ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

V.M. Malkin, E.G. Shapiro

ELEMENTARY EXCITATIONS FOR SOLITONS OF NONLINEAR SCHRÖDINGER EQUATION

PREPRINT 90-78

НОВОСИБИРСК

Institute of Nuclear Physics

V.M. Malkin, E.G. Shapiro

ELEMENTARY EXCITATIONS FOR SOLITONS OF NONLINEAR SCHRÖDINGER EQUATION

PREPRINT 90-78

NOVOSIBIRSK
1990

Elementary Excitations for Solitons of Nonlinear Schrödinger Equation

V.M. Malkin, E.G. Shapiro[*)]

Institute of Nuclear Physics
630090, Novosibirsk, USSR

ABSTRACT

Dynamics of nonlinear wave field having stable localized states (solitons) is essentially dependent on spectra of such states' elementary excitations. The spectra is found below for solitons of two-dimensional Schrödinger equation with focusing cubic nonlinearity. The equation describes a number of physical phenomena, in particular, the self-focusing of radiation in a medium.

[*)] Institute of Automation and Electrometry, 630090, Novosibirsk, USSR.

1. INTRODUCTION

The field described by the equation

$$\left(i\frac{\partial}{\partial t}+\Delta+|\psi|^2\right)\psi=0\,, \tag{1.1}$$

where Δ is a two-dimensional Laplace operator, has localized stationary states of the form:

$$\psi(\vec{r},t)=\varkappa R(\varkappa r)\exp(i\varkappa^2 t+i\varphi)\,. \tag{1.2}$$

The parameter $\varkappa^2$ may be called «the energy for a quantum drop out of the soliton». The designation $R(r)$ is used below for the monotone solution of equation

$$\left(-1+\frac{1}{r}\frac{d}{dr}\,r\,\frac{d}{dr}+R^2\right)R=0. \tag{1.3}$$

The graph of this solution, calculated in paper [1], is reproduced for readers convenience in Fig. 1. The Eq. (1.3) has also oscillating over r localized solutions (see [2—4]), but those are exponentially unstable with respect to small perturbations. As for the Townes soliton [1], its perturbation can grow no faster than the time squared (see [5—7]). With the inclusion of small defocusing nonlinearity, proportional to the fifth order of $|\psi|$, to Eq. (1.1), the squared-time-unstable mode turns into the stable low-frequency mode minor change in form (as well as the soliton itself).

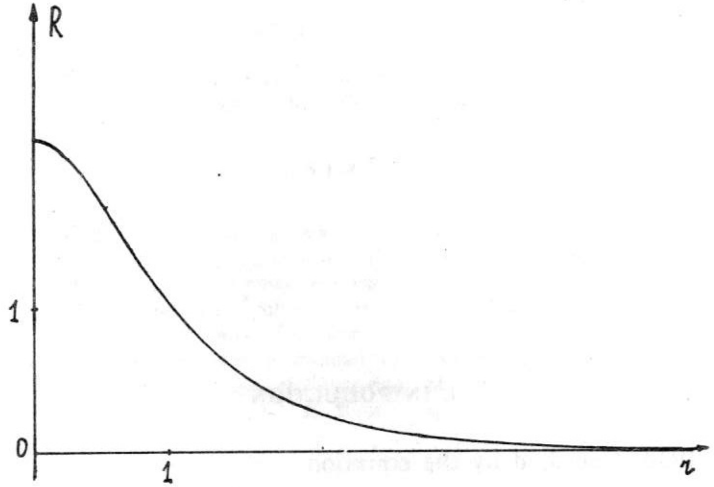


Fig. 1. The Townes soliton $R(r)$.

The dynamics of stabilized so soliton with strongly excited low-frequency mode is of essential interest and depends on the spectrum of other eigenmodes (see [8]). The determination of this spectrum is the purpose of the present paper.

