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DYNAMICAL CHAOS AND BEAM-BEAM MODELS

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ABSTRACT

Some aspects of the nonlinear dynamics are discussed for the simple one-dimensional and two-dimensional models. The main attention is paid to the stochasticity threshold due to the overlapping of nonlinear resonances. The peculiarities of a round beam are investigated in view of using the round beams in storage rings to get high luminosity.

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1. INTRODUCTION

One of the interesting possibility to reach high luminosity in storage rings is to use the round beams with equal β-functions at interactions points. Such approach has been already discussed in literature (see, e. g. [1—3]), nevertheless, up to now the beam-beam dynamics for this specific case seems to be quite unclear. This may be explained by the fact that almost all facilities in operation have the beams which are essentially elliptical in their transverse section. For this reason, the main attention has been paid to the beam-beam dynamics which is related to the specific properties of elliptical beams (see, e. g. [4—6]). Recently, some interest has appeared again to the study of beam-beam dynamics of round or nearly round beams [7a, 7b]. It is related to real projects of the construction of such storage rings. Here we discuss some aspects of nonlinear dynamics of particle-beam interaction for the simple models of the round and flat beams.

2. THE SIMPLEST ONE-DIMENSIONAL BEAM-BEAM MODELS

The first comparison between round and flat beams has been performed using simplest one-dimensional models of a beam-beam interaction (see, e. g., [8—9]). In these models the motion equations for a particle of a weak beam under the influence of a strong beam can be effectively written in the form of two-dimensional mappings. Such a simplification is possible, provided the longitudinal size of a strong beam is small enough. Then, the perturbation from a strong beam can be regarded as an instantoneous kick, therefore, the transverse momentum p and displacement x after one period of perturbation are changing through the relation

$$p_{n+1} = -\frac{x_n}{\beta}\sin\mu + p_n\cos\mu + f(x_n)\cos\mu,$$

$$x_{n+1} = x_n\cos\mu + \beta p_n\sin\mu + \beta f(x_n)\sin\mu. \qquad (2.1)$$

Here μ is the betatron tune advance between two successive kicks, β is the value of β-function in the interaction point (the first derivative of β-function is assumed to vanish in this point, $\dot{\beta}=0$, $\dot{x}=p=dx/ds$, where s is the longitudinal coordinate). The parameter μ is related to the betatron frequency ν by the expression $\mu=2\pi\nu/m_0$ with m_0 being the number of interaction points over the ring. The mapping (2.1) consists of two parts, the free betatron oscillation between two kicks and the kicked perturbation which is given by the nonlinear force $f(x)$.

In what follows we shall use the well known expression for the strength parameter ξ:

$$\xi_{z,r} = \frac{N r_0 \beta\sigma_{z,r}}{2\pi\gamma\sigma_z\sigma_r(\sigma_z+\sigma_r)}, \qquad (2.2)$$

which is one of the most important parameters to characterize the beam-beam interaction. Here N is the number of particles in the bunch, r_0 is the classical radius of electron and γ is the relativistic factor. As it is known, the value ξ is approximately equal to the shift of betatron frequency $\Delta\nu$ for one interaction point for the particle with a small betatron amplitude (far enough from an integer resonance). Then, for the round beam ($\sigma_z=\sigma_r=\sigma$) one can write

$$f_1(x) = -\frac{8\pi\sigma^2\xi}{\beta}\,\frac{1-\exp(-x^2/2\sigma^2)}{x} \qquad (2.3)$$

correspondingly, for the flat beam ($\sigma_z=\sigma$; $\sigma_r\to\infty$) we have

$$f_2(x) = -\frac{4\pi\xi}{\beta}\int_0^x e^{-x^2/2\sigma^2}\,dx. \qquad (2.4)$$

Fig. 1 shows the shapes of $f(x)$ for the round and flat beams

depending on the transverse displacement x. Here, the value of ξ is fixed, therefore, the density in the origin ($x=0$) for the round beam is twice larger as compared to the flat beam. In the following we shall use dimensionless variables $X=x/\sigma$ and $P=p\beta/\sigma$.

The main problem is to find the maximal value of ξ which is restricted by nonlinear effects. The strongest effect comes from the overlapping of the main betatron nonlinear resonances resulting in the fast diffusion of a particle of weak beam in the phase space of

