tau$ in the argument of 'sin' owing to the inequalities $\tau < t - T_1 < T_2 - T_1 \ll \ell_\omega$. As a result, we obtain for the energy losses

$$d\varepsilon^{(2)} = d\varepsilon_1^{(2)} + d\varepsilon_2^{(2)},$$

where

$$d\varepsilon_1^{(2)} = \frac{\alpha\omega d\omega}{4\pi\gamma^2}\left(\frac{\varepsilon'}{\varepsilon} + \frac{\varepsilon}{\varepsilon'}\right) \int_{T_1}^{T_2} dt\, \psi^2(t) \sin\xi(t) \left[\ln\frac{\ell_\omega^2}{\Phi^2(t)} - 2C\right], \tag{A.9}$$

$$d\varepsilon_2^{(2)} = -\frac{\alpha\omega d\omega}{4\pi\gamma^2}\left(\frac{\varepsilon'}{\varepsilon} + \frac{\varepsilon}{\varepsilon'}\right) \int_{T_1}^{T_2} dt \int_0^{t-T_1} \frac{d\tau}{\tau} \psi^2(t) \sin\left[\xi(t) - \frac{\Phi^2(t)}{\tau \ell_\omega}\right]. \tag{A.10}$$

Integration over $t$ in the integral (A.9) can be extended to infinity, $-\infty < t < \infty$, owing to the inequalities $\psi(t < T_1) \ll 1$ and $\tilde{\tau}(t > T_2) \gg 1$. Using $\tilde{\tau}(t)$ as a new variable in Eq.(A.9), and taking into account that

$$\int_0^\infty d\tilde{\tau} \sin\tilde{\tau} \ln\tilde{\tau} = -C \tag{A.11}$$

we find

$$\frac{d\varepsilon_L^{(2)}}{d\omega} = \frac{\alpha}{2\pi}\left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)\int_0^\infty d\tilde{\tau} \sin\tilde{\tau}\left(\ln\Gamma(\tilde{\tau}) - C\right) \tag{A.12}$$

where

$$\Gamma(\tilde{\tau}) = \frac{l_\omega^2 \tilde{\tau}}{\Phi^2(t(\tilde{\tau}))} = \frac{l_\omega \int_{-\infty}^{t} \psi^2(t')dt'}{\left(\int_{-\infty}^{t} \psi(t')dt'\right)^2} . \tag{A.13}$$

We will add $d\varepsilon_2^{(2)}$ [Eq.(A.10)] to the energy losses in the third range $(T_1 \le t_1, t_2 \le T_2)$, where the region $\psi \gg 1$ is dominant also, and where, because $T_2 - T_1 \ll l_\omega$, the term $\tau/l_\omega$ may also be neglected. As a result, we obtain

$$d\varepsilon_2^{(2)} + d\varepsilon^{(3)} = \frac{\alpha\omega d\omega}{4\pi\gamma^2}\left(\frac{\varepsilon}{\varepsilon'}+\frac{\varepsilon'}{\varepsilon}\right)\int_{T_1}^{T_2} dt \int_0^{t-T_1} \frac{d\tau}{\tau}\Big\{\left(\underline{\psi}(t) - \underline{\psi}(t-\tau)\right)^2 \sin\left[\tilde{\tau}(t) - \tilde{\tau}(t-\tau) - \frac{1}{l_\omega \tau}\left(\underline{\Phi}(t) - \underline{\Phi}(t-\tau)\right)^2\right] - \tag{A.14}$$



$$-\psi^2(t)\sin\left[\tilde{\tau}(t) - \frac{1}{l_\omega \tau}\Phi^2(t)\right]\Big\}$$

The integrand in (A.14) vanishes at $\tau > t - T_1$ owing to the compensation of the terms in the curly brackets ($\psi$, $\Phi$, $\tilde{\tau}(t-\tau) \simeq 0$ at $\tau > t - T_1$), so the integral over $\tau$ in (A.14) can be extended to infinity. After this, the lower limit of integration over $t$ can be extended: $T_1 \to -\infty$.