Prior to proceed to the solution of the above problem it is useful to remind the main symmetries of the Eq. (1.1).

2. SYMMETRIES

For the class of functions axisymmetric with respect to the origin in the plane of variables $x = r\cos\theta$, $y = r\sin\theta$ the Eq. (1.1) allows three parameter group of transformations. It consists of the gauge, scale and Talanov transformations. (The latter were given explicitly in [9] though had been contained actually in the earlier paper [10] of the same author.) Application of the above transformations to the soliton of unit size ($\varkappa = 1$) and zero phase ($\varphi = 0$) yields the three parameter family of exact solutions of the Eq. (1.1):

$$\psi(r,t) = \frac{\varkappa}{1 - t/t_*} R\left(\frac{\varkappa r}{1 - t/t_*}\right) \exp\left[i\left(\frac{\varkappa^2 t}{1 - t/t_*} - \frac{1}{4}\frac{r^2}{t_* - t} + \varphi\right)\right]. \qquad (2.1)$$

The two parameter family of stationary solitons (1.2) is obtainable from (2.1) as a particular case corresponding to the infinite time of the singularity formation ($t_* = \infty$). At $t_* \to \infty$ the solution (2.1) differ from (1.2) by small (proportional to $t_*^{-1} \to 0$) term growing as a time squared. As it was mentioned in the Introduction, this unstable mode is stabilized by the inclusion of small defocusing nonlinearity to Eq. (1.1). The relation between eigenmode considered and infinitesimal Talanov transformation was pointed out in [7]. The infinitesimal gauge and scale transformations generates constant and linearly growing with time eigenmodes correspondingly. An increase of the latter mode is caused by a storage of the phase difference solely and does not imply instability of the soliton.

Besides the three parameter group described, the Eq. (1.1) is also invariant with respect to the five parameter group of Galilei transformations, including space translations, rotations and movements with a constant velocity in the plane x, y. The rotational symmetry of the Eq. (1.1) and Townes soliton enables one to introduce an azimuth quantum number $m = 0, \pm 1, \pm 2, \ldots$ and to treat small perturbations with different meaning of m independently. Infinitesimal translations and velocity variations of the soliton generate correspondingly two constant and two linearly growing with time dipole eigenmodes ($|m| = 1$). The behaviour of the latter pair of modes is explained by a growing shift of the soliton as a whole and does not imply instability of its form.

3. BASIC EQUATIONS

In order to simplify furthers formulae, it is conveniently to take the soliton of unit size $\varkappa^{-1} = 1$. (This does not lead to any loss of generality due to the scale invariance of the Eq. (1.1)). For such a soliton the elementary excitation of frequency ω

$$\delta\psi_\omega(\bar{r}, t) = [A_\omega(\bar{r})\, e^{-i\omega t} + B_\omega^*(\bar{r})\, e^{i\omega^* t}]\, e^{it} \qquad (3.1)$$

fits the equations

$$\omega A_\omega = \hat{L}_0 A_\omega - R^2(A_\omega + B_\omega), \qquad (3.2)$$

$$-\omega B_\omega = \hat{L}_0 B_\omega - R^2(A_\omega + B_\omega),$$

where

$$\hat{L}_0 = 1 - \Delta - R^2 . \tag{3.3}$$

At combined replacements $\omega \to -\omega^*$, $A_\omega \rightleftarrows B^*_{-\omega}$ the perturbation (3.1) does not change. Furthermore, Eqs (3.2) are invariant with respect to combined complex conjugation and replacements $\omega^* \to \omega$, $A^*_\omega \to A_\omega$, $B^*_\omega \to B_\omega$. Consequently, if ω is eigenvalue, then $-\omega^*$, $-\omega$, ω^* are also eigenvalues. Thus the investigation of spectrum reduces to the first quadrant of the complex plane ω:

$$\operatorname{Re}\omega \geqslant 0, \quad \operatorname{Im}\omega \geqslant 0 .$$