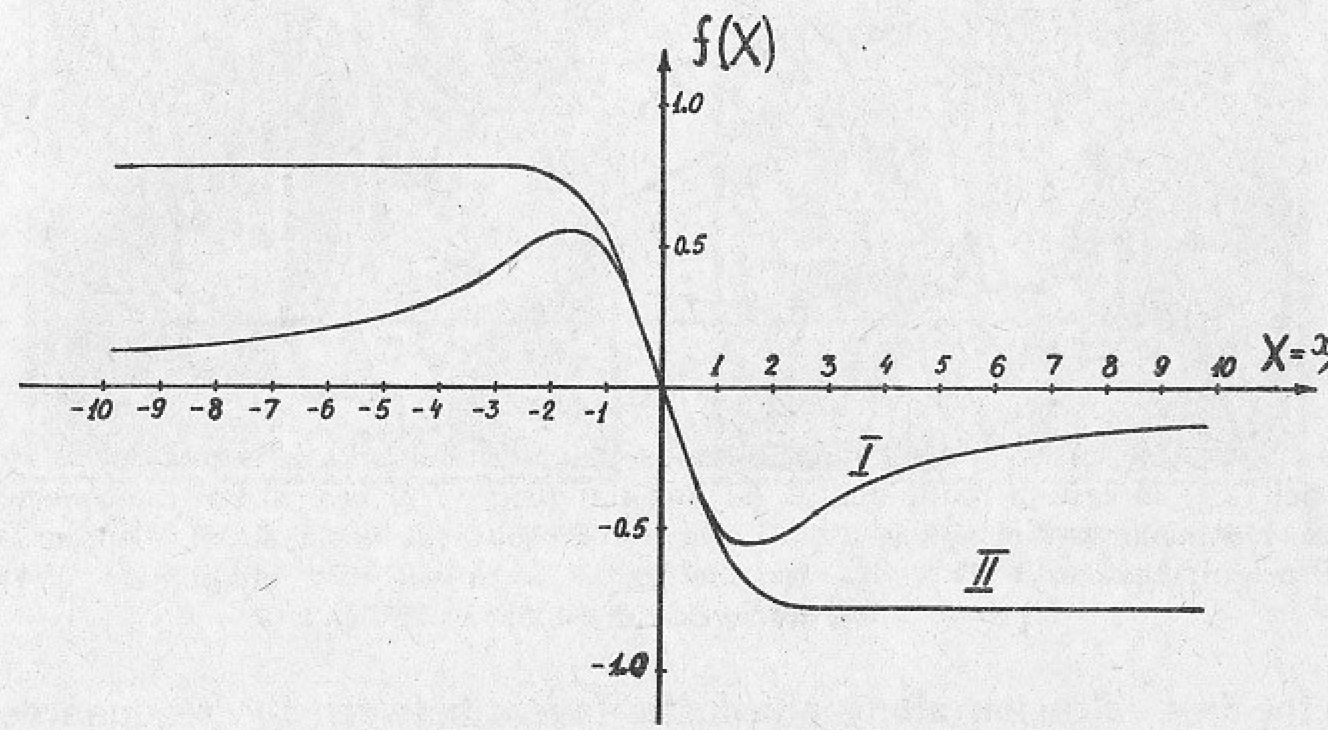

Fig. 1. Nonlinear kick-forces for the round (*I*) and the flat (*II*) beams versus dimensionless displacement $X=x/\sigma$ (see expressions (2.3) and (2.4)).

transverse motion. Numerious simulations have shown (see e. g. [9] and references therein) that the critical value ξ_{cr} for overlapping of main resonances is quite large ($\xi_{cr}\approx 0.2\div 0.3$) in comparison with the values achieved in real experiments ($\xi_{cr}\approx 0.04\div 0.08$). For example, in [8] the dependence ξ_{cr} on the betatron frequency has been numerically investigated both for the round and flat beams. The main result is presented in Fig. 2 where, for comparison, the condition for the linear stability of the origin $x=p=0$ is also shown. For the convenience, the fractional part of μ/π is plotted along the horizontal axis. It allows to use this result for the facilities with any value of unperturbed betatron frequency. The critical value ξ_{cr} has been determined in [8] from the condition for the touching of separatrices of main resonances in the region of $|X|\leqslant 10$. For $\xi>\xi_{cr}$ the strong instability of motion arizes, leading

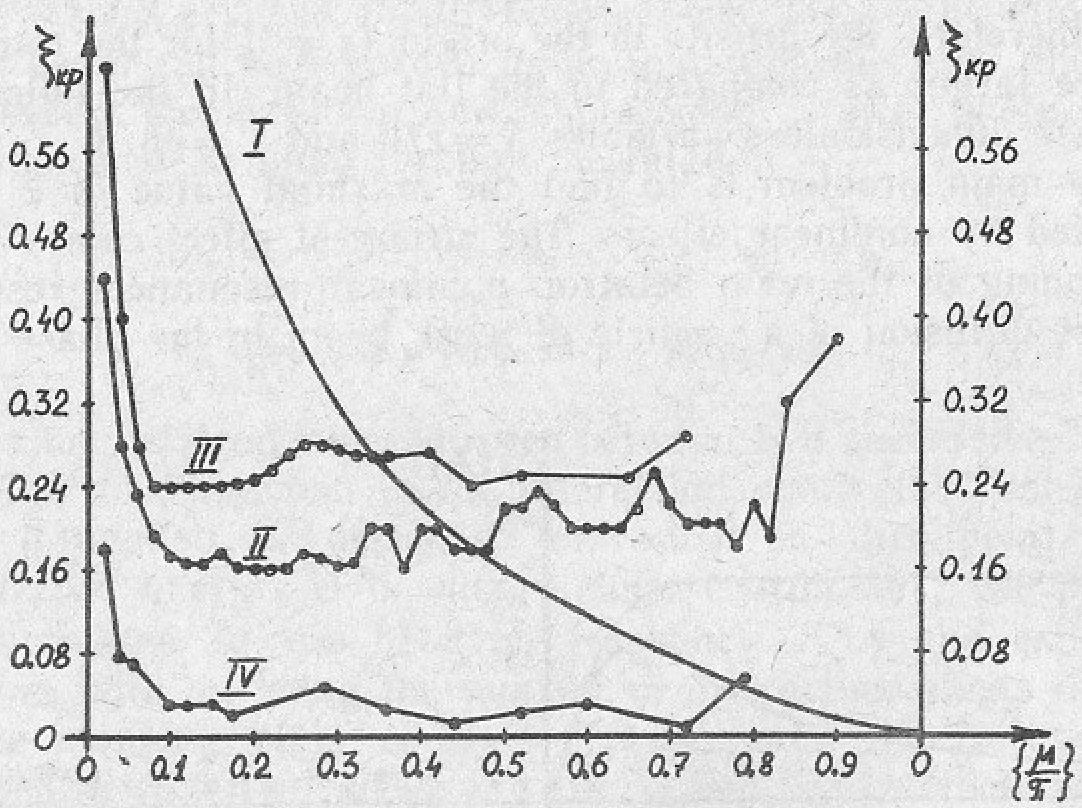

Fig. 2. The stochasticity threshold ξ_{cr} versus tune shift advance μ is plotted for the model (2.1) in case of round (curve *II*) and flat (curve *III*) beams. For the convenience, fractional part of μ/π is used. The curve *I* shows the linear stability border of the origin; the curve *IV* is ξ_{cr} for the round beam with the modulation (3.1); $A_s=1.5\sigma$ and $\nu_s=0.01$.

to the fast diffusion along stochastic layers between the resonances. According to the numerical simulation, about 5-6 resonances start to overlap simultaneuosly with a creation the large region of strong diffusion in the phase space of system.

The most essential conclusions made in [8] are:

1. In the large range of μ the critical value ξ_{cr} is only slightly dependent on μ/π.

2. The obtained values of ξ_{cr} are much larger than the critical values of ξ observed in real experiments.

3. When $\{\mu/\pi\}\ll 1$ the critical value ξ_{cr} is much larger due to the fact that in this region of variation μ/π there exist only resonances of large harmonics.

4. The critical values ξ_{cr} are slightly less for the round beam compared to the flat beam. However, as it was pointed out, the charge density of the round beam is twice larger than for the flat beam for the same value of ξ (see (2. 5)). It was also found that the regions of nonlinear resonances in phase space are larger for the flat beam. It is related to the fact that in the same range of X

the nonlinearity for the flat beam is less than for the round beam (see Fig. 1). For this reason, when nonlinear resonances touch to each other the regions with a strong diffusion for the flat beam are larger than for the round beam. This result is well seen when investigating the structure of phase space of the model (2.1) (see Fig. 3).

