ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

V.G. Dubrovsky, B.G. Konopelchenko

COHERENT STRUCTURES FOR THE ISHIMORI EQUATION. I. LOCALIZED SOLITONS WITH THE STATIONARY BOUNDARIES

PREPRINT 90-76

НОВОСИБИРСК

Coherent Structures
for the Ishimori Equation.
I. Localized Solitons
with the Stationary Boundaries

V.G. Dubrovsky, B.G. Konopelchenko

Institute of Nuclear Physics
630090, Novosibirsk, USSR

ABSTRACT

The initial-boundary value problem for the Ishimori-II equation is studied. The time evolution of the inverse problem data in the case of nontrivial boundaries is found. General formula for exact solutions of the Ishimori-II equation with the nontrivial boundaries is derived. The localized soliton solutions of the Ishimori-II equation with the time independent boundaries is studied in detail. It is shown that there exist essentially the three different types (*ss*, *sb*, *bb*) of the localized solitons for the Ishimori-II equation. Some explicit examples of such solitons are presented.

1. INTRODUCTION

Nonlinear evolution equation in $1+1$, $2+1$ and multidimensions integrable by the inverse spectral transform (IST) method form a wide class of differential equations with number of remarkable properties (see e. g. [1—3]). A main feature of such equations in $1+1$-dimensions is the existence of the soliton solutions which are localized in one dimension. Solutions of the $2+1$-dimensional integrable equations which are localized only in one dimension (plane solitons) have been constructed years ago [4—5]. Then the lumps (rational nonsingular solutions) have been also discovered [5]. Exact solutions of the $2+1$-dimensional integrable equation exponentially localized on the plane but decreasing at time $t\to\pm\infty$ have been constructed in [6]. And only recently the travelling solutions of the soliton type for the $2+1$-dimensional equations exponentially localized in all directions on the plane have been found. For the first time this has been done in the paper [7] by the use of the Backlund transformations. Spectral theory of such localized solitons for the Davey—Stewartson I (DS-I) equation and their connection with the initial-boundary value problem for the DS-I equation have been studied by the different methods in the series of papers [8—13]. The localized solitons (or dromions) of the DS-I equation possess the properties which are different from the properties of the one-dimensional solitons. In particular, they can be driven by the change of the boundaries [11—13]. All this demonstrates the richness of the coherent structure associated with the $2+1$-dimensional integrable equation.

The present paper is the first one from the series of papers devoted to the study of the coherent structures for the Ishimori equation

$$\vec{S}_t(x,y,t)+\vec{S}\times(\vec{S}_{xx}+\alpha^2\vec{S}_{yy})+\varphi_x\vec{S}_y+\varphi_y\vec{S}_x=0\,,$$
$$\varphi_{xx}-\alpha^2\varphi_{yy}+2\alpha^2\vec{S}(\vec{S}_x\times\vec{S}_y)=0\,, \qquad (1.1)$$

where $\vec{S}=(S_1, S_2, S_3)$ is the three-dimensional unit vector $(\vec{S}^2=1)$, φ is the scalar field and $\alpha^2=\pm 1$. Equation (1.1) is the 2+1-dimensional integrable generalization of the Heisenberg ferromagnet equation (isotropic Landau—Lifshitz equation) $\vec{S}_t=\vec{S}\times\vec{S}_{xx}$. It has been introduced by Ishimori in [14]. An important feature of eq. (1.1) is the existence of the classes of the topologically nontrivial and nonequivalent solutions which are classified by the topological charge [14]

$$Q=\frac{1}{4\pi}\iint dx\,dy\,\vec{S}(\vec{S}_x\times\vec{S}_y)\,. \qquad (1.2)$$

This topological invariant is connected with the mapping $\vec{S}^2\to\vec{S}^2$ defined by the unit vector $\vec{S}(x, y, t)$ with the boundary value $\vec{S}_\infty=(0, 0, -1)$. The Ishimori eq. (1.1) is of the great interest also since it is the first example of the integrable nonlinear spin-one field model on the plane.

The applicability of the IST method to the Ishimori eq. (1.1) is based on its equivalence to the commutativity condition $[L_1, L_2]=0$ of the operators [14]:

$$L_1=\alpha\partial_y+P\partial_x\,,$$
$$L_2=i\partial_t+2P\partial_x^2+(P_x+\alpha P_yP-i\alpha^3P\varphi_x+i\varphi_y)\,\partial_x\,, \qquad (1.3)$$

where $P=\vec{S}\cdot\vec{\sigma}$ and $\vec{\sigma}=(\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices and $\partial_x\equiv\partial/\partial_x$, $\partial_y\equiv\partial/\partial_y$, $\partial_t\equiv\partial/\partial_t$. The standard initial value problem for the Ishimori-I $(\alpha=i)$ and Ishimori-II $(\alpha=1)$ equations with the vanishing boundary values has been solved in the papers [15—17] with the use of the $\bar{\partial}$-method $(\alpha=i)$ and the nonlocal Riemann—Hilbert problem method $(\alpha=1)$, respectively. The exact rational-exponential solutions $(\alpha=i)$ and solutions with functional parameters $(\alpha=1)$ have been also constructed in [15, 16].

Our aim is to solve the initial-boundary value problem for the Ishimori-II equation with the nontrivial boundaries. In the characteristic coordinates $\xi=\frac{1}{2}(y+x)$, $\eta=\frac{1}{2}(y-x)$ in virtue of the equalities

$$\varphi_\xi=\int_{-\infty}^{\eta} d\eta'\vec{S}(\vec{S}_\xi\times\vec{S}_{\eta'})+2u_2(\xi,t)\,,$$
$$\varphi_\eta=\int_{-\infty}^{\xi} d\xi'\vec{S}(\vec{S}_{\xi'}\times\vec{S}_\eta)+2u_1(\eta,t)\,, \qquad (1.4)$$

eq. (1.1) at $\alpha=1$ is equivalent to the equation

$$\vec{S}_t+\frac{1}{2}\vec{S}\times(\vec{S}_{\xi\xi}+\vec{S}_{\eta\eta})+\frac{1}{2}\Big(\int_{-\infty}^{\eta} d\eta'\vec{S}(\vec{S}_\xi\times\vec{S}_{\eta'})+2u_2(\xi,t)\Big)\vec{S}_\xi-$$
$$-\frac{1}{2}\Big(\int_{-\infty}^{\xi} d\xi'\vec{S}(\vec{S}_{\xi'}\times\vec{S}_\eta)+2u_1(\eta,t)\Big)\vec{S}_\eta=0\,, \qquad (1.5)$$

where $u_1(\eta, t)$ and $u_2(\xi, t)$ are arbitrary scalar functions. The initial-boundary value problem is to solve eq. (1.5) with given initial value $\vec{S}(\xi, \eta, 0)$ and boundary values $u_1(\eta, t)$ and $u_2(\xi, t)$.

We will use the method proposed by Fokas and Santini for the DS-I equation [10—13]. Firstly we present the solution of the forward and inverse spectral problems for the auxiliary linear problem $L_1\Psi=0$ slightly different from that given in [16]. The consideration of the nontrivial boundaries requires a modification of the operator L_2. The obtained equation $\hat{L}_2\Psi=0$ gives rise to the following linear equation for the Fourier transform $\hat{S}(\xi, \eta, t)$ of the inverse problem data

$$\hat{S}_t-\frac{i}{2}(\hat{S}_{\xi\xi}+\hat{S}_{\eta\eta})+u_2(\xi,t)\,\hat{S}_\xi-u_1(n_1,t)\,\hat{S}_\eta=0\,, \qquad (1.6)$$

which coincides with the linearized eq. (1.5) for $S_+=S_1+iS_2$. For real S separation of variables reduces problem of solution of eq. (1.6) to the solution of the linear equations

$$2iX_t(\xi,t)+X_{\xi\xi}+2iu_2(\xi,t)\,X_\xi=0\,,$$
$$2iY_t(\eta,t)+Y_{\eta\eta}-2iu_1(\eta,t)\,Y_\eta=0\,. \qquad (1.7)$$

Using the exact solutions of eq. (1.7) and the solution of the factorized inverse problem for the auxiliary linear problem $L_1\Psi=0$, we construct the exact solutions of the Ishimori-II eq. (1.5). They are representable in the form

$$\vec{S}(\xi,\eta,t)=-\operatorname{tr}(\vec{\sigma}g\sigma_3 g^{-1}),$$
$$\varphi(\xi,\eta,t)=2i\ln(\det g)+2\partial_\xi^{-1}u_2(\xi,t)+2\partial_\eta^{-1}u_1(\eta',t), \tag{1.8}$$

where

$$g(\xi,\eta,t)=\hat{1}-$$
$$-\begin{pmatrix} \langle X,(1-\rho a\rho^+ b)^{-1}\rho a\rho^+ X^*\rangle, & -\langle Y^*,\rho^+(1-b\rho a\rho^+)^{-1}X^*\rangle \\ \langle X,(1-\rho a\rho^+ b)^{-1}\rho Y\rangle, & -\langle Y^*,\rho^*(1-b\rho a\rho^+)^{-1}b\rho Y\rangle \end{pmatrix} \tag{1.9}$$

where $\langle X,Y\rangle \doteq \sum_i X_i Y_i$, X_i and Y_i are the solutions of eq. (1.7), ρ_{ij} are arbitrary constants and

$$a_{ik}=\int_{-\infty}^{\eta} dn' Y_k^*(\eta',t)\,\partial_{\eta'}Y_i(\eta',t),$$

$$b_{ik}=-\int_{-\infty}^{\xi} d\xi' X_k(\xi',t)\,\partial_{\xi'}X_i^*(\xi_i^*(\xi',t). \tag{1.10}$$

Charge Q for these solutions also is given by the compact formula

$$Q=-\frac{i}{8\pi}\iint_{-\infty}^{+\infty} d_\xi d_\eta\,\partial_\xi\partial_\eta \ln(\det g). \tag{1.11}$$

In the present paper we consider in detail the case of the stationary boundaries $u_1(\eta)$ and $u_2(\xi)$. In this case the linear problems (1.7) are gauge equivalent to the 2×2 matrix Zakharov—Shabat spectral problem with the reduction which gives rises to the modified KdV equation. The inverse problem for that spectral problem has been solved in [18] and corresponding discrete spectrum includes the solitons (s) and the breathers (b). As a result, we have essentially the three different types of exact localized solutions of the Ishimori-II equation: $\vec{S}_{ss}$, $\vec{S}_{sb}$ and $\vec{S}_{bb}$ which correspond to the choice of the functions X and Y as the soliton or breather eigenfunctions. The simplest examples of such solutions are presented explicitly.

