ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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ANNIHILATION OF SLOW e^+e^--PAIR AND POSITRONIUM WIDTH

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НОВОСИБИРСК

Annihilation of Slow e^+e^--Pair
and Positronium Width

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ABSTRACT

The cross section of slow e^+e^--pair annihilation into two and three photons is calculated taking into account relativistic corrections up to second order in relative velocity. The contributions to the cross section from total spin zero and one in the initial state are separated from each other. The result is used for the estimation of a part of second order radiative corrections for the ortopositronium width. The discrepancy between theory and experiment cannot be removed by these corrections.

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The discrepancy with the theoretical prediction for ortopositronium width up to one loop radiative corrections (RC) [2] was found in recent measurements of the positronium width [1].

The second order corrections must be anomalously large to remove this contradiction:

$$\Gamma=\Gamma_0\left(1-10.7\left(\frac{\alpha}{\pi}\right)+N\left(\frac{\alpha}{\pi}\right)^2\right),\quad \Gamma_0=\frac{\alpha^6 mc^2}{\hbar}\cdot\frac{2(\pi^2-9)}{9\pi},\quad N\approx 350. \quad (1)$$

The systematic calculation of RC up to second order has not yet been performed, but some particular results are known. For example, the logarithmic contribution [3]

$$\left(\frac{\alpha}{\pi}\right)^2\Delta N=\frac{1}{3}\alpha^2\ln\alpha\sim\left(\frac{\alpha}{\pi}\right)^2(-16) \quad (2a)$$

is negative, but it can be cancelled by the positive contribution arising from first order corrections in (1):

$$\left(\frac{\alpha}{\pi}\right)^2\left(\frac{10.7}{2}\right)^2\sim\left(\frac{\alpha}{\pi}\right)^2(28.6). \quad (2b)$$

In this work we estimate so called «relativistic corrections» arising from the relative motion of electron and positron which we regard as free in this approximation:

$$\left(\frac{\alpha}{\pi}\right)^2\Delta N=\left(\frac{v}{c}\right)^2(-5.36)=\left(\frac{\alpha}{\pi}\right)^2(-52.9) \quad (3)$$

(where $2v$ is a relative velocity in the c.m. system, $v/c=\beta\approx\alpha$). These corrections turn out to be negative emphasizing a necessity of detailed calculation of second order perturbative RC as well as the same order corrections to a positronium wave function following from the Bethe—Salpeter equation.

In the first part we obtain a contribution to a differential cross section of annihilation into two hard photons arising from virtual and soft photons, and in the second part we consider an annihilation process into three hard photons and give a total cross section in the $\beta\ll 1$ limit. In the third part we separate from a three photon annihilation cross section individual contributions from the total spin zero and one in the initial state of e^+e^--pair and obtain the relativistic corrections to ortopositronium decay rate. We conclude with a brief discussion of other possible sources for second order perturbative corrections.

1. The two photon annihilation cross section

$$e^+(p_+)+e^-(p_-)\to\gamma(q_1)+\gamma(q_2)\,,$$

$$m^2k=2p_-q_1=2p_+q_2=2\varepsilon^2(1+\beta\cos\theta)\,,\quad p_\mp=(\varepsilon,0,0\pm p)\,, \tag{4}$$

$$m^2\tau=2p_-q_2=2p_+q_1=2\varepsilon^2(1-\beta\cos\theta)\,,\quad \beta=\left|\frac{\vec p}{\varepsilon}\right|=\frac{v}{c}\,,$$

which takes into account low order RC was first derived by Harris and Brown followed by other authors. We make use of a result of Berends and Gastmans [4] which is free from misprints. It has the form

$$\frac{d\sigma}{d\Omega_1}=\frac{d\sigma_0}{d\Omega_1}(1+\delta_V+\delta_S)\,,\qquad \frac{d\sigma_0}{d\Omega_1}=\frac{\alpha^2}{S\beta}\left(\frac{1+\beta^2+\beta^2\sin^2\theta}{1-\beta^2\cos^2\theta}-\frac{2\beta^4\sin^4\theta}{(1-\beta^2\cos^3\theta)^2}\right),$$
$$S=(p_++p_-)^2=4\varepsilon^2\,, \tag{5}$$

where δ_V takes into account virtual photon exchange:

$$\delta_V=-\frac{\alpha}{\pi}\left\{\left(2+\frac{1+\beta^2}{\beta}\ln b\right)\ln\frac{\lambda}{m}-\right.$$
$$\left.-\frac{1+\beta^2}{\beta}\left(\ln b\,\ln\beta-2Li_2(-b)+\frac{1}{2}Li_2(b^2)+\frac{\pi^2}{4}\right)+G+\tilde G\right\}, \tag{6}$$

