ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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VACUUM POLARIZATION AND QUADRUPOLE MOMENT OF A HEAVY NUCLEUS

PREPRINT 90-2

НОВОСИБИРСК

Vacuum Polarization and
Quadrupole Moment of a Heavy Nucleus

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ABSTRACT

The contribution of the vacuum polarization to the quadrupole moment of a heavy nucleus is considered. The leading term is obtained exactly in $Z\alpha$, using the electron Green function in the Coulomb field. This term contains the large logarithm of the ratio $\lambda\!\!\!^{-}/R$, where R is the nucleus radius. The spatial distributions of the induced charge and potential are discussed also.

The vacuum polarization gives rise to measurable shifts of energy levels in atoms. Until recently only the spherically-symmetric part of the induced vacuum charge distribution was taken into account in the analysis of such shifts. A lot of papers is devoted to the study of this phenomenon (see, e. g., the review [1] and literature cited therein). In particular, the modification of the Coulomb potential due to the vacuum polarization was considered for a nucleus with the charge $Z|e|$ (e is the charge of the electron, $\alpha = e^2 = 1/137$ is the fine structure constant; we set $\hbar = c = 1$). In the pioneer work [2] the Laplace transform of the product $\rho(r)r^2$, where $\rho(r)$ is the vacuum charge density, was found exactly in $Z\alpha$. In the work [3] the density $\rho(r)$ itself was determined. The $\rho(r)$ behavior at small distances was studied in Ref. [4] by operator methods. The numerical calculations of some particular contributions to the vacuum polarization were undertaken also (see [1]). The induced charge potential was used for the calculation of the shifts of energy levels (see, e. g., [5]).

At the same time, heavy nuclei exist owning large multipole moments. The field of these nuclei may induce corresponding moments in the vacuum. Contrary to the full induced charge, which is zero due to electro-neutralness of the vacuum, higher multipole induced moments can be (and, in fact, are) nonzero. Moreover, the leading contribution to an induced multipole moment in the limit $R/\lambda\!\!\!^{-} \to 0$ is proportional to the large logarithm $\ln(\lambda\!\!\!^{-}/R)$ of the ratio of the electron Compton wavelength $\lambda\!\!\!^{-}$ to the nucleus radius R. We shall discuss the last statement further.

In the previous paper [6] we have calculated the logarithmic contribution to the induced magnetic moment of a nucleus exactly in $Z\alpha$. This contribution grows considerably with $Z\alpha$. When $Z\alpha=\sqrt{3}/2$, the induced magnetic moment formally turns out to be infinite. As will be discussed further, this infinity transforms into one more logarithm $\ln(\lambda\!\!\!^{-}/R)$ when the finiteness of R is taken into account more accurately. The present paper is devoted to the analogous calculation for the electric quadrupole moment.

The quadrupole part of a nucleus electrostatic potential is of the form:

$$\varphi=Q_{ij}\frac{n_i n_j}{2r^3}, \qquad (1)$$

where Q_{ij} is the tensor of the nucleus quadrupole moment, $\vec{n}=\vec{r}/r$. The corresponding electric field induces the charge distribution in the vacuum of electrons:

$$\rho(\vec{r})=-ie\int\frac{d\varepsilon}{2\pi}\,\mathrm{Tr}\,\gamma_0\,G(\vec{r},\vec{r}\,|\,\varepsilon), \qquad (2)$$

where $G(\vec{r},\vec{r}\,'\,|\varepsilon)$ is the electron Green function, which we present as follows:

$$G(\vec{r},\vec{r}\,'\,|\,\varepsilon)=\langle\vec{r}\,|\frac{1}{\gamma_0\left(\varepsilon+\frac{Z\alpha}{r}-e\varphi\right)-\vec{\gamma}\vec{p}-m}|\,\vec{r}\,'\rangle, \qquad (3)$$

where γ_μ are the Dirac matrices. According to the Feynman rules, the contour of integration over the energy ε in (2) goes from $-\infty$ to $+\infty$ below the real axis in the left half-plane of the variable ε and above the axis in the right one. The quadrupole moment due to the vacuum charge distribution (2) is

$$\tilde{Q}_{ij}=\int d\vec{r}\,\rho(\vec{r})\,r^2(3n_i n_j-\delta_{ij}). \qquad (4)$$