The paper is organized as follows. In the second section the solution of the inverse spectral problem for the Ishimori-II equation is presented. The compact formulae in the case of the degenerated

inverse problem data are derived in sect. 3. The time-dependence of the Ishimori-II eigenfunction is found in sext. 4. It is shown that the Fourier transforms of the inverse problem data obey the linear eq. (1.6). It is demonstrated also that the linearized eq. (1.5) coincides with (1.6). The compact formulas for the exact solutions of the Ishimori-II equation via the eigenfunctions of the linear problems (1.7) are derived in sect. 5. Some explicit formulae for the exact localized solutions of the Ishimori-II equation are presented in sect. 6.

2. THE INVERSE SPECTRAL TRANSFORM FORMALISM FOR THE ISHIMORI-II EQUATION

In the characteristic coordinates $\xi=\frac{1}{2}(y-x)$, $\eta=\frac{1}{2}(y-x)$ the first auxiliary linear problem $L_1\Psi=0$ for the Ishimori-II (Ish-II) equation is of the form

$$\begin{pmatrix}\partial_\eta & 0\\ 0 & \partial_\xi\end{pmatrix}\Psi+\frac{1}{2}Q(\partial_\xi-\partial_\eta)\Psi=0. \tag{2.1}$$

We assume that $\vec{S}\to\vec{S}_\infty=(0,0,-1)$ at $\xi,\eta\to\infty$ and $Q\doteq P+\sigma_3$. The spectral parameter λ is introduced by the transition to the function χ defined as [16]

$$\chi(\xi,\eta,\lambda)=\Psi(\xi,\eta)\begin{pmatrix}e^{-i\xi/\lambda} & 0\\ 0 & e^{i\eta/\lambda}\end{pmatrix}. \tag{2.2}$$

The function χ obeys the equation

$$\begin{pmatrix}\partial_\eta & 0\\ 0 & \partial_\xi\end{pmatrix}\chi-\frac{i}{2\lambda}[\sigma_3,\chi]+\frac{1}{2}Q\left(\partial_\xi-\partial_\eta+\frac{i}{\lambda}\right)\chi=0. \tag{2.3}$$

The operator $\hat{G}$ formally inverse to the operator $\hat{L}_0=$ $=\begin{pmatrix}\partial_\eta & 0\\ 0 & \partial_\xi\end{pmatrix}-\frac{i}{2\lambda}[\sigma_3,\cdot]$ acts as follows [16]

$$(\hat{G}\Phi)(\xi,\eta)\doteq\begin{pmatrix}\partial_\eta^{-1}(\Phi_{11}(\xi,\eta')), & \partial_\eta^{-1}\left(e^{\frac{i(\eta-\eta')}{\lambda}}\Phi_{12}(\xi,\eta')\right)\\ \partial_\xi^{-1}\left(e^{\frac{-i(\xi-\xi')}{\lambda}}\Phi_{21}(\xi',\eta)\right), & \partial_\xi^{-1}(\Phi_{22}(\xi',\eta))\end{pmatrix} \tag{2.4}$$

where $\Phi=\begin{pmatrix}\Phi_{11} & \Phi_{12}\\ \Phi_{21} & \Phi_{22}\end{pmatrix}$ is 2×2 matrix. The kernal $G(\xi-\xi', \eta-\eta')$ of the operator $\hat{G}$ is the Green function for the problem (2.3). Since the integrals ∂_ξ^{-1} and ∂_η^{-1} can be choosen as $\partial_\xi^{-1}=\int\limits_{\pm\infty}^{\xi} d\xi'\cdot$ and $\partial_\eta^{-1}=\int\limits_{\pm\infty}^{\eta} d\eta'\cdot$, the Green function G is defined nonuniquelly. This freedom allows us to contrust the bounded analytic Green functions. Here we will use the following Green functions:

$$(G^+(\cdot,\lambda)\Phi)(\xi,\eta)\doteq\begin{pmatrix}\int\limits_{-\infty}^{\eta} d\eta'\Phi_{11}(\xi,\eta'), & \int\limits_{+\infty}^{\eta} d\eta' e^{\frac{i(\eta-\eta')}{\lambda}}\Phi_{12}(\xi,\eta')\\ \int\limits_{-\infty}^{\xi} d\xi' e^{-\frac{i(\xi-\xi')}{\lambda}}\Phi_{21}(\xi',\eta), & \int\limits_{-\infty}^{\xi} d\xi'\Phi_{22}(\xi',\eta)\end{pmatrix},$$

$$(G^-(\cdot,\lambda)\Phi)(\xi,\eta)\doteq\begin{pmatrix}\int\limits_{-\infty}^{\eta} d\eta'\Phi_{11}(\xi,\eta'), & \int\limits_{-\infty}^{\eta} d\eta' e^{\frac{i(\eta-\eta')}{\lambda}}\Phi_{12}(\xi,\eta')\\ \int\limits_{+\infty}^{\xi} d\xi' e^{-\frac{i(\xi-\xi')}{\lambda}}\Phi_{21}(\xi',\eta), & \int\limits_{-\infty}^{\xi} d\xi'\Phi_{22}(\xi',\eta)\end{pmatrix}. \quad (2.5)$$

It is easy to see that the Green function $G^+(\lambda)$ is analytic and bounded at the upper half-plane $\operatorname{Im}\lambda>0$ while the Green function $G^-(\lambda)$ is analytic and bounded at the lower half-plane $\operatorname{Im}\lambda<0$. Note that the Green functions (2.5) differ from the Green functions used in [16] by the signs of the lower limits of integration in the Φ_{11} and Φ_{22} respectively.

With the use of the Green functions G^+ and G^- we define the solutions χ^+ and χ^- of equation (2.3) via the integral equations

$$\chi^\pm(\xi,\eta,\lambda)=1-\left\{G^\pm(\cdot,\lambda)\frac{1}{2}Q\left(\partial'-\bar{\partial}'+\frac{i}{\lambda}\right)\chi^\pm(\cdot,\lambda)\right\}(\xi,\eta), \quad (2.6)$$

where $\partial'\doteq\partial/\partial\xi'$, $\bar{\partial}'\doteq\partial/\partial\eta'$. As far as the Green functions G^+, G^-, the solutions χ^+, χ^- of the integral equations (2.6) are analytic and bounded in the upper and lower half-planes $\operatorname{Im}\lambda>0$ and $\operatorname{Im}\lambda<0$, respectively. Then since $G^+-G^-\neq0$ at $\operatorname{Im}\lambda=0$ one has also $\chi^+-\chi^-\neq0$ at $\operatorname{Im}\lambda=0$.

Thus one can define the function

$$\chi=\begin{cases}\chi^+, & \operatorname{Im}\lambda>0\\ \chi^-, & \operatorname{Im}\lambda<0\end{cases},$$

which is analytic and bounded at whole complex plane and has a jump across the real axis. So we arrive at the standard Riemann—Hilbert problem. As in [16] we assume that the homogeneous integral equations (2.6) have no nontrivial solutions.

To specify the Riemann—Hilbert problem arised one must express, according to the standard procedure, the jump $\chi^+-\chi^-$ at $\operatorname{Im}\lambda=0$ via χ^-. To do this we firstly obtain the integral equation for $\chi^+-\chi^-$. Using (2.6), one gets

$$(\chi^+-\chi^-)(\xi,\eta,\lambda)=\Gamma(\xi,\eta,\lambda)-\left(\bar{G}(\cdot,\lambda)\frac{1}{2}Q\left(\partial'-\bar{\partial}'+\frac{i}{\lambda}\right)(\chi^+-\chi^-)\right)(\xi,\eta), \quad (2.7)$$

where

$$\Gamma(\xi,\eta,\lambda)=$$
$$=\begin{pmatrix}0, & \int\limits_{-\infty}^{+\infty} d\eta' e^{\frac{i(\eta-\eta')}{\lambda}}\frac{1}{2}\left(Q(\xi,\eta')\left(\partial_\xi-\partial_{\eta'}+\frac{i}{\lambda}\right)\chi^+\right)_{12}\\ -\int\limits_{-\infty}^{+\infty} d\xi' e^{-\frac{i(\xi-\xi')}{\lambda}}\frac{1}{2}\left(Q(\xi',\eta)\left(\partial_{\xi'}-\partial_\eta+\frac{i}{\lambda}\right)\chi^-(\xi',\eta)\right)_{21}, & 0\end{pmatrix} \quad (2.8)$$

and

$$(\bar{G}(\cdot,\lambda)\Phi)(\xi,\eta)=\begin{pmatrix}\int\limits_{-\infty}^{\eta} d\eta'\Phi_{11}(\xi,\eta'), & \int\limits_{-\infty}^{\eta} d\eta' e^{\frac{i(\eta-\eta')}{\lambda}}\Phi_{12}(\xi,\eta')\\ \int\limits_{-\infty}^{\xi} d\xi' e^{-\frac{i(\xi-\xi')}{\lambda}}\Phi_{21}(\xi',\lambda), & \int\limits_{-\infty}^{\xi} d\xi'\Phi_{22}(\xi',\eta)\end{pmatrix}. \quad (2.9)$$

Note that the diagonal elements of the quantity Γ (2.8) are equal to zero in contrast to the similar quantity considered in [16]. This is due to the our choice of the Green functions (2.5).

We will look for the expression for $\chi^+-\chi^-$ via χ^- in the form

$$(\chi^+-\chi^-)(\xi,\eta,\lambda)=\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}\chi^-(\xi,\eta;l)\,\Sigma_l(\xi,\eta)\,f(l,\lambda)\,\Sigma_\lambda^{-1}(\xi,\eta), \quad (2.10)$$

where

$$\Sigma_\lambda(\xi,\eta)\doteq\begin{pmatrix}e^{i\xi/\lambda}, & 0\\ 0, & e^{-i\eta/\lambda}\end{pmatrix} \quad (2.11)$$

and $f(l, \lambda)$ is the 2×2 matrix which must be found.