In the invariant subspace $\mathscr{L}_m$ of functions with azimuth quantum number m a regular solution of Eqs (3.2) looks at $r \to 0$ as

$$A_\omega(\vec{r}) = A_\omega^{(m)} r^m + O(r^{m+2}) ,$$
$$B_\omega(\vec{r}) = B_\omega^{(m)} r^m + O(r^{m+2}) . \tag{3.4}$$

All possible asymptoticses of functions $A_\omega(\vec{r})$ and $B_\omega(\vec{r})$ at $r \to \infty$ are given by the formulae

$$A_\omega(\vec{r}) \propto r^{-1/2} \exp[\pm(1-\omega)^{1/2} r] ,$$
$$B_\omega(\vec{r}) \propto r^{-1/2} \exp[\pm(1+\omega)^{1/2} r] . \tag{3.5}$$

A regular in the point $r = 0$ solution of Eqs (3.3) depends, besides ω and m, on two additional parameters $A_\omega^{(m)}$ and $B_\omega^{(m)}$. After identification of solutions proportional to each other only one additional parameter remains. It can be chosen so that one of the asymptoticses (3.5) is excluded. For real $\omega > 1$ only one of four asymptoticses (3.5) does not tend to zero at $r \to \infty$. This asymptotics can be excluded by a proper choice of the additional parameter in regular at $r = 0$ solution. Consequently, the real ray $\omega > 1$ belongs to continuous spectrum. For others values of ω (in semiplane $\operatorname{Re}\omega \geqslant 0$) two of asymptoticses (3.5) do not tend to zero at $r \to \infty$. Therefore all the rest eigenvalues ω belong to the point spectrum.

For arbitrary value of ω from the semiplane $\operatorname{Re}\omega \geqslant 0$ (the ray $\omega > 1$ is excised) the solution of Eq. (3.2) regular at $r = 0$ increases exponentially at $r \to \infty$:

$$A_\omega(\vec{r}) \approx [\, C_{AA}^{(m)}(\omega) A_\omega^{(m)} + C_{AB}^{(m)}(\omega) B_\omega^{(m)} \,]\, r^{-1/2} \exp[(1-\omega)^{1/2} r] ,$$
$$B_\omega(\vec{r}) \approx [\, C_{BA}^{(m)}(\omega) A_\omega^{(m)} + C_{BB}^{(m)}(\omega) B_\omega^{(m)} \,]\, r^{-1/2} \exp[(1+\omega)^{1/2} r] . \tag{3.6}$$

The increase is preventable by a nontrivial choice of parameters $A_\omega^{(m)}$, $B_\omega^{(m)}$ only if the determinant

$$D_m(\omega) = C_{AA}^{(m)}(\omega)\, C_{BB}^{(m)}(\omega) - C_{AB}^{(m)}(\omega)\, C_{BA}^{(m)}(\omega) . \tag{3.7}$$

is equal to zero. Thus, isolated eigenvalues ω are roots of equations

$$D_m(\omega) = 0, \qquad m = 0, \pm 1, \pm 2, \ldots . \tag{3.8}$$

The linear combinations of functions $A_\omega(\vec{r})$ and $B_\omega(\vec{r})$:

$$u_\omega = A_\omega + B_\omega, \quad v_\omega = A_\omega - B_\omega \tag{3.9}$$

fitted equations

$$\omega v_\omega = \hat{L}_1 u_\omega ,$$
$$\omega u_\omega = \hat{L}_0 v_\omega , \tag{3.10}$$

where

$$\hat{L}_1 = \hat{L}_0 - 2R^2 . \tag{3.11}$$

The pair of Eqs (3.10) is conveniently to present as a one matrix equation for two-component function χ_ω:

$$\hat{L}\chi_\omega = \omega\, \chi_\omega \tag{3.12}$$

$$\hat{L} = \begin{pmatrix} 0 & \hat{L}_0 \\ \hat{L}_1 & 0 \end{pmatrix}, \quad \chi_\omega = \begin{pmatrix} u_\omega \\ v_\omega \end{pmatrix}, \tag{3.13}$$