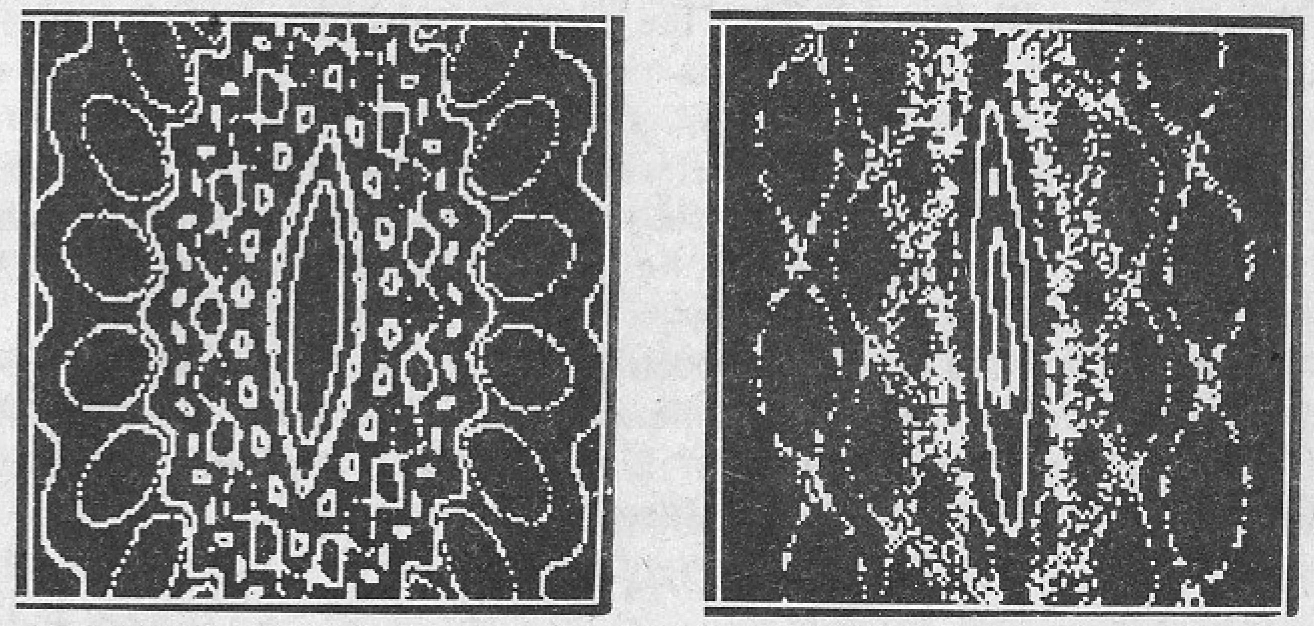

Fig. 3. The structure of phase space for the round and flat beams for the same charge density in the origin. The scales are $|X|\leqslant 30$ and $|P|\leqslant 15$; $\{\mu/\pi\}\approx 0.08$; *a*—round beam, $\xi\approx 0.20$; *b*—flat beam, $\xi\approx 0.40$.

3. THE INFLUENCE OF MODULATIONS ON THE STOCHASTICITY LIMIT

One of the important factors that can decrease ξ_{cr} is the modulation of model parameters. This was known long ago, here we only remind some results which are important for the further discussion. The first investigations [8] with the model (2.1) have revealed that the modulation of betatron motion by a synchrotron one may lead to a significant decrease of the current. One example is given in Fig. 2 (curve *IV*) where possible modulation of the beam-beam interaction point is taken into account

$$x\to u_s=x+A_s\sin(2\pi\nu_s n/m_0)\,. \tag{3.1}$$

Here A_s is the amplitude and ν_s is the frequency of synchrotron oscillations. Such a modulation occurs when dispersion energy function Ψ at the interaction points does not vanish. In this case the orbit of a particle with non-equilibrium energy shifts on the value $\Delta x = \Psi R \cdot \Delta E / E$ [8] where R is the mean radius of a ring. For this reason, synchrotron oscillations result in the time modulation of nonlinear force of a strong beam. As a result, in the model (2.1) the force (2.3) (or (2.4)) has to be changed in accordance with (3.1).

From the data presented in Fig. 2 it is seen that in case of large synchrontron oscillations, $A_s \sim \sigma$, the critical value ξ_{cr} drops significantly up to $\xi_{cr} \approx 0.02 \div 0.04$. This result obtained in a very simple model of beam-beam interaction, indicates that the synchrotron oscillations can play an important role in the restriction of luminosity. Further investigations with the more realistic models have been confirmed this conclusion (see, e. g. [10—12] and references therein).

The more detailed study has been performed in [13] where different types of modulations are compared from the point of view of their influence on the decrease of ξ_{cr}. Analytical approach consists in the analysis of resonance structure of the Hamiltonian

$$H = J\nu_0 + V(J, \varphi, \theta)\, \delta_T(\theta) \tag{3.2}$$

corresponding to the mapping (2.1) in the «action-phase» variables $(x = \sqrt{2J\beta}\cos\varphi;\ p = \sqrt{2J/\beta}\sin\varphi$, see [9]). Here ν_0 is the unperturbed betatron frequency, $\delta_T(\theta)$ is the periodic delta funcion which depends on the phase θ introduced instead of the azimuthal coordinate s; $\theta = 2\pi s/L$ (L is the interaction period). The external perturbation $V(J, \varphi, \theta)$ is determined by the period $T = 2\pi/m_0$ and by the nonlinear force $f(x)$. For the round beam V has the form

$$V_1(J, \varphi, \theta) = -\frac{4\pi\xi}{\beta}\sigma^2 \int_0^1 \frac{1-\exp(-u_s z)}{z}\, dz \tag{3.3}$$

and for the flat beam

$$V_2(J, \varphi, \theta) = -\frac{4\pi\xi}{\beta}\sigma^2 \frac{1}{2} \int_0^1 \frac{1-\exp(-u_s z)}{z^{3/2}}\, dz . \tag{3.4}$$

From the experimental point of view, the most important modulations are:

1. The modulation caused by the non-vanishing value of dispersion function Ψ in the interaction points (see (3.1));

2. The modulation of betatron phase shift μ between the interaction points. Such modulation arises due to the dependence of a rotational period on the particle energy (see, also, [13]):

$$\Delta\mu = \mu - \mu_0 \approx -\alpha \frac{\Delta p_\parallel}{p_\parallel} \frac{\pi}{m_0} \frac{R}{\beta} \approx (\Delta\mu)_0 \sin(\nu_s \Omega_0 t + \delta) = = B \sin\left(\frac{2\pi\nu_s n}{m_0} + \delta\right), \tag{3.5}$$

where $p_\parallel$ and Ω_0 are the transverse momentum and the angular frequency of particle revolution; α is the momentum compaction factor and $\Delta p_\parallel / p_\parallel = (\Delta p_\parallel / p_\parallel)_0 \sin(\nu_s \Omega_0 t + \delta)$. Analogous modulation occurs when magnetic field in a storage ring has periodic pulsations. It should be also noted that the particular case of betatron phase modulation appears, due to inaccurate azimuthal adjustment of the ring component [13]. In the latter case $\Delta\mu$ varies kick-likely which is formally corresponds to $\nu_s = 0$ in (3.5).