We now calculate the difference between the energy losses (A.14) and the asymptotic expression $d\varepsilon_F^{(3)}$ calculated in the constant-field limit:

$$d\varepsilon_F^{(3)} = \frac{\alpha\omega d\omega}{4\pi\gamma^2}\left(\frac{\varepsilon}{\varepsilon'}+\frac{\varepsilon'}{\varepsilon}\right)\int_{-\infty}^{T_2} dt \int_0^\infty \frac{d\tau}{\tau}\tau^2\dot{\psi}^2(t-\tau/2) \times \sin\left[\dot{\psi}^2(t-\tau/2)\tau^3/(12 l_\omega)\right] . \tag{A.15}$$

For this difference, the integral over $t$ converges at the upper limit $T_2$, and does not depend on this limit. The sum of $d\varepsilon_F^{(3)}$ and the integral for $t > T_2$ presents the total integral over $t$ in the magnetic bremsstrahlung limit (because of the applicability of the constant-field limit in this range). Summing all the results, we obtain

$$d\varepsilon_L = d\varepsilon_F + d\varepsilon_b \tag{A.16}$$

where $d\varepsilon_L$ are the total energy losses, $d\varepsilon_F$ is the constant-

-field contribution, and $d\mathcal{E}_b$ presents the end effects:

$$\frac{d\mathcal{E}_F}{d\omega} = \int_{-\infty}^{\infty} \frac{dI_c(\omega,t)}{d\omega}dt \tag{A.17}$$

$$\frac{d\mathcal{E}_b}{d\omega} = \frac{\alpha}{\pi}\Big\{-2\frac{\varepsilon'}{\varepsilon} + \Big(1+\frac{\varepsilon'^2}{\varepsilon^2}\Big)\Big[\int_0^\infty d\zeta \sin\zeta \big(\ln\sqrt{\Gamma_i\Gamma_f} - C\big) + \frac{1}{2}\int_{-\infty}^{\infty}\frac{dt}{l_\omega}\int_0^\infty \frac{d\tau}{\tau}\Big\{\Big[\underline{\psi}\big(t+\tfrac{\tau}{2}\big) - \underline{\psi}\big(t-\tfrac{\tau}{2}\big)\Big]^2 \times$$
$$\times \sin\Big[\frac{1}{l_\omega}\int_{t-\tau/2}^{t+\tau/2}\psi^2(t')dt' - \frac{1}{\tau l_\omega}\Big(\int_{t-\tau/2}^{t+\tau/2}\underline{\psi}(t')dt'\Big)^2\Big] -$$
$$-\psi_i^2(t+\tau/2)\sin\Big[\frac{1}{l_\omega}\int_{-\infty}^{\infty}\psi_i^2(t')dt' - \frac{1}{l_\omega\tau}\Big(\int_{-\infty}^{t+\tau/2}\underline{\psi}_i(t')dt'\Big)^2\Big] -$$
$$-\psi_f^2(t-\tau/2)\sin\Big[\frac{1}{l_\omega}\int_{t-\tau/2}^{\infty}\psi_f^2(t')dt' - \frac{1}{l_\omega\tau}\Big(\int_{t-\tau/2}^{\infty}\underline{\psi}_f(t')dt'\Big)^2\Big] -$$
$$-\dot{\psi}^2(t)\tau^2\sin\big(\dot{\psi}^2(t)\tau^3/(12 l_\omega)\big)\Big\}\Big]\Big\}, \tag{A.18}$$

where

$$\underline{\psi}_i(t) = \int_{-\infty}^{t}\frac{e\underline{F}(t)}{m}dt, \quad \underline{\psi}_f(t) = \int_t^{\infty}\frac{e\underline{F}(t)}{m}dt, \quad \dot{\underline{\psi}} = \frac{e\underline{F}(t)}{m},$$

$$\psi_i(t) = |\underline{\psi}_i(t)|, \text{ etc.}, \quad \Gamma_i = \frac{l_\omega\int_{-\infty}^{t}\psi_i^2(t')dt'}{\Big(\int_{-\infty}^{t}\underline{\psi}_i(t')dt'\Big)^2},$$

$$\Gamma_f = \frac{l_\omega\int_t^{\infty}\psi_f^2(t')dt'}{\Big(\int_t^{\infty}\underline{\psi}_f(t')dt'\Big)^2}.$$



