ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

I.B. Khriplovich and A.S. Yelkhovsky

ON THE RADIATIVE CORRECTIONS $\alpha^2 \ln \alpha$ TO THE POSITRONIUM DECAY RATE

PREPRINT 90-52

НОВОСИБИРСК

On the Radiative Corrections $\alpha^2 \ln \alpha$ to the Positronium Decay Rate

I.B. Khriplovich and A.S. Yelkhovsky

Institute of Nuclear Physics
630090, Novosibirsk, USSR

ABSTRACT

The radiative corrections $\sim\alpha^2 \ln \alpha$ to the positronium decay rate are calculated in the Breit approximation which is shown to be quite adequate for the problem. For orthopositronium the result coincides with the previous one, for parapositronium it differs from the old results.

1. The problem of radiative corrections to the positronium decay rate is quite serious due to the strong disagreement between the theoretical value for orthopositronium (o-Ps) that includes the corrections of the order α and $\alpha^2 \ln \alpha$ [1—4]

$$\Gamma^{o\text{-}Ps}_{theor} = m\alpha^6 \frac{2(\pi^2-9)}{9\pi}\left[1 - 10.28\frac{\alpha}{\pi} - \frac{1}{3}\alpha^2 \ln\frac{1}{\alpha}\right] = 7.03830\ \mu s^{-1} \quad (1)$$

and the recent experimental value [5],

$$\Gamma^{o\text{-}Ps}_{exp} = 7.0516(13)\ \mu s^{-1}. \quad (2)$$

The measurements of the p-Ps decay rate will also in the near future become sensitive enough to feel the corrections $\sim\alpha^2 \ln \alpha$.

In the present paper we rederive in a way, which looks (at least to us) simple and transparent, the corrections $\sim\alpha^2 \ln \alpha$ to the ortho- and parapositronium (p-Ps) decay rates. In the case of o-Ps our result coincides with the known one [3] (see (1)). So, unfortunately, we cannot say anything substantial about the mentioned disagreement between (1) and (2), although some remarks, hopefully pertinent to the problem, will be made below.

On the other hand, our corresponding result for the p-Ps decay rate is new in the sence that it differs from the previous ones [3, 6].

2. As distinct from the previous calculations [2—4, 6], our approach is not based on any fully relativistic two-body wave equation. It has been noted already [2], that nonanalytic in α corrections arise due to the range of momenta between m and $m\alpha$. Our

observation is that the logarithmic nature of the correction allows us to restrict to the range $m\alpha \lesssim p \ll m$, where we can treat relativistic effects as perturbation. Hence we can consider the corrections $\sim\alpha^2 \ln\alpha$ as such corrections to the wave function of positronium, that logarithmically diverge at small distances and are cut off at $r \sim 1/m$ where the annihilation occurs. These effects can be described as $(v/c)^2$ corrections. The theoretical status of the corresponding Breit Hamiltonian (BH) is absolutely safe.

The part of the BH that corresponds to the relativistic corrections to the dispersion law of the particles and to their Coulomb interaction (see, e. g., Ref. [7]),

$$V_1 = -\frac{\mathbf{p}^4}{4m^3} + \frac{\pi\alpha}{m^2}\delta(\mathbf{r}), \tag{3}$$

can be easily transformed to

$$V_1 \to -\frac{\alpha^2}{4mr^2} - \frac{\alpha}{2m^2r^2}\partial_r. \tag{4}$$

In (4) we retain only those terms which are sufficiently singular at $r \to 0$ to influence the behaviour of the wave function at small distances. Let us note that

$$\partial_r R_{n,l=0}\Big|_{r\to 0} = -\frac{1}{a} R_{n,l=0}\Big|_{r\to 0}, \tag{5}$$

where $R_{n,l=0}$ is the radial wave function of the Coulomb state with the principal quantum number n and vanishing orbital angular momentum l. The Bohr radius a in positronium is $a = 2/(m\alpha)$, so that with our accuracy $V_1(r)$ vanishes. The conclusion that this part of relativistic effects does not work in the $\alpha^2 \ln\alpha$ corrections to the Ps decay rate, was made already in [3].