λ is a «fictitious mass» of this photon;

$$b=\frac{1-\beta}{1+\beta}\,,\quad Li_2(y)=-\int_0^y\frac{dx}{x}\ln(1-x)\,,\quad G=G(k,\tau)\,,\quad \tilde G=G(\tau,k)\,; \tag{7a}$$

G looks as

$$\left(4\left(\frac{1}{k}+\frac{1}{\tau}\right)^2-4\left(\frac{1}{k}+\frac{1}{\tau}\right)-\frac{k}{\tau}-\frac{\tau}{k}\right)G(k,\tau)=\left(\frac{4}{k\tau}\frac{\beta(3-\beta^2)}{(1-\beta^2)^2}+\beta\right)\times$$
$$\times\left(\frac{1}{2}\ln b\,\ln\frac{1-\beta^2}{4}-\frac{1}{4}\ln^2 b-Li_2(-b)-\frac{\pi^2}{12}\right)+\left\{\frac{1+\beta^2}{\beta}\left(\frac{1}{k}-\frac{4}{k^2}+\frac{\tau}{2k}+\right.\right.$$
$$\left.+\frac{k}{\tau}+1+\frac{4}{k\tau}\frac{\beta^2}{(1-\beta^2)}\right)\ln b+\frac{1-\beta^2}{\beta}\left(\frac{k}{2\tau}-\frac{3}{\tau}\right)\ln b+\frac{3\tau}{2k^2}(1+k)+$$
$$\left.+\frac{3}{\tau}+1-\frac{7}{k\tau}+\frac{8}{k}-\frac{8}{k^2}-\frac{\tau^2-2k+k^2\tau}{2k^2\tau(k-1)}-\frac{2k^2+\tau}{2\tau(k-1)^2}\right\}\ln k+$$
$$+\frac{1-\beta^2}{4}(\pi^2-\ln^2 b)\left(\frac{2}{k}-\frac{7k}{4}-\frac{3k^2}{4\tau}\right)+\frac{2}{\beta}\left(\frac{1}{2}-\frac{1}{k}\right)\ln b+$$
$$+4\left(\frac{1}{k}+\frac{1}{\tau}\right)^2-\frac{12}{k}-\frac{3k}{2\tau}-\frac{2k}{\tau^2}+\frac{2k+\tau}{2\tau(k-1)}+$$
$$+\frac{2}{k}\left(\frac{\pi^2}{6}-Li_2(1-k)\right)\left(\frac{k^2}{\tau}+\frac{\tau}{k^2}+\frac{k}{\tau}+k+\frac{\tau}{2}+\frac{2}{k}-\frac{3}{\tau}-1\right). \tag{8}$$

Additional soft photon emission with an energy less than $\Delta\varepsilon\ll\varepsilon$ in a c.m. system is taken into account by δ_S:

$$\delta_S=-\frac{\alpha}{\pi}\left\{\left(2+\frac{1+\beta^2}{\beta}\ln b\right)\ln\left(\frac{2\Delta\varepsilon}{\lambda}\right)+\frac{1}{\beta}\ln b+\right.$$
$$\left.+\frac{1+\beta^2}{\beta}\left(Li_2\left(\frac{2\beta}{1+\beta}\right)+\frac{1}{4}\ln^2 b\right)\right\}. \tag{9}$$

A decomposition of δ_S and δ_V in powers of β gives:

$$\frac{d\sigma}{d\Omega_1}=\frac{d\sigma_0}{d\Omega_1}\left\{1+\frac{\alpha}{\pi}\left[-\left(5-\frac{\pi^2}{4}\right)+\right.\right.$$
$$+\frac{\pi^2}{2\beta}(1+\beta^2)+\beta^2\left(\frac{8}{3}\ln\left(\frac{2\Delta\varepsilon}{m}\right)-\pi^2+\frac{16}{3}\ln 2-\frac{16}{9}\right)+$$
$$\left.\left.+\left(\frac{7}{4}\pi^2+3-24\ln 2\right)\beta^2\cos^2\theta_1\right]\right\},\qquad \frac{d\sigma_0}{d\Omega_1}=\frac{r_0^2}{4\beta}(1+\beta^2)\,,\quad r_0=\frac{\alpha}{m}. \tag{10}$$

The second term in square brackets is interpreted as effect of Coulomb interaction of initial particles and is neglected when dealing with the positronium width.