APPENDIX B

## RADIATION END-EFFECTS FOR THE FINITE RANGE OF THE FIELD

Let us consider the radiation of the initial particle for the case $F(z<z_0)=0$. For definiteness, we assume that the particle passes $z=z_0$ at $t=0$. In this case, the second integral in Eq.(A.18) takes the form

$$\frac{1}{l_\omega}\int_0^\infty dt\int_0^{2t}\frac{d\tau}{\tau}\{\ldots\} - \frac{1}{l_\omega}\int_0^\infty dt\int_{2t}^\infty d\tau\,\tau\,\dot{\psi}^2(t)\sin\Big(\frac{\psi^2(t)\tau^3}{12 l_\omega}\Big) =$$
$$= \int_0^\infty\frac{d\tau}{\tau}\int_{\tau/2}^\infty\frac{dt}{l_\omega}\{\ldots\} - 4\int_1^\infty \eta\, d\eta\int_0^\infty \frac{t^2dt}{l_\omega}\dot{\psi}^2(t)\sin\Big(\frac{2\psi^2(t)t^3\eta^3}{3l_\omega}\Big). \tag{B.1}$$

In the first integral in the right-hand side of (B.1) it is convenient to make the substitution $t \to t+\tau/2$, so that $t+\tau/2 \to t+\tau$, $t-\tau/2 \to t \geq 0$. The expression obtained appears to be more suitable for the case under consideration. We will consider the case of the power increase of the field:

$$\underline{F}(t) = \underline{F}_0\xi^n, \quad \xi = t/\sigma, \quad \underline{\psi}(t) = \frac{e\underline{F}_0\sigma}{m(n+1)}\xi^{n+1},$$

$$\zeta(t) = \frac{e^2F_0^2\sigma^3\xi^{2n+3}}{l_\omega m^2(n+1)^2(2n+3)} = \frac{au\,\xi^{2n+3}}{2(n+1)^2(2n+3)}, \tag{B.2}$$

$$u = \frac{\omega}{\varepsilon'}, \quad a = \frac{e^2F_0^2\sigma^3}{l_0 m^2} = \chi^2\Big(\frac{\sigma}{l_0}\Big)^3, \quad \underline{\Phi}(t) = \frac{e\underline{F}_0\sigma^2\xi^{n+2}}{m(n+1)(n+2)}.$$

Using formulae (B.2), we obtain

$$\Gamma(\tau)=\frac{l_\omega^2\tau}{\Phi^2}=\frac{2(n+2)^2}{u(2n+3)}\frac{l_0}{\sigma}\left[\frac{au}{2\tau(n+1)^2(2n+3)}\right]^{\frac{1}{2n+3}}. \qquad (B.3)$$

Taking into account Eq.(A.11), we have

$$-\int_0^\infty d\tau \sin\tau\left(\ln\Gamma(\tau)-C\right)=2\ln(n+2)+\frac{2}{2n+3}\Big[\ln\gamma+ n\ln\frac{l_0}{\sigma}+(n+1)\left(\ln\frac{2}{u}-C\right)-(n+2)\ln(2n+3)- \ln(n+1)\Big]. \qquad (B.4)$$

Then the last integral in (B.1) is easily done:

$$-4\int_1^\infty \eta\, d\eta\int_0^\infty\frac{dt\,t^2}{l_\omega}\dot\psi^2(t)\sin\left(\frac{2\dot\psi^2(t)t^3\eta^3}{3l_\omega}\right)= \qquad (B.5)$$

$$=-\frac{6}{2n+3}\int_1^\infty d\eta/\eta^2=-\frac{6}{2n+3}.$$

In the first integral in (B.1) we will make the substitution $\tau\to\eta t$ after the above transformations. Then integration over $t$ is simple, and the last integral over $\eta$ has the form



$$\mathcal{P}\int_0^\infty\frac{d\eta}{\eta}\left[\frac{(z^{n+1}-1)^2}{z^{2n+3}-1-\frac{2n+3}{(n+2)^2\eta}(z^{n+2}-1)^2}-\frac{1}{z\left(1-\frac{2n+3}{(n+2)^2}\frac{z}{\eta}\right)}-\frac{12}{(2n+3)\eta}\right];\quad z=1+\eta. \qquad (B.6)$$