The next spin-independent term in the BH,

$$V_2 = -\frac{\alpha}{2m^2 r}\left(\mathbf{p}^2 + \frac{1}{r^2}\,\mathbf{r}(\mathbf{r}\mathbf{p})\,\mathbf{p}\right), \tag{6}$$

describes the magnetic electron-positron interaction due to the orbital motion. At $l = 0$ it can be easily transformed to

$$V_2 = -\frac{\alpha}{m^2 r}\left(\mathbf{p}^2 + \frac{1}{r}\,\partial_r\right). \tag{7}$$

Using the same substitution (see (5)), $\partial_r \to -m\alpha/2$, and retaining only the terms singular as r^{-2}, we get

$$V_2 \to -\frac{\alpha^2}{2mr^2}. \tag{8}$$

Now one can show in a straightforward way that the s-wave radial function, instead of being constant at $r \to 0$, behaves as

$$\psi\big|_{r\ll a} \sim \left(\frac{r}{a}\right)^{-\alpha^2/2} \approx 1 - \frac{\alpha^2}{2}\ln(m\alpha r). \tag{9}$$

The corresponding relative correction to the particle density at the distances $r \sim 1/m$, where the annihilation takes place, and therefore to the decay rate itself is

$$\alpha^2 \ln\frac{1}{\alpha}. \tag{10}$$

The spin-orbital part of the BH is irrelevant to our problem since it does not work at all in s-states. The part of the BH that describes the tensor spin-spin interaction does not contribute to the decay rate to the accuracy considered. Indeed, being applied to the s-state, this interaction either annihilates it (in the singlet case) or transforms it (in the triplet one) into the d-state with the same total angular momentum. But the annihilation from a d-state is strongly hampered.

So, we are left with the contact spin-spin interaction

$$V_3 = \frac{\pi\alpha}{m^2}\left[\frac{7}{3}S(S+1) - 2\right]\delta(\mathbf{r}). \tag{11}$$

It originates from both the magnetic spin-spin interaction, and from the one-quantum annihilation contribution which does not vanish in the triplet case only.

We shall solve the corresponding wave equation

$$\left[-\frac{1}{m}\Delta - \frac{\alpha}{r} - E_0 + A\frac{\pi\alpha}{m^2}\delta(\mathbf{r})\right]\psi(\mathbf{r}) = 0;$$

$$A = \frac{7}{3}S(S+1) - 2 = \begin{cases} -2, & S=0 \\ 8/3, & S=1 \end{cases} \tag{12}$$

by iterations. In its turn in the inhomogeneous equation

$$\left(\Delta+\frac{m\alpha}{r}-\frac{m^2\alpha^2}{4}\right)\psi_1(\mathbf{r})=A\frac{\pi\alpha}{m}\,\delta(\mathbf{r})\,\psi_0(0) \qquad (13)$$

we omit the term $-m^2\alpha^2/4$, regular at $r\to 0$, and treat $m\alpha/r$, which is less singular than Δ, as a perturbation. Simultaneously this procedure is also an expansion in α.

The solution of the zeroth order in α,

$$\psi_1^{(0)}=-A\frac{\alpha}{4mr}\,\psi_0(0)$$

at the distances $r\sim 1/m$ of interest to us leads evidently to the correction $\sim\alpha$ to the decay rate. The calculation of the corresponding numerical factor is beyond our accuracy. The first-order solution,

$$\psi_1^{(1)}=A\frac{\alpha^2}{4}\ln(rm\alpha)\,\psi_0(0) \qquad (14)$$

gives evidently at $r\sim 1/m$ the relative correction $A\alpha^2\ln\alpha/4$ to $\psi_0(\mathbf{r})|_{r\sim 1/m}$ and the correction

$$-\frac{A}{2}\alpha^2\ln\frac{1}{\alpha}=\alpha^2\ln\frac{1}{\alpha}\begin{cases}1, & S=0\\ -4/3, & S=1\end{cases} \qquad (15)$$

to $|\psi_0(\mathbf{r})|^2_{r\sim 1/m}$ and to the decay rate.

The total correction to the decay rate includes both (10) and (15) and thus constitutes

$$\alpha^2\ln\frac{1}{\alpha}\begin{cases}2, & S=0\\ -1/3, & S=1\end{cases} \qquad (16)$$

3. One may feel dissatisfied with the above treatment of the spin-spin interaction (11). Have not we buried some contribution $\sim\alpha^2\ln\alpha$ in the linearly divergent, at $r\to 0$, term $\psi_1^{(0)}$? To reject the suspicion we shall calculate the discussed conribution of V_3 in a straightforward way. We shall follow the line of reasoning close to that of Refs [8, 9]. Let us consider the diagram of Figure. Its left vertex corresponds to the interaction (11); in the momentum representation this amplitude is $-A\pi\alpha/m^2$. The right vertex is the annihilation one. In the momentum representation it is, to our accuracy, a constant, correction to which we are looking for. The Coulomb attraction (the dashed vertical line) supplements the imaginary part of the corresponding one-loop diagram by the well-known Coulomb

factor $\pi\alpha/(2v)$ where $2v$ is the relative velocity of the particles. So, the imaginary part of the diagram becomes a constant:

$$\operatorname{Im}M=-A\frac{\pi\alpha}{m^2}\,\frac{v}{16\pi}\,4m^2\frac{\pi\alpha}{2v}=-\frac{\pi\alpha^2}{8}A. \qquad (17)$$