2. A three photon annihilation cross section of unpolarized e^+e^--pair has the form [4, 6]

$$d\sigma=\frac{(4\pi\alpha)^3}{S\beta}\overline{|M|^2}\frac{(2\pi)^4}{2m^2(2\pi)^9}\frac{d^3q_1\,d^3q_2\,d^3q_3}{2\omega_1\,2\omega_2\,2\omega_3}\,\delta^{(4)}(p_+ + p_- - q_1 - q_2 - q_3)\,. \tag{11}$$

The quantity $\overline{|M|^2}$ can be derived from a similar one for the double Compton scattering process studied by Mandl and Skyrme in [5] and looks as

$$\overline{|M|^2}=\frac{m^2}{4}\sum_{spin}|M|^2=-2(ab-c)\,[(a+b)(X+2)+c-ab-8]+$$
$$+2X(a^2+b^2)+8c-\frac{4x}{AB}\Big[(A+B)(1+X)-(aA+bB)\Big(2+z\frac{1-X}{X}\Big)+$$
$$+X^2(1-z)+2z\Big]+2\rho\,(ab+c(1-X))\,, \tag{12}$$

where

$$A=k_1k_2k_3\,,\quad B=k_1'k_2'k_3'\,,\quad a=\sum(k_i)^{-1}\,,\quad b=\sum(k_i')^{-1}\,,\quad c=\sum(k_i\,k_i')^{-1}\,,$$
$$z=\sum k_ik_i'\,,\quad X=\sum k_i\,,\quad \rho=\sum\Big(\frac{k_i}{k_i'}+\frac{k_i'}{k_i}\Big),\quad k_i=\frac{p_-\cdot q_i}{m^2},\quad k_i'=\frac{p_+\cdot q_i}{m^2}, \tag{13}$$
$$k_i=\frac{\varepsilon^2}{m^2}x_i(1-\beta c_i)\,,\quad k_i'=\frac{\varepsilon^2}{m^2}x_i(1+\beta c_i)\,,\quad x_i=\frac{\omega_i}{\varepsilon},\quad c_i=\cos(\widehat{\vec{p}_-,\vec{q}_i})\,.$$

Decomposing (12) in β up to terms of order $\sim\beta^2$ and integrating over photon emission angles with respect to a beam axis ($\vec{p}_-$) with the help of equations [6]

$$\varepsilon^{-2}\int\frac{d^3q_1\,d^3q_2\,d^3q_3}{\omega_1\,\omega_2\,\omega_3}\,\delta^{(4)}(p_+ + p_- - q_1 - q_2 - q_3)=8\pi^2\int\frac{dc_1\,dc_2}{2\pi\sqrt{D_{12}}}\int d^3x\,\delta(\Sigma x-2)\,,$$
$$D_{12}=(z_+ - c_1)\,(c_1 - z_-)\,,\qquad z_\pm=c_2a_{12}\pm(1-a_{12}^2)^{1/2}(1-c_2^2)^2\,,$$
$$a_{12}=1-2(1-x_3)/(x_1x_2)\,, \tag{14}$$
$$\int\frac{dc_1dc_2}{2\pi\sqrt{D_{12}}}\{1,c_1^2,c_2^2,c_1\,c_2\}=\Big\{1,\frac{1}{3},\frac{1}{3},\frac{1}{3}a_{12}\Big\},$$

we obtain the following distribution in final photon energy branches:

$$d\sigma=\frac{4\alpha r_0^2}{3\beta A^3}\{A(s_2-2s_3+s_4)+$$
$$+\beta^2[-16+28A-11A^2+(16-11A)\,s_2-(4+A)s_2^2]\}\,d^3x\delta(\Sigma x-2)\,, \tag{15}$$

where

$$A=x_1x_2x_3\,,\qquad s_i=x_1^i+x_2^i+x_3^i\,,\qquad i=2,3,4\,.$$

To obtain a total three photon annihilation cross section, we have to integrate (15) on the domain

$$\frac{\Delta\varepsilon}{m}\equiv\Delta<x_i<1\,,\quad \sum x_i=2\,,\quad \Delta\ll 1 \tag{15a}$$

and to divide the result by 3! because of Bose-statistic of photons. Performing this we get

$$\sigma^{e^+e^-\to3\gamma}=\frac{2\alpha r_0^2}{3\beta}\Big[\pi^2-9+\beta^2\Big(2\ln\frac{1}{\Delta}+8-\frac{31}{24}\pi^2\Big)\Big]. \tag{16}$$