Substituting (2.10) into the r.h.s. of (2.7), we obtain

$$\chi^{+}-\chi^{-}=\Gamma-\int_{-\infty}^{+\infty}\frac{dl}{l^2}\times$$

$$\times\begin{pmatrix} \int_{-\infty}^{\eta} d\eta' \frac{1}{2}\left(Q\left(\partial'-\tilde{\partial}'+\frac{i}{\lambda}\right)\chi^{-}(l)\,\Sigma_l f(l,\lambda)\,\Sigma_\lambda^{-1}\right)_{11}, & \int_{-\infty}^{\eta} d\eta' e^{\frac{i(\eta-\eta')}{\lambda}} \frac{1}{2}\left(Q\left(\partial'-\tilde{\partial}'+\frac{i}{\lambda}\right)\chi^{-}(l)\,\Sigma_l f(l,\lambda)\,\Sigma_\lambda^{-1}\right)_{12} \\ \int_{-\infty}^{\xi} d\xi' e^{-\frac{i(\xi-\xi')}{\lambda}} \frac{1}{2}\left(Q\left(\partial'-\tilde{\partial}'+\frac{i}{\lambda}\right)\chi^{-}(l)\,\Sigma_l f(l,\lambda)\,\Sigma_\lambda^{-1}\right)_{21}, & \int_{-\infty}^{\xi} d\xi' \frac{1}{2}\left(Q\left(\partial'-\tilde{\partial}'+\frac{i}{\lambda}\right)\chi^{-}(l)\,\Sigma_l f(l,\lambda)\,\Sigma_\lambda^{-1}\right)_{22} \end{pmatrix} \tag{2.12}$$

On the other hand from (2.6) one has

$$\int_{-\infty}^{+\infty}\frac{dl}{l^2}\chi^{-}(l)\,\Sigma_l f(l,\lambda)\,\Sigma_\lambda^{-1}=\chi^{+}-\chi^{-}=\int_{-\infty}^{+\infty}\frac{dl}{l^2}\Sigma_l f(l,\lambda)\,\Sigma_\lambda^{-1}-\int_{-\infty}^{+\infty}\frac{dl}{l^2}\times$$

$$\times\begin{pmatrix} \int_{-\infty}^{\eta} d\eta' \frac{1}{2}\left(Q\left(\partial'-\tilde{\partial}'+\frac{i}{l}\right)\chi^{-}(l)\right)_{11}, & \int_{-\infty}^{\eta} d\eta' e^{\frac{i(\eta-\eta')}{l}} \frac{1}{2}\left(Q\left(\partial'-\tilde{\partial}'+\frac{i}{l}\right)\chi^{-}(l)\right)_{12} \\ \int_{+\infty}^{\xi} d\xi' e^{-\frac{i(\xi-\xi')}{l}} \frac{1}{2}\left(Q\left(\partial'-\tilde{\partial}'+\frac{i}{l}\right)\chi^{-}(l)\right)_{21}, & \int_{-\infty}^{\xi} d\xi' \frac{1}{2}\left(Q\left(\partial'-\tilde{\partial}'+\frac{i}{l}\right)\chi^{-}(l)\right)_{22} \end{pmatrix} \Sigma_l f(l,\lambda)\,\Sigma_\lambda^{-1}. \tag{2.13}$$

Transforming (2.12) with the use the identity

$$\left(\left(\partial'-\tilde{\partial}'+\frac{i}{\lambda}\right)\chi^{-}(l)\,\Sigma_l f(l,\lambda)\,\Sigma_\lambda^{-1}\right)(\xi,\eta)=$$

$$=\left(\left(\partial'-\tilde{\partial}'+\frac{i}{l}\right)\chi^{-}(l)\right)(\xi,\eta)\,\Sigma_l(\xi,\eta)\,f(l,\lambda)\,\Sigma_\lambda^{-1}(\xi,\eta), \tag{2.14}$$

performing the matrix multiplication in (2.13) and subtracting the obtained equations, one gets

$$\int_{-\infty}^{+\infty}\frac{dl}{l^2}\,\Sigma_l f(l,\lambda)\,\Sigma_\lambda^{-1}=$$

$$=\begin{pmatrix} 0, & \int_{-\infty}^{+\infty} d\eta' e^{-\frac{i\eta'}{\lambda}}\frac{1}{2}\left(Q\left(\partial'-\tilde{\partial}'+\frac{i}{\lambda}\right)\chi^{+}\right)_{12} \\ -\int_{-\infty}^{+\infty} d\xi' e^{\frac{i\xi'}{\lambda}}\frac{1}{2}\left(Q\left(\partial'-\tilde{\partial}'+\frac{i}{\lambda}\right)\chi^{-}\right)_{21}, & 0 \end{pmatrix}\Sigma_\lambda^{-1}(\xi,\eta)-\int_{-\infty}^{+\infty}\frac{dl}{l^2}\times$$

$$\times\begin{pmatrix} 0, & 0 \\ \int_{-\infty}^{+\infty} d\xi' e^{\frac{i\xi'}{l}}\frac{1}{2}\left(Q\left(\partial'-\tilde{\partial}'+\frac{i}{l}\right)\chi^{-}\right)_{21}, & 0 \end{pmatrix}\begin{pmatrix} f_{11}, & f_{12} \\ f_{21}, & f_{22} \end{pmatrix}\Sigma_\lambda^{-1}(\xi,\eta). \tag{2.15}$$

Multiplying (2.15) by Σ_λ from the right and acting on (2.15) by

$$\frac{1}{2\pi}\begin{pmatrix} \int_{-\infty}^{+\infty} d\xi\, e^{-i\xi/P}, & 0 \\ 0, & \int_{-\infty}^{+\infty} d\eta\, e^{i\eta/P} \end{pmatrix}$$

from the left, we obtain

$$\begin{pmatrix} f_{11}(p,\lambda), & f_{12}(p,\lambda) \\ f_{21}(p,\lambda), & f_{22}(p,\lambda) \end{pmatrix}=\begin{pmatrix} 0, & T_{12}^{+}(p,\lambda) \\ -T_{21}^{-}(p,\lambda), & 0 \end{pmatrix}-$$

$$-\int_{-\infty}^{+\infty}\frac{dl}{l^2}\begin{pmatrix} 0 & , 0 \\ T_{21}^{-}(p,l), & 0 \end{pmatrix}\begin{pmatrix} f_{11}(l,\lambda), & f_{12}(l,\lambda) \\ f_{21}(l,\lambda), & f_{22}(l,\lambda) \end{pmatrix} \tag{2.16}$$

where

$$T_{12}^{+}(p,\lambda)\doteq\frac{1}{2\pi}\iint_{-\infty}^{+\infty} d\xi\, d\eta\, e^{-\frac{i\xi}{p}-\frac{i\eta}{\lambda}}\frac{1}{2}\left(Q\left(\partial-\tilde{\partial}+\frac{i}{\lambda}\right)\chi^{+}\right)_{12},$$

$$T_{21}^{-}(p,\lambda)\doteq\frac{1}{2\pi}\iint_{-\infty}^{+\infty} d\xi\, d\eta\, e^{\frac{i\eta}{p}+\frac{i\xi}{\lambda}}\frac{1}{2}\left(Q\left(\partial-\tilde{\partial}+\frac{i}{\lambda}\right)\chi^{-}\right)_{21}. \tag{2.17}$$

In the matrix form equation (2.16) looks like

$$f(p,\lambda)+\int_{-\infty}^{+\infty}\frac{dl}{l^2}T^-(p,l)f(l,\lambda)=T^+(p,\lambda)-T^-(p,\lambda)\,, \tag{2.18}$$

where

$$T^+=\begin{pmatrix}0, & T_{12}^+\\ 0, & 0\end{pmatrix},\quad T^-=\begin{pmatrix}0, & 0\\ T_{12}^+, & 0\end{pmatrix}. \tag{2.19}$$

Thus, we have proved that the jump $\chi^+-\chi^-$ indeed is given by (2.10) with the matrix $f(l,\lambda)$ which obeys equation (2.18) (see also [16]).

Note that f is easily expressed via T_{12}^+ and T_{21}^-

$$f(P,\lambda)=\begin{pmatrix}0, & T_{12}^+(p,\lambda)\\ -T_{21}^-(p,\lambda), & -\int_{-\infty}^{+\infty}\frac{dl}{l^2}T_{21}^-(p,l)\,T_{12}^+(l,\lambda)\end{pmatrix}. \tag{2.20}$$

So, we have arrived at the standard regular nonlocal Riemann—Hilbert problem. Its solution is given by the rather standard linear singular integral equation [16]

$$\chi^-(\lambda)=1+\frac{1}{2\pi i}\iint_{-\infty}^{+\infty}\frac{dp}{p^2}\frac{dk}{k^2}\frac{\chi^-(p)\,\Sigma_p f(p,k)\,\Sigma_k^{-1}}{k-\lambda+i0}. \tag{2.21}$$

Equation (2.21) is the inverse problem equation for the linear problem (2.9). The functions $T_{12}^+(\lambda,\mu)$ and $T_{21}^-(\lambda,\mu)$ are the inverse problem data. The reconstruction formula for the potential $P(\xi,\eta,t)$ is obtained by the substitution of the Taylor expansion

$$\chi(\xi,\eta,\lambda)=\chi_0(\xi,\eta)+\lambda\chi_1(\xi,\eta)+\lambda^2\chi_2(\xi,\eta)+\ldots \tag{2.22}$$

near $\lambda=0$ into equation (2.3). The consideration of the terms of the order λ^{-1} gives rise to the formula [16]

$$P(\xi,\eta,t)=\vec{S}(\xi,\eta,t)\,\vec{\sigma}=-g\sigma_3 g^{-1}, \tag{2.23}$$

where

$$g(\xi,\eta,t)=\chi(\xi,\eta,t,\lambda=0)=$$

$$=1+\frac{1}{2\pi i}\iint_{-\infty}^{+\infty}dl\,dk\,\chi^-(\xi,\eta,l)\,\Sigma_l(\xi,\eta)\,f(l,k)\,\Sigma_k^{-1}(\xi,\eta)\,. \tag{2.24}$$

One is able also to obtain the compact formulae for the topological charge (1.2) and the scalar field φ.