Reductions of operators $\hat{L}_i$ $(i = 0; 1)$ and $\hat{L}$ for their invariant subspaces $\mathscr{L}_m$ and $\mathscr{L}_m^2 = \mathscr{L}_m \otimes \mathscr{L}_m$ are designated below by the same symbols as complete operators. In the subspace $\mathscr{L}_m$ the Laplacian looks as

$$\Delta = \frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \frac{m^2}{r^2} \tag{3.14}$$

and $\hat{L}_0$, $\hat{L}_1$ are reduced to ordinary differential operators.

4. POINT SPECTRUM

For the class of isotropic perturbations, i. e. in subspace $\mathscr{L}_0^2$, the operator $\hat{L}$ is digenerate. The zero eigenvalue corresponds to the eigenfunction $\chi^{(1)}$:

$$\hat{L}\chi^{(1)}=0, \quad u^{(1)}=0, \quad v^{(1)}=R, \tag{4.1}$$

generated by the infinitesimal gauge transformation. Besides $\chi^{(1)}$, the root eigenfunction $\chi^{(2)}$ of height 2:

$$\hat{L}\chi^{(2)}=\chi^{(1)}, \quad u^{(2)}=-\frac{1}{2}\frac{d}{dr}rR, \quad v^{(2)}=0, \tag{4.2}$$

generated by the infinitesimal scale transformation, and the root eigenfunction $\chi^{(3)}$ of height 3:

$$\hat{L}\chi^{(3)}=\chi^{(2)}, \quad u^{(3)}=0, \quad v^{(3)}=\frac{1}{8}r^2R, \tag{4.3}$$

generated by the infinitesimal Talanov transformation, are correspond to the zero eigenvalue of operator $\hat{L}$.

The adjoint to $\hat{L}$ operator $\hat{L}^+=\begin{pmatrix} 0 & \hat{L}_1 \\ \hat{L}_0 & 0 \end{pmatrix}$ also has in $\mathscr{L}_0^2$ the root subspace $\{\bar{\chi}^{(i)},\ i=1,\ 2,\ 3\}$ corresponding to the zero eigenvalue. Functions $\bar{\chi}^{(i)}$ $(i=1,\ 2,\ 3)$ are obtainable from $\chi^{(i)}$ by a simple rearrangement of the components:

$$\bar{u}^{(i)}=v^{(i)}, \quad \bar{v}^{(i)}=u^{(i)}, \quad (i=1,2,3).$$

Each eigenfunction χ_ω of the operator $\hat{L}$ corresponding to the nonzero eigenvalue ω is orthogonal to $\bar{\chi}^{(i)}$ $(i=1,\ 2,\ 3)$ at natural definition of scalar product:

$$\langle\bar{\chi}|\chi\rangle=\int d^2\bar{r}(\bar{u}^*u+\bar{v}^*v).$$

In the terms of single-component functions' scalar product—

$$(u,v)=\int d^2\bar{r}\,u^*v$$

—the orthogonality conditions imply the following:

$$(u^{(i)},v_\omega)=0, \quad i=2; \tag{4.4}$$
$$(v^{(i)},u_\omega)=0, \quad i=1,3.$$

Let $\hat{P}$ be the operator of projection on the subspace of functions ortogonal to $v^{(1)}=R$. Action of the operator P on Eqs (3.10) yields

$$\omega\hat{P}v_\omega=\hat{P}\hat{L}_1u_\omega, \quad \omega\hat{P}u_\omega=\hat{P}\hat{L}_0v_\omega. \tag{4.5}$$