To total e^+e^--pair annihilation cross section for the $\beta\ll1$ case is derived by adding (16) to the result of integration (10) over photon emission angles (divided by 2!) and has the form

$$\sigma^{e^+e^-\to2\gamma,3\gamma}=\sigma_0^c+\frac{\alpha r_0^2}{2\beta}\Big[\frac{19}{12}\pi^2-17-\beta^2\Big(\frac{17}{9}\pi^2-\frac{44}{9}\Big)\Big], \tag{17}$$

where

$$\sigma_0^c=(\pi r_0^2(1+\beta^2)/2\beta)\Big(1+\frac{\pi\alpha}{2\beta}(1+\beta^2)\Big)$$

is the Born cross section with Coulomb interaction of initial particles taken into account.

3. At first let us determine spin structure of an electron and positron state which can decay into two or three photons. A charge parity of a state of e^+e^--pair with angular momentum l and spin s decaying into n photons is $C_n=(-1)^{l+s}=(-1)^n$. In the $\beta\ll1$ domain only $l=0$, 1, 2 can give a considerable contribution as is seen from the decomposition in β of characteristic fermion propagator $(1-\beta\cos\theta)^{-1}$ in matrix elements of 2γ and 3γ channels. For the spin zero state the main (two photon) transition mode corresponds to S-state, $l=0$; the three photon channel is suppressed by a factor β^2 and corresponds to $l=1$ state. This fact can also be seen from the form of two photon annihilation cross section (see (10)) which contains infrared divergences in the $\Delta\to0$ limit. These divergences would be cancelled when we take into account extra

hard photon emission, i. e. with the divergences of a singlet initial state part of three photon annihilation cross section.

Now we have to construct a matrix element of a singlet e^+e^- state annihilation into three photons. Let us choose a quantization axis (z) along the beam axis $(\vec{p}_-)$ in the c.m. frame. The electron states with momentum $\vec{p}$ and z-projection of spin $\pm 1/2$ is described by spinors $U_\pm(\vec{p})$, $(M=1)$:

$$U_+(\vec{p})=\begin{vmatrix}\sqrt{+}\\ 0\\ \sqrt{-}\\ 0\end{vmatrix} \qquad U_-(\vec{p})=\begin{vmatrix}0\\ \sqrt{+}\\ 0\\ -\sqrt{-}\end{vmatrix}$$

where

$$\sqrt{\pm}=\sqrt{\varepsilon\pm 1}\,,\quad \varepsilon^2=1+\beta^2. \tag{18a}$$

The positron states with momentum $-\vec{p}$ and z-projection of spin $\pm 1/2$ is described by spinors $V_\pm(-\vec{p})$:

$$V_+(-\vec{p})=i\begin{vmatrix}0\\ \sqrt{-}\\ 0\\ \sqrt{+}\end{vmatrix} \qquad V_-(-\vec{p})=i\begin{vmatrix}-\sqrt{-}\\ 0\\ \sqrt{+}\\ 0\end{vmatrix} \tag{18b}$$

The matrix element of a singlet e^+e^- initial state annihilation into three photons is proportional to a combination

$$M_0\sim\bar{V}_+(-\vec{p})\,\hat{Q}U(\vec{p})+\bar{V}_-(-\vec{p})\,\hat{Q}U_+(\vec{p})=\mathrm{Sp}\,\hat{\pi}_0\hat{Q}\,, \tag{19}$$

where

$$\hat{Q}=\sum_{P(123)}\hat{e}_3\frac{-\hat{p}_++\hat{k}_3-m}{-2p_+\cdot k_3}\,\hat{e}_2\frac{\hat{p}_--\hat{k}_1+m}{-2p_-\cdot k_1}\hat{e}_1=$$

$$=Q_{123}+Q_{321}+Q_{231}+Q_{132}+Q_{312}+Q_{213}\,,$$

$$\hat{\pi}_0=-(1+\gamma_0)\,\gamma_5-\beta\gamma_0\gamma_3\gamma_5+\frac{1}{2}\beta^2\gamma_5\,,$$

$$\gamma_0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \vec{\gamma}=\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix},\quad \gamma_5=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{20}$$

The quantity $\hat{Q}$ corresponds to six Feynman diagrams of three

quantum annihilation. Since we have free electron and positron spinors in the initial state, Q can be presented as

$$Q=\sum_{P(123)}\frac{1}{2p_+\cdot k_3\,2p_-\cdot k_1}(\hat{e}_3\hat{k}_3-2(p_+\cdot e_3))\,\hat{e}_2(-\hat{k}_1\hat{e}_1+2(p_-\cdot e_1))\equiv\sum_{P(123)}Q_{123}\,.$$

It is clear from this form that only $-\gamma_0\gamma_5$ piece of the singlet state projection operator gives a nonzero contribution because the trace of an odd number of γ-matrices equals 0.