Note firstly that

$$\frac{1}{4\pi}\vec{S}(\vec{S}_x\cdot\vec{S}_y)=-\frac{i}{8\pi}\operatorname{tr}(PP_xP_y)\,. \tag{2.25}$$

Using the expression (2.23) for P, one obtains

$$\operatorname{tr}(PP_xP_y)=2\operatorname{tr}(\sigma_3((g^{-1})_y g_x-(g^{-1})_x g_y))\,. \tag{2.26}$$

On the other hand substituting (2.22) into (2.3), one gets in the zero order on λ

$$\sigma_3 g^{-1}g_x-g^{-1}g_y=\frac{i}{2}[g^{-1}\chi_1,\sigma_3]\,. \tag{2.27}$$

Differentiating (2.27) with respect to x and y, one obtains

$$\sigma_3(g^{-1})_y g_x+\sigma_3 g^{-1}g_{xy}-(g^{-1}g_y)_y=\frac{i}{2}\partial_y[g^{-1}\chi_1,\sigma_3]\,,$$

$$(g^{-1}g_x)_x-\sigma_3(g^{-1})_x g_y-\sigma_3 g^{-1}g_{xy}=\frac{i}{2}\sigma_3\,\partial_x[g^{-1}\chi_1,\sigma_3]\,.$$

The summation of these two formulae gives

$$\sigma_3((g^{-1})_y g_x-(g^{-1})_x g_y)=(g^{-1}g_y)_y-(g^{-1}g_x)_x+$$

$$+\frac{i}{2}(\partial_y+\sigma_3\partial_x)[g^{-1}\chi_1,\sigma_3]\,. \tag{2.28}$$

Substituting (2.28) into (2.26), taking into account that $\operatorname{tr}((\partial_y+\sigma_3\partial_x)[g^{-1}\chi_1\ \sigma_3])=0$ and using the well-known formula $\operatorname{tr}(g_x g^{-1})=\partial_x\ln\det g$, we obtain:

$$\operatorname{tr}(PP_xP_y)=2\operatorname{tr}((g^{-1}g_y)_y-(g^{-1}g_x)_x)=$$

$$=2(\partial_y^2-\partial_x^2)\ln\det g=2\partial_\xi\partial_\eta\ln\det g\,. \tag{2.29}$$

So the topological charge Q is

$$Q=-\frac{i}{8\pi}\iint_{-\infty}^{+\infty}d\xi\,d\eta\,\partial_\xi\partial_\eta\ln\det g\,. \tag{2.30}$$

The formula (2.29) gives rise also to the following reconstruction formula for the scalar field φ:

$$\varphi(\xi,\eta,t)=2i\ln\det g+2\partial_\xi^{-1}u_2(\xi',t)+2\partial_\eta^{-1}u_1(\eta',t)\,. \tag{2.31}$$

The formulae (2.21), (2.23), (2.24) and (2.31) completely solve the inverse problem for the linear equation (2.3).

For the real $\vec{S}$ the matrix P is the self-Hermitian and the relations (2.23), (2.24) imply that $g(\xi, \eta, t)$ is the unitary matrix $gg^{+}=1$. So $|\det g|=1$ and therefore the scalar field φ is a real one for the real u_1 and u_2.

Note that the similar expressions for the topological charge and the scalar field can be derived for the Ishimori-I equation ($\alpha=i$):

$$Q=-\frac{1}{4\pi}\iint\limits_{-\infty}^{+\infty} dx\,dy\,(\partial_x^2+\partial_y^2)\ln\det g \tag{2.32}$$

and

$$\varphi(x, y)=2\ln\det g\,. \tag{2.33}$$

The formulae (2.31), (2.33) are in complete agreement with the formula $\varphi=2i\alpha\ln\det g^{-1}$, derived by the different method in [19].

3. DEGENERATED SPECTRAL DATA

To study the initial-boundary value problem for the Ishimori equation and to construct the exact solutions we need, similar to the DS-I equation [10—12], the solution of the inverse problem equations with the degenerated inverse problem data, i. e.

$$T_{12}^{+}(p, \lambda)=\sum_{k=1}^{N_+} S_k(p)\,\tilde{S}_k(\lambda)\,,$$

$$T_{21}^{-}(p, \lambda)=\sum_{k=1}^{N_-} T_k(p)\,\tilde{T}_k(\lambda)\,. \tag{3.1}$$

For this purpose it is convenient to rewrite the matrix equations (2.10) as the system of equations for the columns of χ. One has

$$\begin{pmatrix}\chi_{11}^{+}(\lambda)\\ \chi_{21}^{+}(\lambda)\end{pmatrix}-\begin{pmatrix}\chi_{11}^{-}(\lambda)\\ \chi_{21}^{-}(\lambda)\end{pmatrix}=-\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}\begin{pmatrix}\chi_{12}^{-}(l)\\ \chi_{22}^{-}(l)\end{pmatrix}T_{21}^{-}(l, \lambda)\,e^{-\frac{i\eta}{l}-\frac{i\xi}{\lambda}} \tag{3.2}$$

and

$$\begin{pmatrix}\chi_{12}^{+}(\lambda)\\ \chi_{22}^{+}(\lambda)\end{pmatrix}-\begin{pmatrix}\chi_{12}^{-}(\lambda)\\ \chi_{22}^{-}(\lambda)\end{pmatrix}=\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}T_{12}^{+}(l, \lambda)\,e^{-\frac{i\xi}{l}-\frac{i\eta}{\lambda}}\begin{pmatrix}\chi_{11}^{+}(l)\\ \chi_{21}^{+}(l)\end{pmatrix}. \tag{3.3}$$

Correspondingly the inverse problem equation (2.21) looks like

$$\begin{pmatrix}\chi_{11}^{\pm}(\lambda)\\ \chi_{21}^{\pm}(\lambda)\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}-\left(\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}T_{21}^{-}(l, \lambda)\,e^{-i\left(\frac{\eta}{l}+\frac{\xi}{\lambda}\right)}\begin{pmatrix}\chi_{12}^{-}(l)\\ \chi_{22}^{-}(l)\end{pmatrix}\right)^{\pm}; \tag{3.4}$$

$$\begin{pmatrix}\chi_{12}^{\pm}(\lambda)\\ \chi_{22}^{\pm}(\lambda)\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}+\left(\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}T_{12}^{+}(l, \lambda)\,e^{i\left(\frac{\xi}{l}+\frac{\eta}{\lambda}\right)}\begin{pmatrix}\chi_{11}^{+}(l)\\ \chi_{21}^{+}(l)\end{pmatrix}\right)^{\pm}, \tag{3.5}$$

where $(f(\lambda))^{\pm}$ denoted the projection onto the real axis:

$$(f(\lambda))^{\pm}\doteq\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{d\lambda' f(\lambda')}{\lambda'-(\lambda\pm i0)}\,. \tag{3.6}$$

Substitution of (3.1) into (3.4) and (3.5) gives respectively

$$\begin{pmatrix}\chi_{11}^{\pm}(\lambda)\\ \chi_{21}^{\pm}(\lambda)\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}-\sum_{j=1}^{N_-}\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}T_j(l)\,e^{-\frac{i\eta}{l}}\begin{pmatrix}\chi_{12}^{-}(l)\\ \chi_{22}^{-}(l)\end{pmatrix}\left(\tilde{T}_j(\lambda)\,e^{-\frac{i\xi}{\lambda}}\right)^{\pm}, \tag{3.7}$$

$$\begin{pmatrix}\chi_{12}^{\pm}(\lambda)\\ \chi_{22}^{\pm}(\lambda)\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}+\sum_{i=1}^{N_-}\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}\,S_i(l)\,e^{\frac{i\xi}{l}}\begin{pmatrix}\chi_{11}^{+}(l)\\ \chi_{21}^{+}(l)\end{pmatrix}\left(\tilde{S}_i(\lambda)\,e^{\frac{i\eta}{\lambda}}\right)^{\pm}. \tag{3.8}$$

Let us introduce the quantities

$$F_j\doteq\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}T_j(l)\,e^{-\frac{i\eta}{l}}\begin{pmatrix}\chi_{12}^{-}(l)\\ \chi_{22}^{-}(l)\end{pmatrix}, \tag{3.9}$$

$$G_j\doteq\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}S_j(l)\,e^{-\frac{i\xi}{l}}\begin{pmatrix}\chi_{11}^{+}(l)\\ \chi_{21}^{+}(l)\end{pmatrix}. \tag{3.10}$$

Equations (3.7), (3.8) imply now the following system of equations for F_j and G_j:

$$G_j=\begin{pmatrix}1\\ 0\end{pmatrix}\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}S_j(l)\,e^{\frac{i\xi}{l}}-\sum_{k=1}^{N_-}F_k\int\limits_{-\infty}^{+\infty}\frac{d\lambda}{l^2}S_j(\lambda)\,e^{\frac{i\xi}{\lambda}}\left(\tilde{T}_k(\lambda)\,e^{-\frac{i\xi}{\lambda}}\right)^{+},$$

$$F_j=\begin{pmatrix}0\\ 1\end{pmatrix}\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}T_j(l)\,e^{-\frac{i\eta}{l}}+\sum_{k=1}^{N_+}G_k\int\limits_{-\infty}^{+\infty}\frac{d\lambda}{\lambda^2}T_j(\lambda)\,e^{-\frac{i\eta}{\lambda}}\left(\tilde{S}_k(\lambda)\,e^{\frac{i\eta}{\lambda}}\right)^{-}. \tag{3.11}$$

If one denotes

$$\sigma_j(\xi) \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{dl}{l^2} S_j(l)\, e^{\frac{i\xi}{l}},$$

$$\tau_j(\eta) \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{dl}{l^2} T_j(l)\, e^{-\frac{i\eta}{l}},$$

$$\alpha_{jk} \doteq \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda^2} T_j(\lambda)\, e^{-\frac{i\eta}{\lambda}} \left(\tilde{S}_k(\lambda)\, e^{\frac{i\eta}{\lambda}}\right)^-,$$

$$\beta_{jk} \doteq -\int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda^2} S_j(\lambda)\, e^{\frac{i\xi}{\lambda}} \left(\tilde{T}_k(\lambda)\, e^{-\frac{i\xi}{\lambda}}\right)^+ \tag{3.12}$$

then the system (3.11) is represented in the compact form

$$F_i - \sum_{k=1}^{N_+} \alpha_{ik} G_k = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tau_i(\eta),$$

$$G_j - \sum_{k=1}^{N_-} \beta_{jk} F_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma_i(\xi). \tag{3.13}$$