3. The modulation of perturbation strength due to azimuthal dependence of a β-function in the gap where the beam-beam interaction occurs

$$\xi = \xi_0 \left(1 + \frac{l^2}{\beta_0^2}\right)^{1/2} \tag{3.6}$$

Here β_0 is the minimum value of β-function at the interaction point and l is the azimuthal deviation. Since for the round beam $\xi_{x,z} \sim \beta_{x,z}/\sigma^2$ and $\sigma \sim \sqrt{\beta}$ it is seen that under the condition $\beta_x = \beta_z$ the dependence of longitudinal coordinate dissappears, therefore, the modulation (3.6) is absent. For the analysis of the influence of modulation (3.6) the expression

$$\xi = \xi_0 \left\{1 + A_0 \cos^2\left(\frac{2\pi\nu_s n}{m_0} + \delta\right)\right\}^{1/2} \tag{3.7}$$

is commonly used where $A_0 = (S_0/\beta_0)^2$ with S_0 being the amplitude of oscillations of a particle with the non-equilibrium energy.

It is known that the decrease of critical value ξ_{cr} in the presence of modulations is caused by the appearence of additional nonlinear resonances, which are located around the main betatron resonances in the frequency space (so-called side-band resonances). As an example, let us consider the resonance condition for the round beam taking into account the modulation of type (3.1):

$$\nu=\nu_0+\Delta\nu(a)=\frac{m_0k+(p-q+2m)\nu_s}{2n+p+q}. \quad (3.8)$$

Here n, m, p, q, k are integers; k and n determine the main (betatron) resonance and p, q, m characterize the modulation resonances. In the absence of synchrotron oscillations, $p=q=m=0$, therefore, the distance between the neighboring main resonances n and $n+1$ equals $\Delta\nu_n=\frac{m_0}{2n(n+1)}$. Note that if the modulation does not occur but the constant beam shift takes place ($\nu_s=0$; $A_s\neq0$) additional resonances between the main ones arize, too. The condition $p+q=\pm1$ corresponds to the nearest resonance of this kind, and the distance between this resonance and the main one, n, equals $(\Delta\nu)_s=\frac{m_0}{2n(2n+1)}$ which is twice less than between the main resonances (for $n\gg1$). When synchrotron oscillations are taken into account, $\nu_s\ll1$, $A_s\neq0$, the distance from the main resonance and the nearest side-band resonance is

$$(\Delta\nu)_{ns}\approx\frac{\nu_s}{n}\ll\nu_0 .$$

The amplitude of side-band resonances is known to decrease with an increase of p, q, m (at $A_s\ll\sigma$). However, since the distance between these side-band resonances is very small they can overlap at the value of ξ_{cr} which is less than it is necessary for the overlapping of the main resonances. In this case, the overlapping of side-band resonances creates a slow diffusion. If many of such resonances overlap simultaneously, it can lead to the diffusion between the main resonances and, therefore, to the noticeable decrease of ξ_{cr}.

To obtain some analytical estimates of the critical value ξ_{cr}, one need to know the dependence of nonlinear frequency shift $\Delta\nu(a)$ on the transverse energy. For $A_s\ll\sigma$ this quantity $\Delta\nu(a)$ is approximately the same as in the absence of modulations [13]:

$$\Delta\nu_1(a)=\frac{m_0\xi}{a}\{1-e^{-a}I_0(a)\}, \quad (3.9a)$$

$$\Delta\nu_2(a)=\frac{m_0\xi}{2a}\{1-e^{-a}I_0(a)\}+\frac{m_0\xi}{4\sqrt{a}}\int_0^a\frac{dz}{z^{3/2}}[1-e^{-z}I_0(z)] \quad (3.9b)$$

for the round and the flat beam, correspondingly. Here $a=(x_m/2\sigma)^2$ is the dimensionless transverse energy of a particle and $I_0(a)$ is the modified Bessel function. At $a\ll1$ we get the known linear shift $\Delta\nu\approx m_0\xi$, while for $a\gg1$ the shift $\Delta\nu$ vanishes as

$$\Delta\nu_1\approx\frac{m_0\xi}{a};\quad \Delta\nu_2\approx\frac{2m_0\xi}{\sqrt{2\pi a}}. \quad (3.10)$$

One interesting conclusion was made [13] from the above expressions. It is seen that for the same energy (or displacement) the order n of resonances for the round beam is larger than for the flat beam. This allows to expect that modulations are more dangerous for the flat beam. This conclusion is in a good agreement with numerical simulations performed in [8, 13].

Detailed analysis of the resonance structure of Hamiltonian (3.2) for the modulations (3.1), (3.5) and (3.6) have shown [9, 13] that the modulation of the betatron phase shift (see (3.5)) seems to be the most dangerous in comparison with other types of modulation. This conclusion is based on the fact that for the modulation (3.5) the amplitudes of side-band resonances are of the same order in some frequency range unlike the modulation (3.1) where the amplitude of side-band harmonics decreases apart from the main resonance. Nevertheless, in real situation the result depends on the specific values of parameters. For example, for the case of VEPP-2M the β-funcion at the interaction point was quite large and numerical data indicate [13] that the tune shift modulation (3.5) turns out to be less important compared to the modulation (3.1) which is caused by the presence of Ψ-function.

It was also found in [13] that the joint effect of a few modulations is not a trivial one. In particular, the result of the influence of two modulations (3.1) and (3.5) have been numerically investigated. According to [13] the modulation (3.1) with $A_s=0.46$ and $\nu_s=0.01$ causes in the decrease of ξ_{cr} from 0.2 to 0.1, i. e. by a factor 2. To compare with, the modulation (3.5) with $B=0.005$ changes ξ_{cr} from 0.28 to 0.19, i. e, approximately by a factor of 1.5. The combined effect of these two modulations has found to decrease ξ_{cr} to the value $\xi_{cr}=0.05$, i. e. approximately by a factor of 6. Hence, both these modulations decrease the stochasticity threshold ξ_{cr} independently unlike other cases where one of the most important modulation essentially determines ξ_{cr} (see details in [13]). This result is explained by the difference in the resonance structure of perturbation for these modulations. More precisely, each of side-band resonances, caused by the modulation (3.5), turns out to be splitted by the sets of additional side-band resonances in the presence of the modulation (3.1). This simplifies very much the overlapping of resonances.