Evidently, this tensor is proportional to Q_{ij}: $\tilde{Q}_{ij}=qQ_{ij}$. Expanding the Green function G with respect to φ and taking the linear term we get from (2) and (4) the following expression for the coefficient q:

$$q=\frac{-i\alpha}{10\pi}\int d\varepsilon\int\frac{d\vec{r}\,d\vec{r}\,'}{r^3}(r')^2P_2(x)\,\mathrm{Tr}\,\gamma_0 G_c(\vec{r},\vec{r}\,'\,|\,\varepsilon)\gamma_0\,G_c(\vec{r}\,',\vec{r}\,|\,\varepsilon), \qquad (5)$$

where G_c is the electron Green function in the Coulomb field,

$x=\vec{n}\,\vec{n}\,'$, and $P_2(x)=(3x^2-1)/2$ is the Legendre polynomial. As will be shown below, the first term of the expansion of the renormalized quantity q with respect to $Z\alpha$ is proportional to $(Z\alpha)^2$. Therefore we have to subtract from the integrand for q in (5) the value of this integrand at $Z=0$. In the following such a subtraction is implicitly assumed and we take it into account in the explicit form in the final result. After this subtraction we, nevertheless, have to regularize the integral in (5) since it diverges logarithmically at small distances. We perform regularization introducing the ultraviolet cut-off equal to the nucleus radius R. All further calculations are carried out with the logarithmic accuracy. Therefore one can set the electron mass m in (5) to be equal to zero, cutting off the large distance radial integration at $\lambda\!\!\!^{-}$. Making in such a way we obtain the coefficient at the logarithm exactly in $Z\alpha$.

Using the analytic properties of the Green function we deform the contour of integration over ε in (5) so that it coincides finally with the imaginary axis. Using the formula (19) of Ref. [7] we obtain the following expression for the Green function at $m=0$ and $\varepsilon=iE$:

$$G_c(\vec{r},\vec{r}\,'\,|\pm i\,|E|)=-\frac{1}{4\pi r\,r'}\int_0^\infty ds\,\exp[\pm 2iZ\alpha s-|E|(r+r')\,\mathrm{cth}(s)]\times$$

$$\times\sum_{l=1}^{\infty}\Big\{\gamma_0(1-\vec{\gamma}\vec{n}\cdot\vec{\gamma}\vec{n}')\left[\pm i\frac{y}{2}I'_{2\nu}(y)+Z\alpha\,\mathrm{cth}(s)\,I_{2\nu}(y)\right]B(x)\pm$$

$$\pm(1+\vec{\gamma}\vec{n}\cdot\vec{\gamma}\vec{n}\,')\gamma_0 il\,A(x)\,I_{2\nu}(y)-$$

$$-i\left[\frac{|E|(r-r')}{2\mathrm{sh}^2(s)}\,(\vec{\gamma},\vec{n}+\vec{n}\,')B(x)+lA(x)\,\mathrm{cth}(s)\,(\vec{\gamma},\vec{n}-\vec{n}\,')\right]I_{2\nu}(y)\Big\}. \qquad (6)$$

Here $I_{2\nu}(y)$ is the modified Bessel function of the first kind, $\vec{n}=\vec{r}/r$, $\vec{n}\,'=\vec{r}\,'/r$, $y=2|E|\sqrt{rr'}/\mathrm{sh}\,(s)$, $x=\vec{n}\,\vec{n}'$,

$$A(x)=\frac{d}{dx}(P_l(x)+P_{l-1}(x)),\qquad B(x)=\frac{d}{dx}(P_l(x)-P_{l-1}(x)),$$

P_l are the Legendre polynomials, $\nu=\sqrt{l^2-(Z\alpha)^2}$. Proceeding from the integration over r to that over $r|E|$, one can easily perform the integration over E which gives the logarithm mentioned above. It is convenient to represent (5) in the following form:

$$q=\frac{\alpha}{30\pi}\ln\left(\frac{\lambda\!\!\!^{-}}{R}\right)f(Z\alpha)\,. \tag{7}$$

Taking trace over γ—matrices and integrating over directions $\vec{n}$ and $\vec{n}\,'$, we obtain:

$$f=24\int_0^\infty\frac{dr}{r^3}\int_0^\infty(r')^2\,dr'\int_0^\infty ds\int_0^\infty ds'\times$$

$$\times\sum_{\sigma=\pm1}\exp\left[2i\sigma Z\alpha T-(r+r')\left(\operatorname{cth}(s)+\operatorname{cth}(s')\right)\right]\times$$

$$\times\sum_{l,l'=1}^{\infty}\left\{-\left[\frac{y}{2}I'_{2\nu}(y)-i\sigma Z\alpha\operatorname{cth}(s)\,I_{2\nu}(y)\right]\cdot\left[\begin{matrix}s\to s'\\ l\to l'\end{matrix}\right]+\right. \tag{8}$$