This system is easily solvable and the solution is of the form

$$F_j = \sum_{p=1}^{N_-} (1-A)^{-1}_{jp} \begin{pmatrix} \sum_{k=1}^{N_+} \alpha_{pk} \sigma_k \\ \tau_p \end{pmatrix},$$

$$G_j = \sum_{p=1}^{N_+} (1-B)^{-1}_{jp} \begin{pmatrix} \sigma_p \\ \sum_{k=1}^{N_-} \beta_{pk} \tau_k \end{pmatrix}, \tag{3.14}$$

where A and B are $N_- \times N_-$ and $N_+ \times N_+$ matrices of the form

$$A_{ij} = \sum_{k=1}^{N_+} \alpha_{ik} \beta_{kj}, \quad B_{lm} = \sum_{k=1}^{N_-} \beta_{lk} \alpha_{km}. \tag{3.15}$$

Khowledge of F_j and G_j allows us to find the matrix $\chi^-(\lambda)$. Namely, (3.7) and (3.8) imply

$$\begin{pmatrix} \chi_{11}^-(\lambda) \\ \chi_{21}^-(\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sqrt{2\pi} \sum_{j=1}^{N_-} F_j \left(\tilde{T}_j(\lambda)\, e^{-\frac{i\xi}{\lambda}}\right)^-,$$

$$\begin{pmatrix} \chi_{12}^-(\lambda) \\ \chi_{22}^-(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sqrt{2\pi} \sum_{i=1}^{N_+} G_i \left(\tilde{S}_i(\lambda)\, e^{\frac{i\eta}{\lambda}}\right)^-. \tag{3.16}$$

Finally, one finds $\vec{S}(\xi, \eta, t)$ by the formula (1.8) via $\chi^-(\xi, \eta, \lambda=0)$.

Thus, the inverse problem for the Ishimori-II equation with the degenerated inverse data, as usually, is solved explicitly. The corresponding potential $p(\xi, \eta, t)$ depends on the $2(N_+ + N_-)$ arbitrary functions of one variable.

Note that in the degenerated case the Fourier transforms of the inverse problem data

$$\hat{S}(\xi, \eta) = \iint_{-\infty}^{+\infty} \frac{dl}{l^2} \frac{d\lambda}{\lambda^2} T_{21}^-(l, \lambda)\, e^{-\frac{i\eta}{l} - \frac{i\xi}{\lambda}},$$

$$\hat{T}(\xi, \eta) = \iint_{-\infty}^{+\infty} \frac{dl}{l^2} \frac{d\lambda}{\lambda^2} T_{12}^+(l, \lambda)\, e^{\frac{i\xi}{l} + \frac{i\eta}{\lambda}} \tag{3.17}$$

have the factorized form too:

$$\hat{S}(\xi, \eta) = 2\pi \sum_{i=1}^{N_-} \tau_i(\eta)\, \tilde{\tau}_i(\xi),$$

$$\hat{T}(\xi, \eta) = 2\pi \sum_{i=1}^{N_+} \sigma_i(\xi)\, \tilde{\sigma}_i(\eta), \tag{3.18}$$

where $\tau_i(\eta)$ and $\sigma_i(\xi)$ are given by the formulas (3.12) and

$$\tilde{\tau}_i(\xi) = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda^2} \lambda \tilde{T}_i(\lambda)\, e^{-\frac{i\xi}{\lambda}},$$

$$\tilde{\sigma}_j(\eta) = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda^2} \lambda \tilde{S}_j(\lambda)\, e^{\frac{i\eta}{\lambda}}. \tag{3.19}$$

4. TIME EVOLUTION OF THE INVERSE PROBLEM DATA

In the case of the nontrivial boundary conditions

$$\varphi_\eta \xrightarrow[\xi\to-\infty]{} 2u_1(\eta,t)\,, \qquad \varphi_\xi \xrightarrow[\eta\to-\infty]{} 2u_2(\xi,t) \tag{4.1}$$

the second operator L_2, associated with the Ishimori-II equation must be modified, in order to the linear problems

$$\hat{L}_1\Psi=0\,,$$
$$\hat{L}_2\Psi=0 \tag{4.2}$$

will be compatible and equivalent to the Ishimori equation. The operator L_1 remains unchanged $\hat{L}_1=L_1$. The modified operator $\hat{L}_2$ can be constructed analogously to the DS-I equation case [10—11]. Namely, we will look for this operator in the form

$$\hat{L}_2=L_2+\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}\gamma(k,l)\,, \tag{4.3}$$

where L_2 is given by (1.3) and the kernel $\gamma(k,l)$ must be found.

So the time evolution of the eigenfunction Ψ is defined by the equation

$$L_2\Psi+\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}\gamma(k,l)\,\Psi(l)=L_2\Psi+v\,. \tag{4.4}$$

Since the Ishimori equation is equivalent to the condition $[L_1, L_2]=0$ and $L_1\Psi=0$, then equation (4.4) implies

$$L_1 v=0 \tag{4.5}$$

i. e. v and Ψ are the solutions of the same differential equation. The relation between v and Ψ can be found by comparing the integral equations satisfied by these functions.

Let us choose Ψ as

$$\Psi(\xi,\eta,\lambda)=\chi^-(\xi,\eta,\lambda)\begin{pmatrix} e^{i\xi/\lambda} & 0\\ 0 & e^{-i\eta/\lambda}\end{pmatrix}, \tag{4.6}$$

where χ obeys the integral equation (2.6). This function Ψ obeys

the integral equation

$$\Psi(\xi,\eta,\lambda)=\begin{pmatrix} e^{i\xi/\lambda} & 0\\ 0 & e^{-i\eta/\lambda}\end{pmatrix}-\left\{\hat{G}^-(\cdot,\lambda)\frac{1}{2}Q\left(\partial'-\bar{\partial}'+\frac{i}{\lambda}\right)\Psi(\cdot)\right\}(\xi,\eta)\,, \tag{4.7}$$

where

$$\hat{G}^-\cdot=(G^-\cdot\Sigma_\lambda^{-1})\,\Sigma_\lambda\,. \tag{4.8}$$

The first column of Ψ obeys the equation

$$\begin{pmatrix}\Psi_{11}\\ \Psi_{21}\end{pmatrix}=\begin{pmatrix} e^{i\xi/\lambda}\\ 0\end{pmatrix}-\left\{\hat{G}^-(\cdot,\lambda)\frac{1}{2}Q\left(\partial'-\bar{\partial}'+\frac{i}{\lambda}\right)\Psi\right\}(\xi,\eta)\,. \tag{4.9}$$

The quantity v obeys the same equation (4.5) as Ψ, hence, the first column of v obeys the same integral equation as $\begin{pmatrix}\Psi_{11}\\ \Psi_{21}\end{pmatrix}$. The only difference is the inhomogeneous term, i. e.

$$\begin{pmatrix}v_{11}\\ v_{21}\end{pmatrix}=\begin{pmatrix}v_{11\infty}\\ v_{21\infty}\end{pmatrix}-\left\{\hat{G}^-(\cdot,\lambda)\frac{1}{2}Q\left(\partial'-\bar{\partial}'+\frac{i}{\lambda}\right)v\right\}(\xi,\eta)\,. \tag{4.10}$$

The quantities $v_{11\infty}$ and $v_{21\infty}$ are found from equation (4.4). Taking into account (1.3) and (1.4) and considering the limit $\eta\to-\infty$, one finds

$$v_{11\infty}=-\frac{i}{2\lambda^2}e^{\frac{i\xi}{\lambda}}+\frac{i}{2\lambda}u_2(\xi,t)\,e^{\frac{i\xi}{\lambda}}\,,\qquad v_{21\infty}=0\,. \tag{4.11}$$

Comparing now the integral equations (4.9) and (4.11), one gets

$$\begin{pmatrix}v_{11}(\lambda)\\ v_{21}(\lambda)\end{pmatrix}=-\frac{i}{2\lambda^2}\begin{pmatrix}\Psi_{11}(\lambda)\\ \Psi_{21}(\lambda)\end{pmatrix}+\frac{i}{2\lambda}\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}\gamma\left(\frac{1}{\lambda}-\frac{1}{l}\right)\begin{pmatrix}\Psi_{11}(l)\\ \Psi_{21}(l)\end{pmatrix}, \tag{4.12}$$

where

$$\gamma(p)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}d\xi\,e^{i\xi p}\,u_2(\xi,t)\,. \tag{4.13}$$

Now let us consider equation for the second column

$$\begin{pmatrix}\hat{\Psi}_{12}\\ \hat{\Psi}_{22}\end{pmatrix}=\begin{pmatrix}\chi_{12}^+\\ \chi_{22}^+\end{pmatrix}e^{-i\eta/\lambda}$$

It is of the form

$$\begin{pmatrix}\hat{\Psi}_{12}\\ \hat{\Psi}_{22}\end{pmatrix}=\begin{pmatrix}0\\ e^{-i\eta/\lambda}\end{pmatrix}-\left\{\hat{G}^{-}(\cdot,\lambda)\frac{1}{2}Q\left(\partial'-\bar{\partial}'+\frac{i}{\lambda}\right)\Psi\right\}(\xi,\eta)\,. \quad (4.14)$$

The second column of v obeys equation (4.10) with another inhomogeneous term $\hat{v}$. Its form can be found by the consideration of equation (4.4) in the limit $\xi\to-\infty$. One gets

$$v_{12\infty}=0\,,$$

$$v_{22\infty}=\frac{i}{2\lambda^2}e^{-\frac{i\eta}{\lambda}}+\frac{i}{2\lambda}u_1(\eta,t)e^{-\frac{i\eta}{\lambda}}. \quad (4.15)$$