It is useful to check*) that M_0 vanishes in the $\beta\to 0$ limit. Actually, in this limit M_0 is a sum of three quantities of $R=\mathrm{Sp}\,(Q_{123}+Q_{321})\,\gamma_0\gamma_5$ type. Writing $R=\mathrm{Sp}\,C\,(Q_{123}+Q_{321})\,CC\gamma_0\gamma_5C$, where $C=\gamma_0\gamma_2$ is a charge conjugation matrix and using properties $C\gamma_\mu C=-(\gamma_\mu)^T$, $C\gamma_0\gamma_5C=(\gamma_0\gamma_5)^T$, $\mathrm{Sp}\,0^T=\mathrm{Sp}\,0$, we see that $R=-R=0$.

Taking trace (19), summing a squared modulus of matrix element over photons polarization states and integrating over angular variables of final photons with the help of (14) we obtain**) the following spectral distribution for the spin singlet state:

$$d\sigma^{{}^1P_0\to 3\gamma}=\frac{\alpha r_0^2\,d^3x\,\delta(\Sigma_x-2)}{3!\cdot 4A^3}\beta\left[-3S_{32}-\frac{31}{2}A^2+\right.$$

$$\left.+\frac{15}{2}AS_{21}+5S_{31}-2S_3-10A\,S_2-4A\,S_{22}\right],$$

$$S_{22}=(x_1x_2)^2+(x_1x_3)^2+(x_2x_3)^2,$$

$$S_{21}=x_1x_2(x_1+x_2)+x_1x_3(x_1+x_3)+x_2x_3(x_2+x_3)\,,$$

$$S_{31}=x_1x_3(x_1^2+x_3^2)+x_2x_3(x_2^2+x_3^2)+x_1x_2(x_1^2+x_2^2)\,,$$

$$S_{32}=x_1^2x_2^2(x_1+x_2)+x_1^2x_3^2(x_1+x_3)+x_2^2x_3^2(x_2^2+x_3^2)\,. \tag{21}$$

We have $(3!)^{-1}$ factor in (21) because of identical photons. The integration of (21) on the domain (15a) gives

$$\sigma^{{}^1P_0\to 3\gamma}=\frac{2\alpha r_0^2\beta}{3}\left[2\ln\frac{1}{\Lambda}-3+\frac{7\pi^2}{24}\right]. \tag{22}$$

Subtracting (22) from (16) we get the triplet state annihilation cross section into three photons:

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*) We thank V.S. Fadin for discussing this point.

**) REDUCE system was used in calculations.

$$\sigma^{{}^3S_1\to 3\gamma}=\frac{2\alpha r_0^2}{3\beta}(\pi^2-9)\left[1-\beta^2\,\frac{19\pi^2-132}{12\pi^2-108}\right]. \qquad (23)$$

Estimation of the second term in square brackets gives a size of «relativistic corrections» to ortopositronium width which we quote above (see (3)). For the annihilation cross section from singlet state we have (summing (10) and (22))

$$\sigma^{{}^1S_0,\,{}^1P_0\to 2\gamma,\,3\gamma}=\sigma_0^c\left\{1-\frac{\alpha}{\pi}\left(5-\frac{\pi^2}{4}\right)-\frac{\alpha\beta^2}{\pi}\left(\frac{43}{9}-\frac{\pi^2}{36}\right)\right\}. \qquad (24)$$

Estimation of the five photon decay mode contribution to the positronium decay rate $\lambda_5/\lambda_3\sim 10^{-6}$ shows that this mechanism cannot explain a discrepancy between theory and experiment. Corrections of order $(\alpha/\pi)^2\cdot 350$ are hardly expected from the two-loop approximation — they are not natural for QED. Laying aside a question about exotic particle and interaction manifestations we mention a problem of the more precise solution of the Bethe — Salpeter equation [9], and possible incorrectness in $(\alpha/\pi)^2$ order presentation of the positronium decay amplitude in the form [7] $M=\int\chi(p)M_0(p)\,dp$. These questions need further analysis. A confirmation of the experimental result [1] is also desirable.

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