The second integral in (B.6) is defined as a principal value:

$$-\mathcal{P}\int_0^\infty\frac{d\eta}{1+\eta}\frac{(n+2)^2}{(n+1)^2\eta-(2n+3)}=\mathcal{P}\int_0^\infty d\eta\left[\frac{1}{1+\eta}-\frac{(n+1)^2}{(n+1)^2\eta-(2n+3)}\right]=\int_0^\infty d\eta\left[\frac{1}{1+\eta}-\frac{(n+1)^2}{(n+1)^2\eta+2n+3}\right]=$$

$$=\ln\frac{2n+3}{(n+1)^2} \qquad (B.7)$$

Taking into account all the results obtained, we then have Eq.(18).

## APPENDIX C

### RADIATION END-EFFECTS FOR THE FIELD DECREASING AT INFINITY

When $F(z)\to 0$ for $z\to\pm\infty$, it is convenient to make the substitution $t\to t-\tau/2$ in Eq.(A.18) for the contribution to radiation of the incident particle, and $t\to t+\tau/2$ for the outgoing particle. For definiteness, we will consider the radiation of a final particle $(f)$ when there is a power decrease of the field over a specific length $\sigma$ :

$$\underline{F}(t)=\underline{F}_0\,\xi^n,\ \xi=\sigma/t,\ \underline{\psi}(t)=\int_t^\infty \frac{e\underline{F}(t')dt'}{m}=$$

$$=\frac{e\underline{F}_0\sigma}{m}\,\frac{\xi^{n-1}}{n-1},\ \underline{\Phi}(t)=\int_t^\infty \underline{\psi}(t')dt'=\frac{e\underline{F}_0\sigma^2}{m}\,\frac{\xi^{n-2}}{(n-1)(n-2)},$$

$$\zeta(t)=\frac{1}{l_\omega}\int_t^\infty \psi^2(t')dt'=\frac{au}{2}\,\frac{\xi^{2n-3}}{(n-1)^2(2n-3)},\ a=\chi\left(\frac{2\sigma}{l_0}\right)^3\gg 1. \quad (C.1)$$

Putting these functions into (A.13) we obtain

$$\Gamma(\zeta)=\frac{2(n-2)^2}{u(2n-3)}\,\frac{l_0}{\sigma}\left[\frac{2\zeta(n-1)^2(2n-3)}{au}\right]^{\frac{1}{2n-3}},$$

$$\int_0^\infty d\zeta\,\sin\zeta\,(\ln\Gamma(\zeta)-C)=2\ln(n-2)+\frac{2}{2n-3}\Big[n\ln\frac{l_0}{\sigma}-$$

$$-\ln\chi+(n-1)\Big(\ln\frac{2}{u}-C\Big)-(n-2)\ln(2n-3)+\ln(n-1)\Big]. \quad (C.2)$$



In the second integral in (A.18) we make the above mentioned substitution $t\to t+\tau/2$, and then $\tau=\eta t$, so that $t-\tau/2\to t,\ t+\tau/2\to t(1+\eta)$. Carrying out the integration over $t$, we obtain the following integral over $\eta$ :

$$\mathcal{P}\int_0^\infty \frac{d\eta}{\eta}\Bigg[\frac{(1-f^{n-1})^2}{1-f^{2n-3}-\frac{2n-3}{(n-2)^2\eta}(1-f^{n-2})^2}-$$

$$-\frac{1}{1-\frac{2n-3}{(n-2)^2\eta}}-\frac{12}{(2n-3)\eta}\Bigg]=\mathcal{E}_n,\ f=\frac{1}{1+\eta}. \quad (C.3)$$

Here the second integral is defined as a principal value. The expression for $\mathcal{E}_n$ [ formula (C.3)] can be transformed into the form

$$\mathcal{E}_n=\int_0^\infty \frac{d\eta}{\eta}\Bigg[\frac{(z^{n-1}-1)^2}{z^{2n-2}-z-\frac{2n-3}{(n-2)^2\eta}(z^{n-1}-z)^2}-$$

$$-\frac{1}{1+\frac{2n-3}{(n-2)^2\eta}}-\frac{12}{(2n-3)\eta}\Bigg],\quad z=1+\eta \quad (C.4)$$