Comparison of the integral equations for the second columns of $\hat{\Psi}$ and v, as a result, gives

$$\begin{pmatrix}v_{12}\\ v_{22}\end{pmatrix}=\frac{i}{2\lambda^2}\begin{pmatrix}\hat{\Psi}_{12}\\ \hat{\Psi}_{22}\end{pmatrix}+\frac{i}{2\lambda}\int_{-\infty}^{+\infty}\frac{dl}{l^2}\hat{\gamma}\left(\frac{1}{\lambda}-\frac{1}{l}\right)\begin{pmatrix}\hat{\Psi}_{12}(l)\\ \hat{\Psi}_{22}(l)\end{pmatrix}, \quad (4.16)$$

where

$$\hat{\gamma}(p)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\eta\, u_1(\eta,t)\,e^{-i\eta p}. \quad (4.17)$$

So, we have found all components of the matrix v. Thus, the time evolution of the quantities

$$\Psi(\lambda)=\begin{pmatrix}\Psi_{11}\\ \Psi_{21}\end{pmatrix}=\begin{pmatrix}\chi_{11}^{-}\\ \chi_{21}^{-}\end{pmatrix}e^{\frac{i\xi}{\lambda}}$$

and

$$\hat{\Psi}(\lambda)=\begin{pmatrix}\chi_{12}^{+}\\ \chi_{22}^{+}\end{pmatrix}e^{-\frac{i\eta}{\lambda}}$$

is defined by the following equations

$$\Psi_t=\frac{i}{2}p(\partial_\xi-\partial_\eta)^2\Psi+\frac{i}{4}R(\partial_\xi-\partial_\eta)\,\Psi-\frac{i}{2\lambda^2}\Psi+$$

$$+\frac{i}{2\lambda}\int_{-\infty}^{+\infty}\frac{dl}{l^2}\gamma\left(\frac{1}{\lambda}-\frac{1}{l}\right)\Psi(l) \quad (4.18)$$

and

$$\hat{\Psi}_t=\frac{i}{2}P(\partial_\xi-\partial_\eta)^2\hat{\Psi}+\frac{i}{4}R(\partial_\xi-\partial_\eta)\,\hat{\Psi}+\frac{i}{2\lambda^2}\hat{\Psi}+$$

$$+\frac{i}{2\lambda}\int_{-\infty}^{+\infty}\frac{dl}{l^2}\gamma\left(\frac{1}{\lambda}-\frac{1}{l}\right)\hat{\Psi}(l)\,, \quad (4.19)$$

where

$$R\doteq P_\xi-p_\eta+(P_\xi+P_\eta)\,P+i(1-P)\int_{-\infty}^{\eta}d\eta'\,\vec{S}(\vec{S}_\xi\times\vec{S}_{\eta'})+$$

$$+i(1-P)\int_{-\infty}^{\xi}d\xi'\,\hat{S}(\vec{S}_{\xi'}\times\vec{S}_\eta)\,. \quad (4.20)$$

Similar to the DS-I equation, the nontriviality of the boundaries essentially changes the evolution of the eigenfunction Ψ in time.

Equations (4.18) and (4.19) allow us to find the time evolution of the inverse problem data.

The consideration of equation (4.18) at $\xi\to-\infty$ gives rise to the following equation

$$\sigma_t(\lambda,\eta,t)=\frac{i}{2}\sigma_{\eta\eta}+u_1(\eta,t)\,\sigma_\eta-\frac{i}{2\lambda^2}\sigma+\frac{i}{2\lambda^2}\int_{-\infty}^{+\infty}\frac{dl}{l^2}\gamma\left(\frac{1}{\lambda}-\frac{1}{l}\right)\sigma(l,\eta,t) \quad (4.21)$$

for the quantity

$$\sigma(\lambda,\eta,t)\doteq\int_{-\infty}^{+\infty}\frac{dl}{l^2}T_{21}^{-}(l,\lambda,t)\,e^{-\frac{i\eta}{l}}=\lim_{\xi\to-\infty}\Psi_{21}\,. \quad (4.22)$$

Analogously for the quantity

$$\tau(\lambda,\xi,t)\doteq\int_{-\infty}^{+\infty}\frac{dl}{l^2}T_{12}^{+}(l,\lambda)\,e^{-\frac{i\xi}{l}}=\lim_{\eta\to-\infty}\hat{\Psi}_{12} \quad (4.23)$$

one obtains from (4.19)

$$\tau_t(\xi,\lambda,t)=-\frac{i}{2}\tau_{\xi\xi}-u_2(\xi,t)\,\tau_\xi+\frac{i}{2\lambda^2}\tau+\frac{i}{2\lambda}\int_{-\infty}^{+\infty}\frac{dl}{l^2}\gamma\left(\frac{1}{\lambda}-\frac{1}{l}\right)\tau(l,\xi)\,. \quad (4.24)$$

Further, for the full Fourier transforms of the inverse problem data

$$\hat{S}(\xi,\eta,t)\doteq -i\iint_{-\infty}^{+\infty}\frac{dl}{l^2}\frac{d\lambda}{\lambda^2}\lambda T_{21}^{-}(l,\lambda)\,e^{-\frac{i\eta}{l}-\frac{i\xi}{\lambda}}=-i\int_{-\infty}^{+\infty}\frac{dl}{\lambda^2}\lambda\sigma(\lambda,\eta,t)\,e^{-\frac{i\xi}{\lambda}}, \quad (4.25)$$

$$\hat{T}(\xi,\eta,t)\doteq -i\iint_{-\infty}^{+\infty}\frac{d\lambda}{\lambda^2}\frac{dl}{l^2}\lambda T_{12}^{+}(l,\lambda)\,e^{\frac{i\xi}{l}+\frac{i\eta}{\lambda}}=-i\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda^2}\lambda\tau(\lambda,\xi,t)\,e^{\frac{i\eta}{\lambda}} \quad (4.26)$$

one gets the simpler evolution equations. They are

$$\hat{S}_t(\xi,\eta,t)-\frac{i}{2}(\hat{S}_{\xi\xi}+\hat{S}_{\eta\eta})+u_2(\xi,t)\hat{S}_{\xi}-u_1(\eta,t)\,\hat{S}_{\eta}=0 \quad (4.27)$$

and

$$\hat{T}_t(\xi,\eta,t)+\frac{i}{2}(\hat{T}_{\xi\xi}+\hat{T}_{\eta\eta})+u_2(\xi,t)\,\hat{T}_{\xi}-u_1(\eta,t)\,\hat{T}_{\eta}=0. \quad (4.28)$$

Thus, the time evolution of the full Fourier transforms of the inverse problem data for the Ishimori-II equation is given by the simple linear differential equations which contain the boundaries $u_1(\eta,t)$ and $u_2(\xi,t)$ as the potentials.

Equations (4.27) and (4.28) play a fundamental role for all further constructions. Note that they contain the first order derivatives over ξ and η in contrast to the DS-I equation case.

Equations (4.27) and (4.28) are closely connected with the Ishimori-II equation in the weak fields limit. Indeed, in the weak limit $\delta S_1\doteq\tilde{S}_1\ll 1$, $\delta S_2\doteq\tilde{S}_2\ll 1$ and $\delta S_3=0$ (since $\vec{S}_{\infty}\cdot\delta\vec{S}=0$). Neglecting the nonlinear terms, one obtains from (1.5) the system of two equations

$$\tilde{S}_{+t}-\frac{i}{2}(\tilde{S}_{+\xi\xi}+\tilde{S}_{+\eta\eta})+u_2(\xi,t)\,\tilde{S}_{+\xi}-u_1(\eta,t)\,\tilde{S}_{+\eta}=0,$$

$$\tilde{S}_{-t}+\frac{i}{2}(\tilde{S}_{-\xi\xi}+\tilde{S}_{-\eta\eta})+u_2(\xi,t)\,S_{-\xi}-u_1(\eta,t)\,\tilde{S}_{-\eta}=0, \quad (4.29)$$

where $\tilde{S}_{\pm}\doteq\tilde{S}_1\pm i\tilde{S}_2$.

So, equations (4.27), (4.28) which define the time evolution of the inverse problem data coincide with the linearized Ishimori-II equation.

The exact interrelation between the inverse problem data and $\tilde{S}_+$ and $\tilde{S}_-$ follow from the definitions (2.17) and (4.25), (4.26). In the weak fields limit the formulae (2.17) become

$$T_{12}^{+}(p,\lambda)\simeq\frac{1}{2\pi}\iint_{-\infty}^{+\infty}d\xi\,d\eta\,e^{-\frac{i\xi}{p}-\frac{i\eta}{\lambda}}\frac{1}{2}\tilde{S}_{-}\left(\partial_{\xi}-\partial_{\eta}+\frac{i}{\lambda}\right)\chi_{22}^{+},$$

$$T_{21}^{-}(p,\lambda)\simeq\frac{1}{2\pi}\iint_{-\infty}^{+\infty}d\xi\,d\eta\,e^{\frac{i\eta}{p}+\frac{i\xi}{\lambda}}\frac{1}{2}\tilde{S}_{+}\left(\partial_{\xi}-\partial_{\eta}+\frac{i}{\lambda}\right)\chi_{11}^{-}. \quad (4.30)$$

Then from the integral equations (2.6) it follows that in the weak limit

$$\chi_{12}^{+}\simeq-\int_{+\infty}^{\eta}d\eta'\,e^{\frac{i(\eta-\eta')}{\lambda}}\frac{1}{2}\tilde{S}_{-}\left(\partial_{\xi}-\partial_{\eta'}+\frac{i}{\lambda}\right)\chi_{22}^{+},$$

$$\chi_{22}^{+}\simeq 1-\int_{-\infty}^{\xi}d\xi'\,\frac{1}{2}\tilde{S}_{+}\left(\partial_{\xi'}-\partial_{\eta}+\frac{i}{\lambda}\right)\chi_{12}^{+},$$

$$\chi_{11}^{-}\simeq 1-\frac{1}{2}\int_{-\infty}^{\eta}d\eta'\,\tilde{S}_{-}\left(\partial_{\xi}-\partial_{\eta'}+\frac{i}{\lambda}\right)\chi_{21}^{-},$$

$$\chi_{21}^{-}\simeq-\int_{+\infty}^{\xi}d\xi'\,e^{-\frac{i(\xi-\xi')}{\lambda}}\frac{1}{2}\tilde{S}_{+}\left(\partial_{\xi'}-\partial_{\eta}+\frac{i}{\lambda}\right)\chi_{11}^{-} \quad (4.31)$$

So

$$\chi_{12}^{+}\simeq 0,\quad \chi_{22}^{+}\simeq 1,\quad \chi_{11}^{-}\simeq 1,\quad \chi_{21}^{-}\simeq 0. \quad (4.32)$$

Substituting (4.32) into (4.30), one gets

$$T_{12}^{+}(p,\lambda)\simeq\frac{1}{2\pi}\iint_{-\infty}^{+\infty}d\xi\,d\eta\,e^{-\frac{i\eta}{\lambda}-\frac{i\xi}{p}}\frac{i}{2\lambda}\tilde{S}_{-},$$

$$T_{21}^{-}(p,\lambda)\simeq\frac{1}{2\pi}\iint_{-\infty}^{+\infty}d\xi\,d\eta\,e^{\frac{i\eta}{p}+\frac{i\xi}{\lambda}}\frac{i}{2\lambda}\tilde{S}_{+}. \quad (4.33)$$

Combination of (4.33) and (4.25), (4.26) gives rise to the following interrelation between $\hat{S}$, $\hat{T}$ and $\tilde{S}_+$, $\tilde{S}_-$:

$$\hat{S}(\xi,\eta,t)=\pi\tilde{S}_{+}(\xi,\eta,t),$$

$$\hat{T}(\xi,\eta,t)=\pi\tilde{S}_{-}(\xi,\eta,t). \quad (4.34)$$

The coincidence of the equation which define the time evolution of the Fourier transform of the inverse problem data and linearized soliton equation takes place also for the DS-I equation [10—12]. This is a general phenomenon.