$$\left.+\left[\frac{(r-r')^2}{4\operatorname{sh}^2(s)\operatorname{sh}^2(s')}-ll'(-)^{l+l'}\left(1-\operatorname{cth}(s)\operatorname{cth}(s')\right)\right]I_{2\nu}(y)\,I_{2\nu'}(y')\right\}\Delta_{ll'},$$

where $y=2\sqrt{rr'}/\operatorname{sh}(s)$, $y=2\sqrt{rr'}/\operatorname{sh}(s')$, $T=s+s'$, $\nu=\sqrt{l^2-(Z\alpha)^2}$ and

$$\Delta_{ll'}=\frac{l^3-l}{4l^2-1}\delta_{l,l'}+\frac{6l(l+1)}{(4l^2-1)(2l+3)}\delta_{l',l+1}+\frac{3l(l+1)(l+2)}{(2l+1)(2l+3)}\delta_{l',l+2}. \tag{9}$$

We used in (8) the symmetry with respect to the permutation $l\leftrightarrow l'$. So, here we have «nondiagonal» transitions of two types: with $l'=l+1$ and $l'=l+2$, together with «diagonal» ones ($l'=l$).

After integration by parts in terms, proportional to $i\sigma\cdot Z\alpha$, we introduce the variables $T=s+s'$, $y=2\sqrt{rr'}/\operatorname{sh}(s)$, $y=2\sqrt{rr'}/\operatorname{sh}(s')$ and $u=\sqrt{r'/r}$. The integral takes the form:

$$f=12\int_0^\infty y\,dy\int_0^\infty y'\,dy'\int_0^\infty dT\int_0^\infty u^4\,du\,\cos(2Z\alpha T)\times$$

$$\times\exp\left[-\frac{1}{2}\left(u+\frac{1}{u}\right)D\right]\sum_{l,l'=1}^{\infty}\Delta_{ll'}I_{2\nu}(y)\,I_{2\nu'})(y')\left\{-\frac{1}{D^3}-\right.$$

$$\left.-\frac{\left(u+\frac{1}{u}\right)}{2D^2}+\frac{\left(u-\frac{1}{u}\right)^2}{4D}+\frac{4ll'(-)^{l+l'}}{D^3}\left[2+\left(\frac{y}{y'}+\frac{y'}{y}\right)\operatorname{ch}(T)\right]\right\}, \tag{10}$$

where $D=[y^2+(y')^2+2yy'\operatorname{ch}(T)]^{1/2}$.

The further way of integration is similar to that in Ref. [6]. Finally we represent f in the form

$$f(Z\alpha)=12\int_0^1\frac{dx}{x^4}(1-x)^2\left(\frac{4}{x}-1\right)\sum_{l,l'=1}^{\infty}\{\Phi(x,Z\alpha)-\Phi(x,0)\}\,\Delta_{ll'},$$

$$\Phi(x,Z\alpha)=\left(1+\frac{ll'}{3}(-)^{l+l'}\right)B^{2}_{2\nu,2iZ\alpha}(x)\,B^{2}_{2\nu',2iZ\alpha}(x)+$$

$$+\frac{ll'}{6}(-)^{l+l'}\operatorname{Re}\left[B^{1}_{2\nu,1+2iZ\alpha}(x)\,B^{3}_{2\nu',1+2iZ\alpha}(x)+B^{3}_{2\nu,1+2iZ\alpha}(x)\,B^{1}_{2\nu',1+2iZ\alpha}(x)\right], \tag{11}$$