The sum of (C.2) and (C.4) gives in Eq.(22) the term with the factor $(1+\varepsilon'^2/\varepsilon^2)$, depending on the field type. For $n=2$, integrals (C.2) and (C.4) are both logarithmically divergent, but there exists a finite limit for their sum. For its calculation, let us take into account that

$$\int_0^\infty d\eta\left(\frac{1}{1+\eta}-\frac{1}{\eta+\frac{2n-3}{(n-2)^2}}\right)=\ln\frac{2n-3}{(n-2)^2},$$

$$\lim_{n\to 2}\frac{(z^{n-2}-1)^2}{(n-2)^2}=\left(\lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}z^{\varepsilon}\right)^2=\ln^2 z. \quad (C.5)$$

Then at $n \to 2$,

$$(\text{C.2}) + (\text{C.4}) = 2\left[2\ln\frac{l_0}{\sigma} - \ln\chi + \ln\frac{2}{u} - C\right] +$$

$$+\int_0^\infty \frac{d\eta}{1+\eta}\left[\frac{1}{1-\eta^{-2}(1+\eta)\ln^2(1+\eta)} - 1 - \frac{12(1+\eta)}{\eta^2}\right]. \quad (\text{C.6})$$

For the case where $n \to \infty$, we will make the substitution $\eta \to 2\eta/n$ ; then $z \to 1$, $z^n \to e^{2\eta}$, and

$$\mathcal{E}_\infty = \int_0^\infty d\eta\left[\frac{1}{\eta\,\mathrm{cth}\,\eta - 1} - \frac{1}{1+\eta} - \frac{3}{\eta^2}\right]. \quad (\text{C.7})$$

In the case where there is exponential decreasing of the field, for the final electron contribution to the radiation we have

$$\underline{F}(t) = \underline{F}_0 e^{-t/\sigma} = \underline{F}_0 e^{-\xi},\ \underline{\Psi}(t) = \frac{e\underline{F}_0\sigma}{m} e^{-\xi},$$

$$\underline{\Phi}(t) = \frac{e\underline{F}_0\sigma^2}{m} e^{-\xi},\ \tau(t) = \frac{au}{2} e^{-2\xi},$$

$$\Gamma(\tau) = l_0/\sigma u. \quad (\text{C.8})$$



In this case, both integrals over $\tau$ and $t$ in Eq.(A.18) are taken as elementary and we obtain Eq.(23).

For an external field of the Gaussian type $\underline{F} = \underline{F}_0 e^{-\xi^2/2}$, we bear in mind that $\tau(\xi) \simeq 1$ for large $\xi \simeq \sqrt{\ln(au)}$. Then, doing integrals by the Laplace method, we obtain with an accuracy of up to $\sim 1/\ln a$,

$$\underline{\Psi}(\xi) = \underline{\Psi}_0 e^{-\xi^2/2}/\xi,\ \underline{\Phi}(\xi) = \underline{\Phi}_0 e^{-\xi^2/2}/\xi^2,$$

$$\tau(\xi) = \frac{au}{2}\frac{e^{-\xi^2}}{\xi^3},\ \Gamma = l_0\xi/u,$$

$$\int_0^\infty d\tau \sin\tau \ln\xi \simeq \frac{1}{2}\ln\ln(au) + O\left(\frac{1}{\ln a}\right) \quad (\text{C.9})$$

Taking the second integral in Eq.(A.18) we make the substitution $\tau \to \tau/t$, and omit the terms $\sim 1/\xi^2$ (accuracy of $\sim 1/\ln a$ ). After these operations, the integrals coincide with the case of exponential field decrease.

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КВАНТОВАЯ ТЕОРИЯ ИЗЛУЧЕНИЯ В НЕОДНОРОДНЫХ ВНЕШНИХ ПОЛЯХ

Препринт
№ 88-168

Работа поступила - 2 ноября 1988 г.

Ответственный за выпуск - С.Г.Попов
Подписано к печати 23.II.1988г. МН 08606
Формат бумаги 60х90 I/I6 Усл.I,8 печ.л., I,5 учетно-изд.л.
Тираж 200 экз. Бесплатно. Заказ № I68.

Ротапринт ИЯФ СО АН СССР, г.Новосибирск, 90