The formulae (4.33) and (4.34) allow us to describe the constrainst on the inverse problem data which correspond to the real field $\vec{S}(\xi, \eta, t)$.

Indeed, for real $\vec{S}$ one has $S_+ = S_-^*$. Then the relations (4.33) and (4.34) imply

$$\lambda T_{21}^-(l, \lambda) = -l (T_{12}^+(\lambda, l))^* \tag{4.35}$$

and

$$\hat{S}(\xi, \eta, t) = \hat{T}^*(\xi, \eta, t),$$
$$\hat{S}(\xi, \eta, t) = \hat{T}^*(\xi, \eta, t). \tag{4.36}$$

Iterating the formulae (2.17), we conclude that the constraints (4.35), (4.36) take place also for the nonweak real $\vec{S}$.

5. EXACT FORMULAE FOR THE GENERAL COHERENT STRUCTURES OF THE ISHIMORI-II EQUATION

The linear equations (4.27), (4.28) are the key equations in the whole method of the study of the coherent structures for the Ishimori-II equation. In what follows we will consider the case of the real functions $\vec{S}(\xi, \eta, t)$. In virtue of (4.36), (4.35), in this case it is sufficient to study the single equation:

$$i\hat{S}_t + \frac{1}{2}(\hat{S}_{\xi\xi} + \hat{S}_{\eta\eta}) + iu_2(\xi, t)\, \hat{S}_\xi - iu_1(\eta, t)\, \hat{S}_\eta = 0. \tag{5.1}$$

Our aim here is to derive the general formulae for the exact solutions of the Ishimori-II equation via the solutions of equation (5.1).

Solutions of equation (5.1) can be constructed by the method of the separation of the variables

$$\hat{S}(\xi, \eta, t) = X(\xi, t)\, Y(\eta, t) \tag{5.2}$$

similar to the DS-I equation [10—12]. In our case the functions X and Y obey the equations

$$iX_t + \frac{1}{2} X_{\xi\xi} + iu_2(\xi, t)\, X_\xi = 0, \tag{5.3}$$

$$iY_t - \frac{1}{2} Y_{\eta\eta} - iu_1(\eta, t)\, Y_\eta = 0. \tag{5.4}$$

The combination of the exact formulae (3.14) — (3.16) and the factorized solutions of equation (5.1) will allow us to construct the localized solitons of the Ishimori-II equation.

The general factorized solution of (5.1) is of the form

$$\hat{S}(\xi, \eta, t) = 2\pi \sum_{ij} \rho_{ij}\, X_i(\xi, t)\, Y_j(\eta, t), \tag{5.5}$$

where ρ_{ij} are constants and $X_i(\xi, t)$, $Y_j(\eta, t)$ are the solutions of equations (5.3) and (5.4) respectively.

The equivalent form of $\hat{S}$ is

$$\hat{S}(\xi, \eta, t) = 2\pi \sum_i \tau_i(\eta, t)\, \tilde{\tau}_i(\xi, t) \tag{5.6}$$

with

$$\tau_i(\eta, t) = \sum_j \rho_{ij}\, Y_j(\eta, t),$$
$$\tau_i(\xi, t) = X_i(\xi, t). \tag{5.7}$$

For such a $\hat{S}(\xi, \eta, t)$ one has

$$T_{21}^-(l, \lambda', t) = -\iint_{-\infty}^{+\infty} \frac{d\xi\, dn}{2\pi} \sum_{i,j} X_{i\xi}(\xi, t)\, Y_j(\eta, t)\, \rho_{ij}\, e^{\frac{i\xi}{\lambda} + \frac{i\eta}{l}} =$$
$$= \sum_i T_i(l, t)\, \tilde{T}_i(\lambda, t), \tag{5.8}$$

where

$$T_i(l, t) = \sum_j \rho_{ij} \int_{-\infty}^{+\infty} \frac{d\eta}{\sqrt{2\pi}} e^{\frac{i\eta}{l}}\, Y_j(\eta, t),$$

$$\tilde{T}_i(\lambda, t) = \frac{i}{\lambda} \int_{-\infty}^{+\infty} \frac{d\xi}{\sqrt{2\pi}} e^{\frac{i\xi}{\lambda}}\, X_i(\xi, t). \tag{5.9}$$

Using (4.35), one also gets

$$T_{12}^+(l, \lambda, t) = \sum_{ij} \rho_{ij}^* \frac{l}{\lambda} \int_{-\infty}^{+\infty} \frac{d\xi}{\sqrt{2\pi}} X_{i\xi}^* e^{-\frac{i\xi}{l}} \int_{-\infty}^{+\infty} \frac{d\eta}{\sqrt{2\pi}} Y_j^*(\eta, t)\, e^{-\frac{i\eta}{\lambda}} =$$
$$= \sum_i S_i(l, t)\, \tilde{S}_i(\lambda, t), \tag{5.10}$$

where

$$S_i(l,t)=\int\limits_{-\infty}^{+\infty}\frac{d\xi}{\sqrt{2\pi}}X_i^*(\xi,t)\,e^{-\frac{i\xi}{l}},$$

$$\tilde{S}_k(\lambda,\tau)=\sum_j\rho_{kj}\int\limits_{-\infty}^{+\infty}\frac{d\eta}{\sqrt{2\pi}}\,\frac{i}{\lambda}\,Y_j^*(\eta,t)\,e^{-\frac{i\eta}{\lambda}}. \tag{5.11}$$

The formulae (5.8) — (5.11) give the concrete expressions for the degenerated inverse problem data. Using these formulae, one can obtain the concrete expressions for the other needed quantities from the section 3. Taking into account that

$$\left(\tilde{T}_k(\lambda,t)\,e^{-\frac{i\xi}{\lambda}}\right)^{\pm}=\frac{i}{\sqrt{2\pi}\,\lambda}\int\limits_{\pm\infty}^{\xi}d\xi'\,X_k(\xi',t)\,e^{\frac{i}{\lambda}(\xi'-\xi)} \tag{5.12}$$

and

$$\left(\tilde{S}_k(\lambda,t)\,e^{\frac{i\eta}{\lambda}}\right)^{\pm}=-\frac{i}{\sqrt{2\pi}}\sum_j\rho_{kj}^*\int\limits_{\pm\infty}^{\eta}d\eta'\,e^{\frac{i(\eta-\eta')}{\lambda}}\,Y_j(\eta',t), \tag{5.13}$$

one gets

$$\sigma_i(\xi,t)=X_i^*(\xi,t),$$

$$\tau_j(\eta,t)=\sum_k\rho_{jk}Y_k(\eta,t),$$

$$\alpha_{jk}=\sum_{l,m}\rho_{jl}\rho_{km}^*a_{lm}=(\rho a\rho^+)_{jk},$$

$$a_{lm}=\int\limits_{-\infty}^{\eta}d\eta'\,Y_m^*(\eta',t)\,\partial_{\eta'}Y_l(\eta',t),$$

$$\beta_{ik}=b_{ik}=-\int\limits_{-\infty}^{\xi}d\xi'\,X_k(\xi',t)\,\partial_{\xi'}X_i^*(\xi',t), \tag{5.14}$$

$$\left(\tilde{T}_i(\lambda,t)\,e^{-\frac{i\xi}{\lambda}}\right)^{-}\xrightarrow[\lambda\to 0]{}\frac{1}{\sqrt{2\pi}}X_k(\xi,t),$$

$$\left(\tilde{S}_k(\lambda,t)\,e^{\frac{i\eta}{\lambda}}\right)^{-}\xrightarrow[\lambda\to 0]{}\frac{1}{\sqrt{2\pi}}\sum_j Y_j^*(\eta,t)\,(\rho^+)_{jk}.$$

Using these expressions and the formulae (3.14), we obtain

$$F_j=\sum_k(1-\rho a\rho^+b)_{jk}^{-1}\begin{pmatrix}\sum_p(\rho a\rho^+)_{kp}X_p^*(\xi,t)\\ \sum_l\rho_{kl}Y_l(\eta,t)\end{pmatrix}, \tag{5.15}$$

$$G_j=\sum_k(1-b\rho a\rho^+)_{jk}^{-1}\begin{pmatrix}X_k^*(\xi,t)\\ \sum_{l,m}b_{kl}\rho_{lm}Y_m(\eta,t)\end{pmatrix}. \tag{5.16}$$