Taking into account that $\hat{P}\hat{L}_0=\hat{L}_0=\hat{L}_0\hat{P}$ and $\hat{P}u_\omega=u_\omega$, one can rewrite Eqs (4.5) as

$$\omega\tilde{v}_\omega=\hat{L}_1u_\omega, \quad \omega u_\omega=\hat{L}_0u_\omega \tag{4.6}$$
$$(\tilde{v}_\omega=\hat{P}v_\omega, \quad \hat{L}_1=\hat{P}\hat{L}_1\hat{P}).$$

All functions in Eqs (4.6) belong to subspace $\hat{P}\mathscr{L}_0$ already. In this subspace the operator $\hat{L}_0$ is positively definded (which follows from the monotonicity of the function R corresponding to zero eigenvalue of the operator $\hat{L}_0$ in the space $\mathscr{L}_0$). The operator $\hat{L}_1$ is degenerate. Its zero eigenvalue corresponds to the eigenfunction $u^{(2)}$ (As easily seen from the explicit formula (4.2), $u^{(2)}$ is orthogonal to R, so that $\hat{P}u^{(2)}=u^{(2)}$). The function $u^{(2)}$ has only one zero. Functions without zeros do not orthogonal to R and do not belong to subspace $\hat{P}\mathscr{L}_0$. Therefore it is clear that the function $u^{(2)}$ corresponds to smallest eigenvalue of the operator $\hat{L}_1$, i. e. $\hat{L}_1$ is nonnegatively definded. By the formula

$$\omega^2=\frac{(u_\omega,\hat{L}_1u_\omega)}{(u_\omega,\hat{L}_0^{-1}u_\omega)}, \tag{4.7}$$

following from Eqs (4.6), the above properties of operators $\hat{L}_0$ and $\hat{L}_1$ guarantee that all eigenvalues ω of the operator $\hat{L}$ in subspace $\mathscr{L}_0^2$ are real.

For the class of dipole perturbations, i. e. in subspaces $\mathscr{L}^2_{\pm1}$, the operator $\hat{L}$ is also degenerate. Zeros eigenvalues correspond to eigenfunctions $\chi_\pm^{(1)}$:

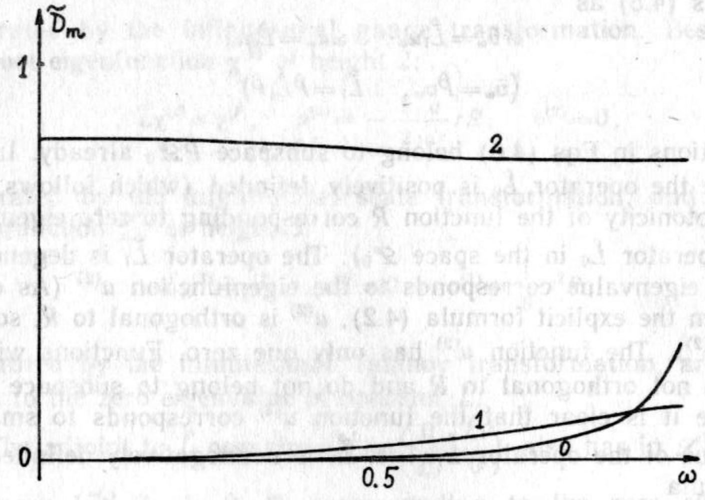
$$\hat{L}\chi_\pm^{(1)}=0, \quad u_\pm^{(1)}=\frac{dR}{dr}e^{\pm i\theta}, \quad v_\pm^{(1)}=0, \tag{4.8}$$

generated by infinitesimal translations in the plane x, y. Corresponding root eigenfunctions $\chi_\pm^{(2)}$:

$$\hat{L}\chi_\pm^{(2)}=\chi_\pm^{(1)}, \quad u_\pm^{(2)}=0, \quad v_\pm^{(2)}=-\frac{1}{2}rR\,e^{\pm i\theta} \tag{4.9}$$

are generated by infinitesimal Galilei transformations.