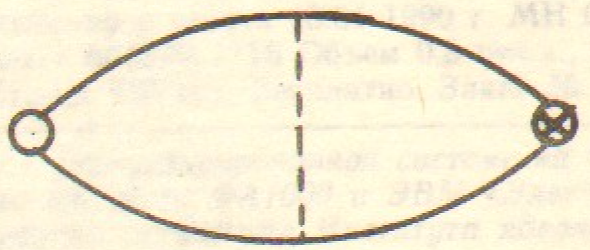
The dispersion integral

$$\operatorname{Re}M=\frac{1}{\pi}\int_0^\infty\frac{dE\,\operatorname{Im}M}{E-E_0}\bigg|_{E_0\approx-\frac{m\alpha^2}{4}}\approx-\frac{A}{4}\alpha^2\ln\frac{1}{\alpha} \qquad (18)$$

is in fact the correction to the annihilation amplitude we are looking for. The correction to the decay rate is evidently twice as large and coincides with (15).

4. Our result for o-Ps coincides with that found earlier. As for the disagreement with earlier results for p-Ps, its origin can be easily elucidated in the case of Ref. [3]. The prescription of the latter paper is stated there explicitly: «For parapositronium, the lifetime calculation is the same as for orthopositronium except that the single-photon annihilation kernels are exactly zero». Indeed, under such a prescription we would get for p-Ps constant A the value $8/3-2=2/3$ instead of -2, and for the factor at the relative correction $\alpha^2\ln\frac{1}{\alpha}$ to the decay rate the value $1-1/3=2/3$, presented in [3]. Our value 2 for this factor arises at the transition from o-Ps to p-Ps, if one not only omits the annihilation kernel, but also modifies correspondingly the contact magnetic interaction.

Concerning the strong disagreement between the theoretical (1) and experimental (2) results for o-Ps, it may be due, to our opinion, to a large and still uncalculated numerical coefficient at $(\alpha/\pi)^2$ correction to the decay rate. Some hint to it can be found in the large, ~ 10, factor at α/π correction. By the way, some large (but far from sufficient) contribution to the $(\alpha/\pi)^2$ correction can



Diagrammatic representation of the $\alpha^2\ln\alpha$ contribution due to the spin-spin contact interaction.

be found directly from this number. Indeed, it arises from the $-5\alpha/\pi$ correction to the annihilation amplitude which, being squared, gives also $25(\alpha/\pi)^2$ correction to the decay rate.

Aknowledgements. Our attention to the problem has been attracted by A. Rich. Great stimulating interest has been shown together with him by A. AlRamadan, R. Conti and D. Gidley. In particular, they informed us of the plans to measure precisely p-Ps decay rate. We also had useful theoretical discussions with S. Brodsky, V.L. Chernyak, M.I. Eides, V.S. Fadin and Yu. Tomozawa. To all of them we owe our sincere gratitude. One of us (I. Kh.) highly appreciates the kind hospitality extended to him at the Theoretical Physics Institute, University of Minnesota, where part of this work has been done.

REFERENCES

1. *A. Ore and J.L. Powell.* Phys. Rev., 75 (1949) 1696.
2. *W.E. Caswell, G.P. Lepage and J.R. Sapirstein.* Phys. Rev. Lett., 38 (1977) 488.
3. *W.E. Caswell and G.P. Lepage.* Phys. Rev., A20 (1979) 36.
4. *G.S. Adkins.* Ann. Phys., 146 (1983) 78.
5. *C.I. Westbrook, D.W. Gidley, R.S. Conti and A. Rich.* Phys. Rev. Lett., 58 (1978) 1328.
6. *Yu. Tomozawa.* Ann. Phys., 128 (1980) 463.
7. *V.B. Berestetsky, E.M. Lifshitz and L.P. Pitaevsky.* Quantum Electrodynamics (Nauka, Moscow, 1980).
8. *R. Barbieri, P. Christillin, and E. Remiddi.* Phys. Rev. A8 (1973) 2266.
9. *A. Owen.* Phys. Rev. Lett. 30 (1973) 887.

I.B. Khriplovich and A.S. Yelkhovsky

On the Radiative Corrections $\alpha^2 \ln \alpha$ to the Positronium Decay Rate

И.Б. Хриплович, А.С. Елховский

Радиационные поправки $\alpha^2 \ln \alpha$ к вероятности распада позитрония

Ответственный за выпуск С.Г.Попов

Работа поступила 30 марта 1990 г.
Подписано в печать 18.04 1990 г. МН 08553
Формат бумаги 60×90 1/16 Объем 0,8 печ.л., 0,7 уч.-изд.л.
Тираж 250 экз. Бесплатно. Заказ № 52

*Набрано в автоматизированной системе на базе фотонаборного автомата ФА1000 и ЭВМ «Электроника» и отпечатано на ротапринте Института ядерной физики СО АН СССР,
Новосибирск, 630090, пр. академика Лаврентьева, 11.*