where

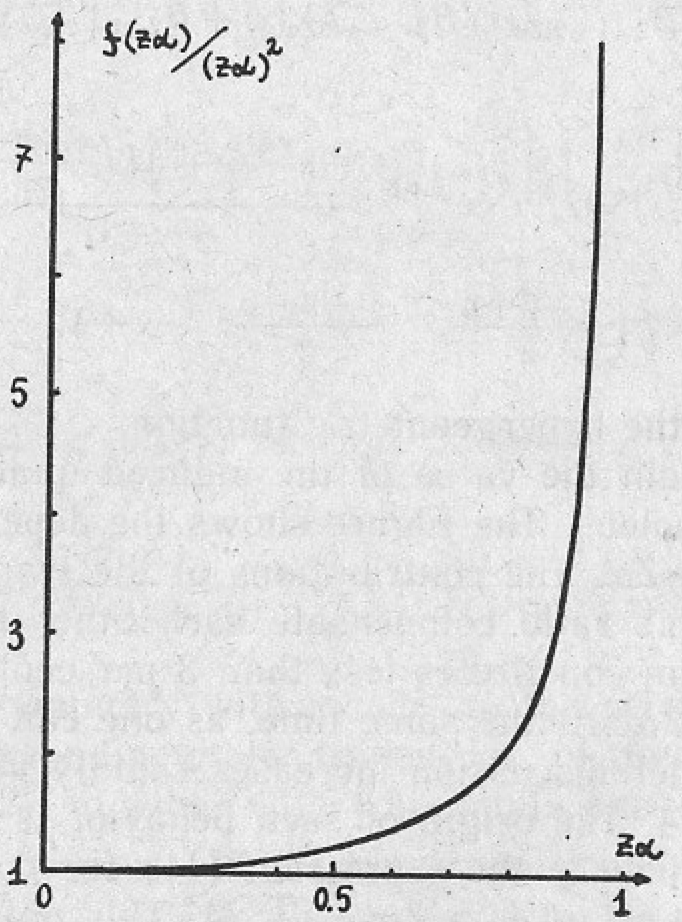
$$B^{\beta}_{\nu,\mu}(x)=2^{\beta-2}\cdot x^{(\nu+\beta)/2}\frac{\Gamma\left(\frac{\nu+\beta+\mu}{2}\right)\Gamma\left(\frac{\nu+\beta-\mu}{2}\right)}{\Gamma(\nu+1)}\times$$

$$\times F\left(\frac{\nu+\beta+\mu}{2},\quad\frac{\nu+\beta-\mu}{2};\quad\nu+1;\quad x\right),$$

$F(a,\,b;\,c;\,x)$ is the hypergeometric function.

Thus we obtain the value of the induced quadrupole moment in the field of a nucleus. The Figure shows the dependence of the ratio $f(Z\alpha)/(Z\alpha)^2$ on $Z\alpha$. The contributions of the transitions with $\Delta l=0$ and $\Delta l=2$ to this ratio compensate each other to the considerable degree. Their sum constitutes less than 3 per cent and slowly varies with respect to $Z\alpha$. At the same time, as one can see from the Figure, the ratio under discussion increases rapidly in the vicinity of the point $Z\alpha=\sqrt{15}/4$. The origin of such behavior is connected with the presence of the pole in the expression (11) for f at $\nu+\nu'=2$, which corresponds to $l=1$, $l'=2$, $Z\alpha=\sqrt{15}/4$. This pole arises due to the singularity in the matrix element of the interaction with the quadrupole potential calculated with the Dirac wavefunctions in the Coulomb field. Indeed, these wavefunctions behave as $r^{\nu-1}$, where $\nu=\sqrt{l^2-(Z\alpha)^2}$, at small distances, and the matrix element of interaction with the potential is proportional to $(\nu+\nu'-2)^{-1}$. Of course, in the vicinity of the point $Z\alpha=\sqrt{15}/4$ the finite size of a nucleus should be taken into account more accurately. One can verify that the formula (11) is valid at $(\nu+\nu'-2)>1/\ln(\lambda\!\!\!^{-}/R)$ and that just at the point $Z\alpha=\sqrt{15}/4$ the divergent integral $\int_0 dx/x$ should be exchanged by $(1/2)\ln(\lambda\!\!\!^{-}/R)$. Thus the net result at the point $Z\alpha=\sqrt{15}/4$ contains the contribution proportional to $\ln^2(\lambda\!\!\!^{-}/R)$.

As we can see, the appearance of the logarithm $\ln(\lambda\!\!\!^{-}/R)$ in the results for induced multipole moments is the quite common fact. It can be easily understood on the dimensional grounds. Really, we calculate the dimensionless quantity (q in the present work, g in Ref. [6]) as some integral, whose integrand is the homogeneous functon of integration parameters (remind, that we work in the limit $m=0$). This integrand is the product of massless propagators and potentials, which are the homogeneous functions of their arguments. A dimensionless integral from a homogeneous function of



The dependence of $f(Z\alpha)/(Z\alpha)^2$ on $Z\alpha$.

dimensionful arguments has to be logarithmically divergent. As was discussed earlier, this divergence is cutting off at R in the ultraviolet region and at $\lambda\!\!\!^{-}$ in the infrared one.