The radial part dR/dr of functions $u_\pm^{(1)}$, corresponding to zero eigenvalue of the operator $\hat{L}_1$ in subspaces $\mathscr{L}_{\pm1}$, has no zeros in the range $0<r<\infty$. This guarantee nonnegative definiteness of the ope-



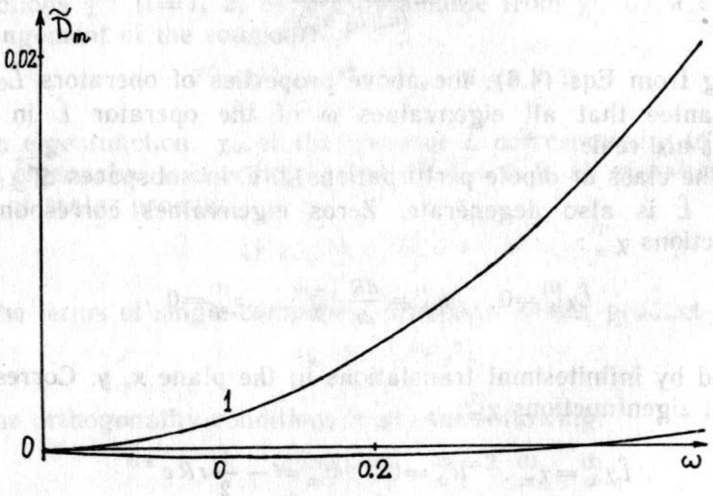


Fig. 2. Graphs for functions $\tilde{D}_m(\omega)=\tilde{D}_{-m}(\omega)$ in the interval (0, 1) numbered by meaning of m.

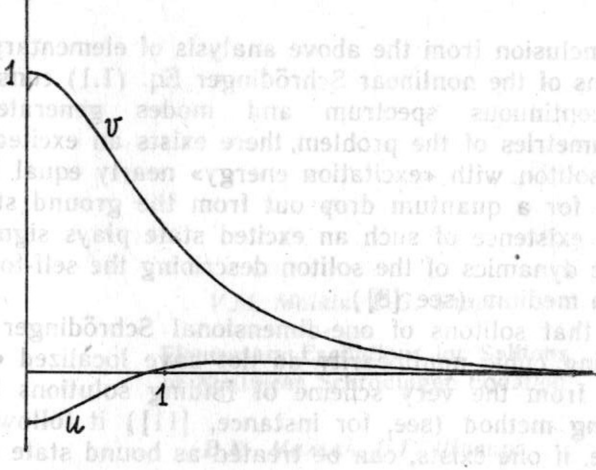


Fig. 3. Two-component «wave function» for quanta in excited state of the soliton.

rator $\hat{L}_1$ in subspaces $\mathscr{L}_{\pm 1}$ and positive definiteness of the operator $\hat{L}_0$ there. It is clear as well that at $|m|\geqslant 2$ both of operators $\hat{L}_0$ and $\hat{L}_1$ are positively definded in subspaces $\mathscr{L}_m$. By the relation (4.7), taking place in all subspaces $\mathscr{L}_m$, it follows that all eigenvalues ω of the operator $\hat{L}$ are real. This allows one to restrict the numerical study of the point spectrum by the interval $0<\omega<1$.

Due to the mirror symmetry of the soliton, functions $D_m(\omega)$ (as well as sets of the $\hat{L}$-operator's eigenvalues in subspaces $\mathscr{L}^2_m$) do not depend on the sign of m. At $m\gg 1$ functions $D_m(\omega)$ are calculated analytically. Picking out the analytical dependence one can get on from $D_m(\omega)$ to functions

$$\tilde{D}_m(\omega)=\frac{2\pi(1-\omega^2)^{\frac{m}{2}+\frac{1}{4}}}{2^{2m}(m!)^2}D_m(\omega)\,, \tag{4.10}$$

tending to unit at $m\rightarrow\infty$. Numerically constructed graphs for functions $\tilde{D}_m(\omega)$ at small values of m are shown in Fig. 2. As it is seen from the figure, there is only one eigenvalue

$$\omega\approx 0.321 \tag{4.11}$$

inside the interval (0.1). Components u_ω and v_ω of corresponding eigenfunction χ_ω normalized by the condition $v_\omega(0)=1$ (which is possible since $m=0$) are shown in Fig. 3.