In spite of the fact that above analysis is made for the simplest one-dimensional model, these data may be used in real situations when one of the transverse direction is the most important in the restriction of the luminosity. Such situations are quite common in case of elliptical beams with a large aspect ratio, $\varkappa \equiv \sigma_x/\sigma_z \gg 1$, where σ_x and σ_z are transverse sizes of a beam with a bi-Gaussian distribution of charge density $\rho = \rho_0 \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2}\right)$ (see, e. g. [4—6]). In the case of nearly round beam, $\varkappa \approx 1$, the situation seems to be more complicated.

4. STOCHASTISITY THRESHOLD FOR THE ROUND BEAM ON THE MAIN COUPLING RESONANCE

The critical value ξ_{cr} found in the previous section for one-dimensional models is expected to be overestimated because of not taking into account additional resonances due to other transverse coordinate. At least, this is true for the elliptical beam; as for the round beam, it is easy to show that the motion remains one-dimensional when operating on the main coupling resonance $\nu_x = \nu_z$ (with the additional condition $\beta_x = \beta_z$). Nevertheless, this situation is not realistic provided some deviations in the values of ν_x, ν_z or for other reasons (see further). Here we discuss some results of the investigation made in [1] to clear up the importance of the second degree of freedom in case of round beam.

The model of particle-beam interaction has the following approximations:

1) the bunch is short (thin lens approximation);

2) the interaction points are spaced in one period of the magnetic system;

3) the damping is not taken into account as well as quantum fluctuations, i. e. only fast effects are considered compared to the damping time;

4) the linear coupling is taking into account corresponding to the presence of skew quadrupoles.

We use here the dimensionless variables $X = x/\sigma$ and $P_x = p_x\beta_x/\sigma$, correspondingly, $Z = z/\sigma$; $P_z = p_z\beta_z/\sigma$. Then the kick perturbation is described by the mapping

$$X_2 = X_1; \quad P_{x_2} = P_{x_1} + \mathcal{F}_{x_1},$$
$$Z_2 = Z_1; \quad P_{z_2} = P_{z_1} + \mathcal{F}_{x_2}, \tag{4.1}$$

where

$$\mathcal{F}_x = -4\pi\xi_x X \mathcal{F}(R); \quad \mathcal{F}_z = -4\pi\xi_z Z \mathcal{F}(R);$$
$$\mathcal{F}(R) = \frac{1-\exp(-R^2/2)}{R^2/2}; \quad R^2 = X^2 + Z^2. \tag{4.2}$$

The parameter ξ is equal $\xi_{x,z} = \frac{Nr_0\beta_{x,z}}{4\pi\gamma\sigma^2}$ (see (2.2)), therefore, the condition $\xi_x/\beta_x = \xi_z/\beta_z$ is assumed.

The transformation for the free betatron rotation between the interaction points and the action of the skew quadrupole has the form

$$X_3 = X_2 \cos\mu_x + P_{x_2}\sin\mu_x; \quad Z_3 = Z_2\cos\mu_z + P_{z_2}\sin\mu_z;$$
$$P_{x_3} = -X_2\sin\mu_x + P_{x_2}\cos\mu_x; \quad P_{z_3} = -Z_2\sin\mu_z + P_{z_2}\cos\mu_z;$$
$$X_4 = X_3; \quad Z_4 = Z_3;$$
$$P_{x_4} = P_{h_3} + M\beta_x Z_3; \quad P_{z_4} = P_{z_3} + M\beta_z X_3; \tag{4.3}$$

where $\mu_{x,z}$ is the betatron phase advance between the interaction points, $\mu_{x,z} = 2\pi\nu_{x,z}/m_0$, M is the strength of skew quadrupole providing the linear coupling between the transverse oscillations.

The linear stability condition in the presence of additional skew quadrupole coupling can be shown to have the form [1]:

$$|b_x + b_z \pm \sqrt{(b_x - b_z)^2 + g^2}| < 4,$$
$$g^2 = 4M^2\beta_x\beta_z \sin\mu_x \sin\mu_z, \tag{4.4}$$
$$b_x = 2\cos\mu_x - 4\pi\xi_x \sin\mu_x; \quad b_z = 2\cos\mu_z - 4\pi\xi_z\sin\mu_z.$$

Near the main coupling resonance ($\mu_x \approx \mu_z \approx \mu$) this relation (4.4) can be written in the more simple form (for $\beta_x = \beta_z = \beta$ and $\xi_x = \xi_z = \xi$)

$$2\,\mathrm{ctg}\,\frac{\mu}{2} > 4\pi\xi \mp M\beta > -2\,\mathrm{tg}\,\frac{\mu}{2}; \quad \text{for} \quad \sin\mu > 0;$$

$$-2\,\mathrm{tg}\,\frac{\mu}{2} > 4\pi\xi \mp M\beta > 2\,\mathrm{ctg}\,\frac{\mu}{2}; \quad \text{for} \quad \sin\mu<0;$$

This expression shows (see Fig. 4)) that in case of $M\neq 0$ the linear stability border becomes stronger as compared to the one-dimensional case.