Now we can easily answer the question about the dependence of the potential, induced by the vacuum charge and currents, on r (just this function enters the calculations of energy shifts in atoms). On the same dimensional grounds, the induced charge density is at $R \ll r \ll \lambda\!\!\!^{-}$:

$$\rho(\vec{r}) = \frac{\alpha}{48\pi^2} f(Z\alpha) \frac{Q_{ij} n_i n_j}{r^5}, \tag{12}$$

where $f(Z\alpha)$ is defined in (7). Just this charge density gives rise to the induced quadrupole moment qQ_{ij}, with q from (7), being integrated according to (3) in the range $R \ll r \ll \lambda\!\!\!^{-}$. When $r \gg \lambda\!\!\!^{-}$, the induced density is

$$\rho(\vec{r}) = \frac{4\alpha(Z\alpha)^2}{15\pi^2} \lambda\!\!\!^{-4} \frac{Q_{ij} n_i n_j}{r^9}. \tag{13}$$

The simplest way to obtain this asymptotics is to use the Euler—Heisenberg lagrangian for the electromagnetic field (see, e. g., [8, sect. 129]). The corresponding potential calculated with the logarithmic accuracy has the form:

$$\varphi(\vec{r}) = \frac{\alpha}{30\pi} f(Z\alpha) \frac{Q_{ij} n_i n_j}{2r^3} \ln\left(\frac{r}{R}\right) \quad \text{when} \quad R \ll r \ll \lambda\!\!\!^{-}, \tag{14}$$

and

$$\varphi(\vec{r}) = \frac{\alpha}{30\pi} f(Z\alpha) \frac{Q_{ij} n_i n_j}{2r^3} \ln\left(\frac{\lambda\!\!\!^{-}}{R}\right) \quad \text{when} \quad r \gg \lambda\!\!\!^{-}. \tag{15}$$

Let us discuss now the lowest-order contribution to the induced potentials. Using the standard relation between the bare charge distribution and the induced one in the lowest order [8, sect. 114], it can be easily shown that in the momentum representation the induced magnetic dipole potential due to the bare nucleus magnetic moment $\vec{\mu}$ is:

$$\vec{A}(\vec{k}) = -i\,\vec{\mu} \times \vec{k}\, \frac{\mathscr{P}(-\vec{k}^2)}{(-\vec{k}^2)} \cdot \frac{4\pi}{\vec{k}^2} \tag{16}$$

and the induced electric quadrupole potential is

$$\varphi(\vec{k}) = -\frac{1}{6} Q_{ij} k_i k_j \frac{\mathscr{P}(-\vec{k}^2)}{(-\vec{k}^2)} \cdot \frac{4\pi}{\vec{k}^2} \tag{17}$$

where $\mathscr{P}(k^2)$ is the lowest-order polarization operator. Hence, in the spatial representation we have correspondingly

$$\vec{A}(\vec{r}) = -\vec{\mu} \times \vec{\nabla} \frac{\Phi(r)}{Z|e|} \tag{18}$$

and

$$\varphi(\vec{r}) = \frac{1}{6} Q_{ij} \nabla_i \nabla_j \frac{\Phi(r)}{Z|e|} \tag{19}$$

Here $\Phi(r)$ is the Uehling potential [9]. At small distances $r \ll \lambda\!\!\!^{-}$

this potential behaves like $(2\alpha/3\pi)\ln(\lambdabar/r)$ times the Coulomb potential of a nucleus, and like $(\alpha/4\sqrt{\pi})\exp(-2mr)/(mr)^{3/2}$ times the Coulomb potential, at large distances. Thus, at $r\gg\lambdabar$ the lowest-order contribution to the induced potentials decays exponentially and hence does not operate in the calculations of the induced moments. At the same time, the potentials (18) and (19) have considerable values at small distances.

Discuss now briefly the possibilities of the experimental observation of the effect. The r dependence of the induced potential found by us enables one to calculate the shifts of energy levels in atoms with the logarithmic accuracy. In heavy mu-mesoatoms these shifts are nontrivial functions of quantum numbers for the states with the size of the muonic cloud less than $\lambdabar$. Measuring the hyperfine intervals for different levels, and taking the ratio of these intervals, one can exclude the value of the bare quadrupole moment Q_{ij}, which cannot be derived with enough accuracy from the present theory. The value of this ratio, extracted from the experimental data, can be compared with our results.

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Поляризация вакуума и квадрупольный момент тяжелого ядра

Ответственный за выпуск С.Г.Попов

Работа поступила 27 декабря 1989 г.
Подписано в печать 8.01.90 г. МН 08021
Формат бумаги 60×90 1/16 Объем 0,9 печ.л., 0,8 уч.-изд.л.
Тираж 250 экз. Бесплатно. Заказ № 2

Набрано в автоматизированной системе на базе фотонаборного автомата ФА1000 и ЭВМ «Электроника» и отпечатано на ротапринте Института ядерной физики СО АН СССР,
Новосибирск, 630090, пр. академика Лаврентьева, 11.