5. CONCLUSION

The main conclusion from the above analysis of elementary excitation for solitons of the nonlinear Schrödinger Eq. (1.1) consists in that, beside continuous spectrum and modes generated by well-known symmetries of the problem, there exists an excited state of the Townes soliton with «excitation energy» nearly equal to 1/3 of the «energy» for a quantum drop out from the ground state of the soliton. The existence of such an excited state plays significant role in adiabatic dynamics of the soliton describing the self-focusing of radiation in a medium (see [8]).

Noteworthy that solitons of one-dimensional Schrödinger equation with focusing cubic nonlinearity do not have localized excited states. In fact, from the very scheme of finding solutions by the inverse scattering method (see, for instance, [11]) it follows that the excited state, if one exists, can be treated as bound state of two solitons moving with equal velocities. An arbitrarily small perturbation, destroyed the velocities equality, leads to diverging of the solitons. In contrast to that, a small perturbation can not cause the decay of the soliton for two others solitons, since such a decay is forbidden with excess by conservation lows for Hamiltonian and number of quanta.

REFERENCES

1. *Chiao R.Y., Garmire E., Townes C.H.* Phys. Rev. Lett., 1964, v.13, p.479.
2. *Jankauskas Z.K.* Izv. VUZov. Radiofizika, 1966, v.9, p.412 [Sov. Radiophys., 1966, v.9, p.261].
3. *Haus H.A.* Applied Phys. Lett., 1966, v.8, p.128.
4. *Zakharov V.E., Sobolev V.V., Synakh V.S.* ZhETF, 1971, v.60, p.136 [Sov. Phys., JETP, 1971, v.33, p.77].
5. *Laedke E.W., Spatschek K.H., Stenflo L.* Jorn. Math. Phys., 1983, v.24, p.2764.
6. *Weinstein M.J.* Commun. Math. Phys., 1983, v.87, p.567.
7. *Kuznetsov E.A., Turitsyn S.K.* Phys. Lett. A, 1985, v.112, p.273.
8. *Malkin V.M.* On the Analytical Theory for Self-focusing of Radiation in a Medium (to be published).
9. *Talanov V.I.* ZhETF Pis'ma, 1970, v.11, p.303 [JETP Lett., 1970, v.11, p.199].
10. *Talanov V.I.* Izv. Radiofizika, 1966, v.9, p.410.
11. Theory of Solitons: Inverse Scattering Method/*Zakharov V.E., Manakov S.V., Novikov S.P., Pitaevski L.P.*/M.: Nauka, 1980, v.1, § 10.

V.M. Malkin, E.G. Shapiro

Elementary Excitations for Solitons of Nonlinear Schrödinger Equation

В.М. Малкин, Е.Г. Шапиро

Элементарные возбуждения солитонов нелинейного уравнения Шредингера

Ответственный за выпуск С.Г.Попов

Работа поступила 29 июня 1990 г.
Подписано в печать 9.07 1990 г. МН 02390
Формат бумаги 60×90 1/16 Объем 1,1 печ.л., 0,9 уч.-изд.л.
Тираж 200 экз. Бесплатно. Заказ № 94

Набрано в автоматизированной системе на базе фотонаборного автомата ФА1000 и ЭВМ «Электроника» и отпечатано на ротапринте Института ядерной физики СО АН СССР,
Новосибирск, 630090, пр. академика Лаврентьева, 11.