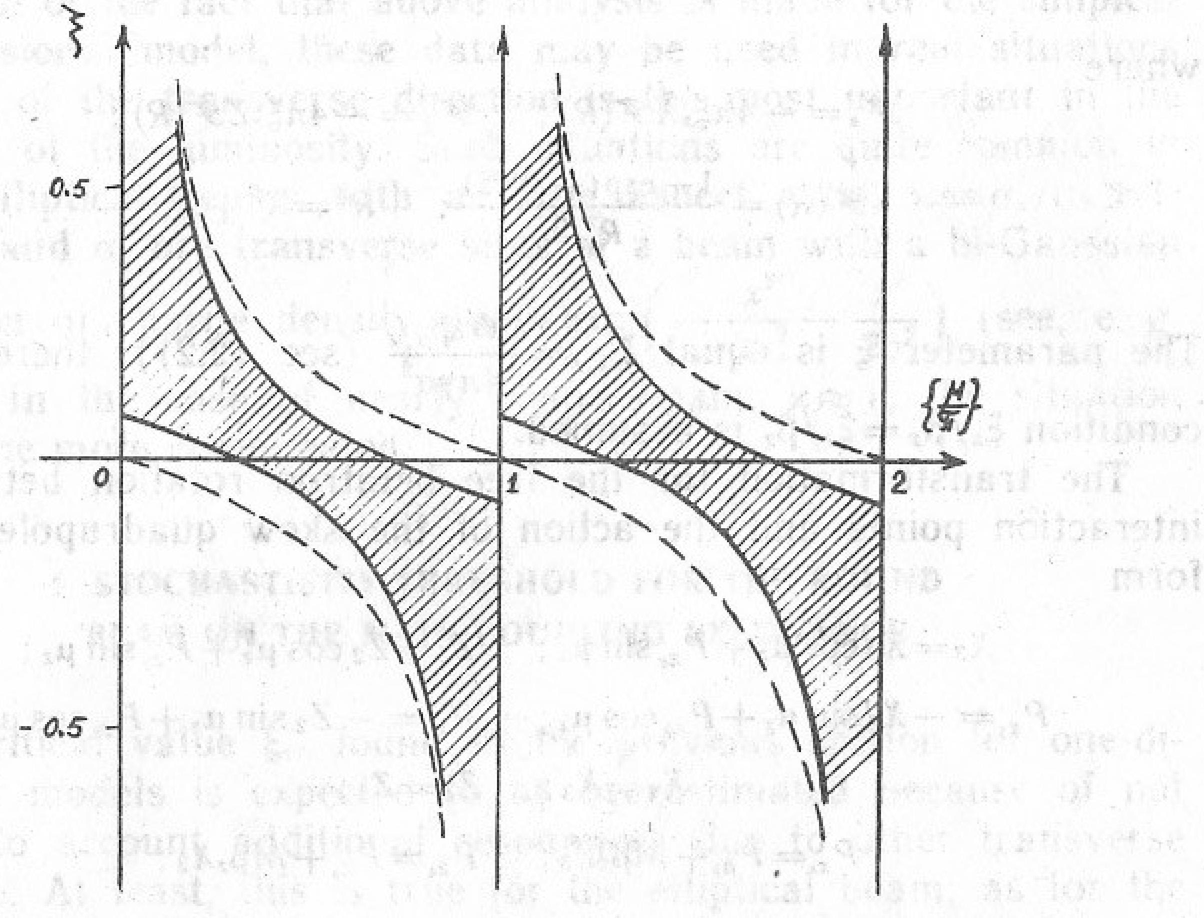

Fig. 4. Linear stability border (4.5) for the round beam with linear coupling $M\beta=1$ (dashed area). The dotted curves correspond to $M=0$; ($\xi<0$ corresponds to the beams of the same charge).

As it is known, the study of four-dimensional mapping is much more difficult in comparison with two-dimensional mappings of type (2.1). The main problem is that the resonances in four-dimensional phase space are not visible in a two-dimensional projection. There is special approach (see, e. g. [14]) which consists of the construction of the so-called Poincare section but it is quite complicated. In [1] another rough procedure has been used where the time dependence of transverse energy

$$W=\sqrt{X^2+Z^2+P_x^2+P_z^2} \tag{4.6}$$

have been examined. In the case when the stochasticity threshold does not exceed, this quantity shows restricted ocsillations. In the

opposite case of stochastic diffusion the energy W is increasing in time. This fact has been used in [13] to define the critical value ξ_{cr} as the lowest value of ξ for which the energy W is increased by $\Delta W\geqslant 2$ compared to the initial energy. To reduce the fluctuations, W was averaged over some time $\Delta t=1000$ (in the number of kicks) with the total time of the motion of a particle $t_m=10^5$ (this corresponds to the damping time in VEPP-2M). Typical dependence of the averaged W is presented in Fig. 5 for a few different initial conditions of a particle.

First we consider the case when the coupling is only due to the interaction of a particle with a strong beam ($M=0$). If $\beta_x=\beta_z=\beta$, therefore, $\xi_x=\xi_z=\xi$, then the only difference from the one-dimensional case is related to non-equal values ν_x, ν_z, which corresponds to real situation in storage rings because of technical reasons. Assuming that $\varepsilon\equiv\nu_z-\nu_x\ll 1$ the energy exchange between transverse degrees of freedom appears, which may lead to the decrease of ξ_{cr} because of the interaction of beam coupling resonance ($\nu_x=\nu_z$) with one-dimensional resonances over x and z directions. The main distinction of the nonlinear coupling resonance (beam coupling resonance, $\xi\neq 0$; $M=0$) from the pure linear resonance $\xi=0$; $M\neq 0$) is a strong dependence of motion on initial conditions. In particular, the degree of exchange between x and z depends on how the initial energy is shared between two degrees of freedom at a given detuning ε. For this reason, it is necesserily to consider the different initial conditions X_0, Z_0, P_{x_0} P_{z_0} to get clear conclusions.

The result of numerical investigation is presented in Fig. 6 where ξ_{cr} is plotted versus the detuning ε when linear coupling is absent, $M=0$. The curve *1* corresponds to the case $X_0\geqslant 2$; $Z_0\approx 0.02\ll 1$; $P_{x_0}=P_{z_0}=0$, when the initial energy is mainly concentrated in one degree of freedom. The other case (curve *2*) represents the initial conditions under which the initial energy is shared equally between two degrees of freedom ($X_0\approx Z_0\geqslant 1$; $P_{x_0}=P_{z_0}=0$). As it was noted, the decrease of ξ_{cr} at large detuning is caused by the two-dimensional character of motion. However, if the detuning is smaller than the size of nonlinear beam resonance, clearly seen in Fig. 6, it is natural to expect that the stochasticity threshold is mainly determined by one-dimensional resonance structure. The dependence of ξ_{cr} on the detuning ε (Fig. 6) can be qualitatively explained by the peculiarities of coupling resonance. Indeed, as it is seen from Fig. 6, for the initial conditions, corresponding to the curve *1* the resonance dependence is sharply asymmetric with the

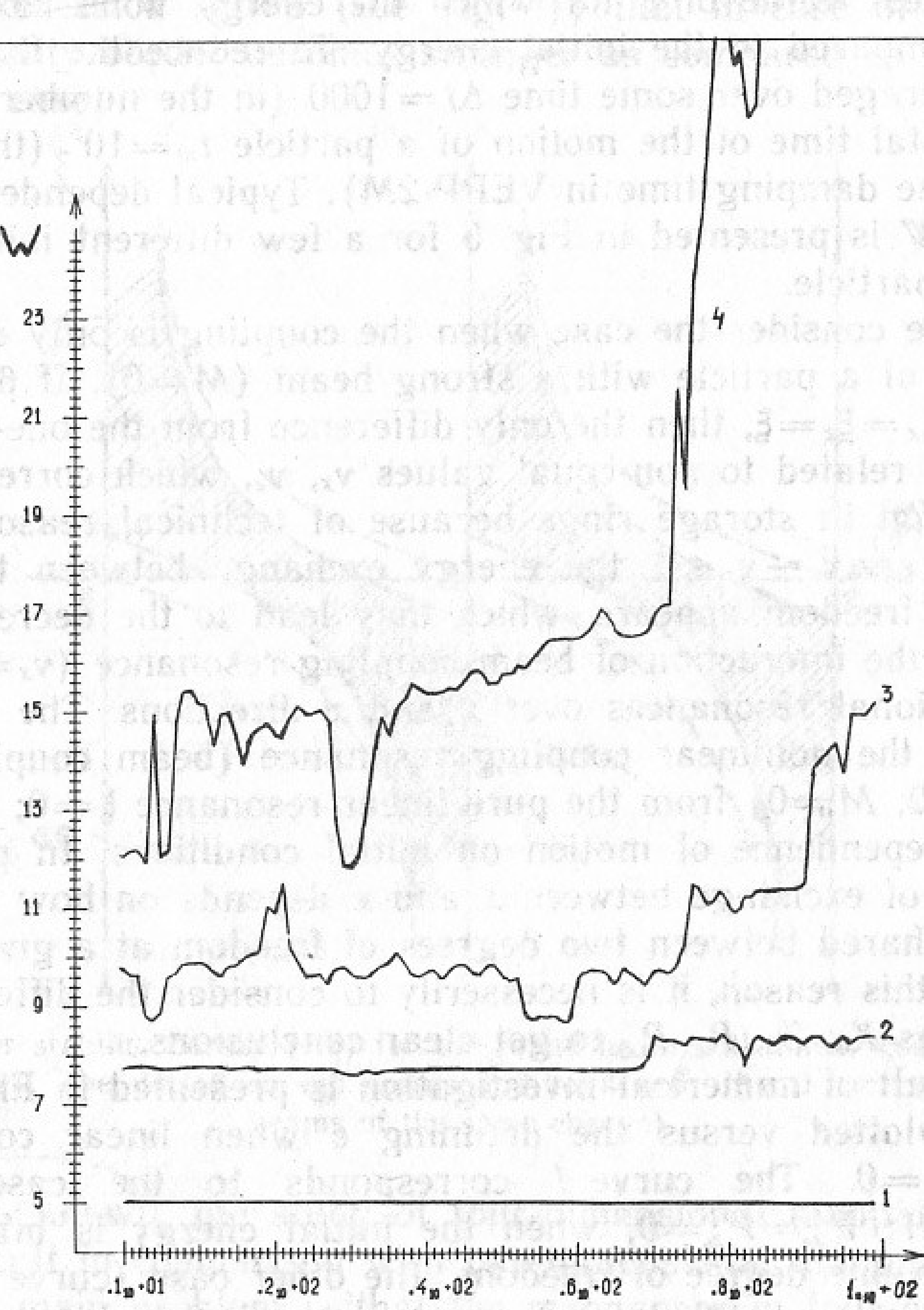

Fig. 5. The dependence of transverse energy W on time for the parameters $\{\mu_x/\pi\}=0.0785$; $\{\mu_z/\pi\}=0.0815$; $\xi_x=0.18$; $\xi_z=0.18$. Initial conditions are $X_0=1.42$; $Z_0=1.40$ (curve *1*); $X_0=2.82$; $Z_0=2.80$ (curve *2*); $X_0=4.22$; $Z_0=4.20$ (curve *3*); $X_0=5.62$; $Z_0=5.60$ (curve *4*) In all cases $P_{x_0}=P_{z_0}=0$.

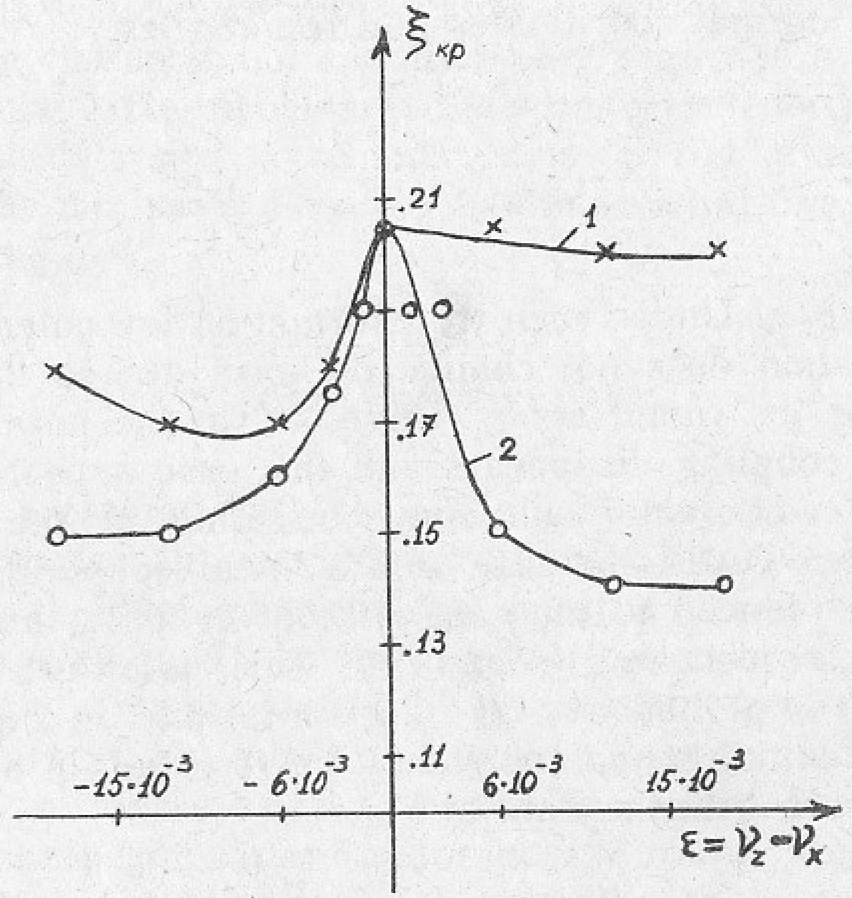

Fig. 6. The dependence of ξ_{cr} on the detuning $\varepsilon=\nu_z-\nu_x$ for $M=0$; $\{\mu_z/\pi\}=0.08+\varepsilon/2$; $\{\mu_x/\pi\}=0.08-\varepsilon/2$. The initial conditions are $2.0\leqslant X_0\leqslant 6.0$; $Z_0=0.02$; $P_{x_0}=P_{z_0}=0$ (curve *1*); $1.4\leqslant X_0\leqslant 4.2$; $Z_0=X_0+0.02$; $P_{x_0}=P_{z_0}=0$.

absence of energy exchange between x and z for $\nu_z>\nu_x$ and $X_0\gg Z_0$. This may be understood by the fact that in this case the shift of betatron frequency over x is much smaller than over z. Hence, for $\nu_z>\nu_x$ (the right-hand of curve *1*) the operating point $(\nu_z+\Delta\nu_z,\ \nu_x+\Delta\nu_x)$ shifts away from the coupling resonance remaning the motion to be almost one-dimensional one. The contrary is the case $\nu_z<\nu_x$ when the operating point shifts to the coupling resonance resulting in two-dimensional character of motion. In the case when the initial energy is shared between x and z, the betatron frequences shift along the coupling resonance and the resonance curve *2* appears to be symmetric.

Another question is how the linear coupling affects the critical value of ξ_{cr} when the unperturbed betatron frequences are equal, $\nu_x=\nu_z=\nu$. The result of numerical simulation is shown in Fig. 7 where $M\neq0$. The curves *1* and *2* correspond, as in Fig. 6, to the different initial conditions. It should be noted that additional linear resonances $\nu_x\pm\nu_z=k$ with $k\neq0$ also appear. However, the influence of these resonances near the main coupling resonance $\nu_x=\nu_z$ can be neglected. To discuss this case with $M\neq 0$, it is convenient to pass to the normal coordinates and frequencies:

$$Y_1=\frac{1}{\sqrt{2}}(X+Z)\,;\qquad Y_2=\frac{1}{\sqrt{2}}(X-Z)\,;$$

$$\nu_1=\nu+\frac{\Delta\nu_{min}}{2};\qquad \nu_2=\nu-\frac{\Delta\nu_{min}}{2};\qquad (4.7)$$

with $\Delta\nu_{min}=M\beta/\pi$. Under such transformation the potential V of the beam perturbation does not change its form due to the symmetric expressions for the round beam. Therefore, in the normal coordinates the linear coupling disappears and this case appears to be similar to the previous one with some rescaling of the frequencies, $\nu_1\neq\nu_2$. In other words, the case with a linear coupling ($M\neq 0$) for $\nu_x=\nu_z$ can be reduced to the case without coupling but with different normal frequencies ν_1 and ν_2. This is why the quantity $M\beta/\pi=\nu_1-\nu_2$ is plotted over the horisontal axis in Fig. 7. Correspondingly, the initial conditions should be rescaled, in accordance to (4.7), to compare with the data of Fig. 6.

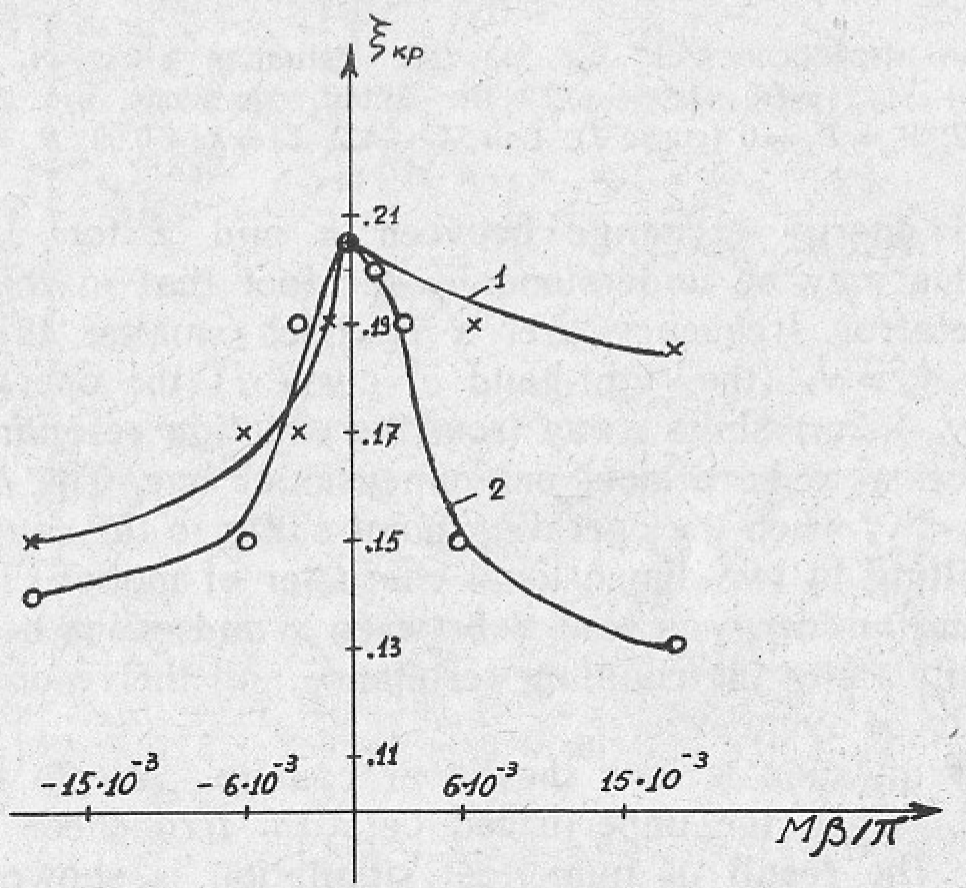

Fig. 7. The dependence of ξ_{cr} on linear coupling $M\beta/\pi$ for $\{\mu_{x,z}/\pi\}=0.08$. The initial conditions are $2.0\leqslant X_0\leqslant 6.0$; $Z_0=0.02$; $P_{x_0}=P_{z_0}=0$ (curve *1*); $1.4\leqslant X_0\leqslant 4.2$; $Z_0=X_0+0.02$; $P_{x_0}=P_{z_0}=0$ (curve *2*).

From the comparison Fig. 6 with Fig. 7 a good qualitative correspondence is seen which supports the above analysis. It should be noted that the chosen initial conditions are regarded as typical ones, representing the extreme cases of the transverse energy sharing. As a result, these data give some indications for the values ε and M to expect that the motion is close to one-dimensional one with a relatively high value of ξ_{cr}.

5. CONCLUDING REMARKS

The data presented here show that detuning ε and linear coupling M needed for the stochasticity limit ξ_{cr} to be determined by one-dimensional effects are not too small and can be achieved in modern storage rings. The more detailed discussion of the perspectiveness of using a magnetic structure with equal betatron frequences and β-functions at the interaction points is presented in [1, 2, 7]. Nevertheless, all these results should be regarded as preliminary ones. The beam-beam dynamics for nearly round beams is studied much less than for ellyptical beams with the large aspect ratio $\varkappa\gg 1$. One of the interesting question is about the diffusion along a coupling resonance which is the most important for the round beams. As it was shown in [15—16], modulation of the parameters of a model gives rise to the thick layers surrounding the main coupling resonance. Though these layers are small compared to the size of resonance itself and does not produce strong diffusion across the resonance, it leads to the diffusion along the resonance which may be dangerous in some situations. In any case, it is a serious problem for the proton machines where the life time is very large and all the weak diffusion processes are to be taken into account including the Arnold diffusion.

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