Further the formulae (3.16) give

$$g=\chi^-(\lambda=0)=$$

$$=\begin{pmatrix}1-\langle X,(1-\rho a\rho^+b)^{-1}\rho a\rho^+X^*\rangle, & \langle Y^*,\rho^+(1-b\rho a\rho^+)^{-1}X^*\rangle\\ -\langle X,(1-\rho a\rho^+b)^{-1}\rho Y\rangle, & 1+\langle Y^*,\rho^+(1-b\rho a\rho^+)^{-1}b\rho Y\rangle\end{pmatrix}, \tag{5.17}$$

where we denote $\langle X,Y\rangle=\sum_i X_iY_i$. At last, rewriting the reconstruction formulae (2.23), (2.31) in the components $\chi(\lambda=0)$, we get

$$S_+=-\frac{2\chi_{22}(0)\,\chi_{21}(0)}{\chi_{11}(0)\,\chi_{22}(0)-\chi_{12}(0)\,\chi_{21}(0)},$$

$$S_-=\frac{2\chi_{11}(0)\,\chi_{12}(0)}{\chi_{11}(0)\,\chi_{22}(0)-\chi_{12}(0)\,\chi_{21}(0)}, \tag{5.18}$$

$$S_3=-\frac{\chi_{11}(0)\,\chi_{22}(0)+\chi_{12}(0)\,\chi_{21}(0)}{\chi_{11}(0)\,\chi_{22}(0)-\chi_{12}(0)\,\chi_{21}(0)}.$$

and

$$\varphi(\xi,\eta,t)=2i\ln(\chi_{11}(0)\,\chi_{22}(0)-\chi_{12}(0)\,\chi_{21}(0))+$$

$$+2\int\limits_{-\infty}^{\xi}d\xi'\,u_2(\xi',t)+2\int\limits_{-\infty}^{\eta}d\eta'\,u_1(\eta',t). \tag{5.19}$$

The formulae (5.18), (5.19), (5.17) are the main result of the present paper. They allow us to construct explicit exact solutions of the Ishimori-II equation with the nontrivial boundaries using the exact solutions of the linear equations (5.3), (5.4). Emphasize that the indices in (5.15), (5.16), (5.17) may be also a continuous one. They correspond to any set of the exact solutions of equations (5.3), (5.4). At this point the situation is similar to the DS-I equation [10—12].

An essential difference between the Ishimori-II and DS-I equations is that in our case the linear equations (5.3), (5.4) contain the first order derivatives over ξ and η and the pure imaginary potentials $iu_1(\eta, t)$ and $iu_2(\xi, t)$.

For the DS-I equation the corresponding linear equation is the one-dimensional nonstationary Schrödinger equation [10—12]. This last equation is associated with the IST integration of the Kadomtsev—Petviashvili (KP) equation [10—12], [1—4]. In our case the linear problem (5.3) (or (5.4)) is associated with the IST integrability of the modified KP equation. Indeed, the modified KP equation [20]

$$u_t + u_{xxx} + 6u^2 u_x \mp 12\partial_x^{-1} u_{yy} + 12 u_x \partial_x^{-1} u_y = 0 \tag{5.20}$$

is equivalent to the commutativity condition $[L_1, L_2] = 0$ of the operators [21]

$$L_1 = 2i\partial_y + \partial_x^2 \pm 2iu\partial_x,$$
$$L_2 = \partial_t + 4\partial_x^3 \pm 12iu\partial_x^2 + (\pm 6iu_x \pm 12(\partial_x^{-1} u_y) - 6u^2)\,\partial_x + C. \tag{5.21}$$

This fact becomes not so suprising if one takes into account the formal gauge equivalence between Ishimori and DS equations mentioned in [15, 16] and the gauge equivalence between KP and mKP equations studied in [20]. Indeed, it is easy to see that equations (5.3), (5.4), convert into the nonstationary Schrödinger equations

$$2i\tilde{X}_t + \tilde{X}_{\xi\xi} + (u_2^2 + iu_{2\xi} + 2\partial_\xi^{-1} u_{2t}(\xi', t))\,\tilde{X} = 0,$$
$$2i\tilde{Y}_t + \tilde{Y}_{\eta\eta} + (u_1^2 - iu_{1\eta} - 2\partial_\eta^{-1} u_{1t}(\eta', t))\,\tilde{Y} = 0. \tag{5.22}$$

under the gauge transformations

$$\tilde{X} = X e^{i\int_{-\infty}^{\xi} d\xi' u_2(\xi', t)}, \qquad \tilde{Y} = Y e^{-i\int_{-\infty}^{\eta} d\eta' u_1(\eta', t)}. \tag{5.23}$$

We see that the solution of the general initial-boundary value problem for the Ishimori-II equation is connected with the solution of the initial value problem for the modified KP equation. This last problem will be considered in the separate paper.

6. TIME INDEPENDENT BOUNDARIES

In this paper we restrict ourselves by the case of the time independent boundaries $u_1 = u_1(\eta)$, $u_2 = u_2(\xi)$. In this case we will be

able to use the previously known results for the special Zakharov—Shabat spectral problem.

For the stationary boundaries $u_1(\eta)$ and $u_2(\xi)$ equation (5.1) admits the full separation of variables

$$\tilde{S}(\xi, \eta, t) = T(t)\, X(\xi)\, Y(\eta). \tag{6.1}$$

The functions T, X and Y obey the following linear equations:

$$T_t(t) + \frac{i}{2}(\lambda^2 + \lambda'^2)\, T = 0,$$
$$X_{\xi\xi} + 2iu_2(\xi)\, X_\xi + \lambda^2 X(\xi) = 0; \tag{6.2}$$
$$Y_{\eta\eta} - 2iu_1(\eta)\, Y_\eta + \lambda'^2 Y(\eta) = 0, \tag{6.3}$$

where λ and λ' are parameters.

The one-dimensional spectral problems (6.2) — (6.3) can be investigated by the different methods. Here we will use their equivalence to the specialized Zakharov—Shabat spectral problem. Firstly we note that the spectral problems (6.2) and (6.3) are equivalent to the problems

$$\tilde{X}_{\xi\xi} + (\lambda^2 + u_2^2(\xi) - iu_{2\xi}(\xi))\,\tilde{X} = 0 \tag{6.4}$$

and

$$\tilde{Y}_{\eta\eta} + (\lambda'^2 + u_1^2(\eta) + iu_{1\eta}(\eta))\,\tilde{Y} = 0 \tag{6.5}$$

via the gauge transformations

$$X(\xi) \doteq \tilde{X}(\xi)\, e^{-i\int_{-\infty}^{\xi} d\xi' u_2(\xi')}, \qquad Y(\eta) \doteq \tilde{Y}(\eta)\, e^{i\int_{-\infty}^{\eta} d\eta' u_1(\eta')}. \tag{6.6}$$

So, the problems (6.2) and (6.3) are gauge equivalent to the spectral problems of the Schrödinger type

$$\Phi^{\pm}_{\rho\rho} + (\lambda^2 + u^2(\rho) \pm iu_\rho(\rho))\,\Phi^{\pm}(\rho, \lambda) = 0 \tag{6.7}$$

with the very special potential

$$V(\rho) = -(u^2(\rho) \pm iu_\rho(\rho)), \tag{6.8}$$

where $u(\rho)$ is a real function.

Then, it is not difficult to see that the spectral problem (6.7) can be rewritten in the following matrix form

$$\Psi_{1\rho}+iu\Psi_1=\lambda\Psi_2\,,$$

$$\Psi_{2\rho}-iu\Psi_2=-\lambda\Psi_1\,. \tag{6.9}$$

Indeed, the elimination of Ψ_2 or Ψ_1 from (6.9) gives rise to the scalar equations

$$\Psi_{1\rho\rho}+(\lambda^2+u^2+iu_\rho)\,\Psi_1=0\,,$$

$$\Psi_{2\rho\rho}+(\lambda^2+u^2-iu_\rho)\,\Psi_2=0\,. \tag{6.10}$$

So one can identify

$$\Phi^+(\rho)=\Psi_1(\rho)\,,\qquad \Phi^-(\rho)=\Psi_2(\rho)\,. \tag{6.11}$$

At last, transiting to the variables

$$v_1 \doteq \frac{\Psi_1+i\Psi_2}{2}\,,\qquad v_2 \doteq \frac{\Psi_1-i\Psi_2}{2i} \tag{6.12}$$

we arrive at the specialized Zakharov — Shabat spectral problem

$$i\begin{pmatrix}\partial_\rho, & -u\\ -u, & -\partial_\rho\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\lambda\begin{pmatrix}v_1\\ v_2\end{pmatrix}. \tag{6.13}$$

It follows from all these formulae that

$$\tilde{X}(\xi)=-v_2(\xi)-iv_1(\xi)\,,\qquad \tilde{Y}(\eta)=-v_1(\eta)-iv_2(\eta)\,, \tag{6.14}$$

where v_1 and v_2 are the solutions of the spectral problem (6.13).

So we able to use the exact results associated with the problem (6.13) for the construction of the exact solutions of the problems (6.2), (6.3). Of course, for the calculation of the function $X(\xi)$ we must use $u_2(\xi)$ as the potential u is (6.13) and for calculating the function $Y(\eta)$ one must use $u_1(\eta)$ as u in (6.13).

The spectral problem (6.13) has been studied in detail in the papers [18, 22] in connection with the IST integration of the modified KdV equation. It has been shown in [18, 22] that the spectrum of the problem (6.13) contains the continuous part defined on the real axis $\operatorname{Im}\lambda=0$ and the discrete part which consists from the points located symmetrically with respect to the imaginary axis $\operatorname{Re}\lambda=0$. The points laying on the imaginary axis correspond to the soliton potentials u and each pair of the points $\pm\alpha+i\beta$ correspond to the breathers. Since the problem (6.13) is not selfadjoint then the discrete spectrum of (6.13) contains also a multiple points which associated with the multi-poles of the reflection coefficients [23].

The eigenfunctions v_n and the potentials u which correspond to the discrete spectrum are found as usually [1—4] explicitly in the closed form.

General soliton-breather eigenfunction which corresponds to N_1 solitons and N_2 breathers is of the form [18, 22]

$$v_n(\rho)\doteq v(\rho,\lambda_n)=\sum_m (1+M^2)^{-1}_{nm}\left\{\begin{pmatrix}0\\1\end{pmatrix}e^{i\lambda_m\rho}+\begin{pmatrix}1\\0\end{pmatrix}\sum_k M_{mk}\,e^{i\lambda_k\rho}\right\} \tag{6.15}$$

where $(N_1+2N_2)\times(N_1+2N_2)$ matrix M looks like

$$M_{nm}=\frac{e^{i(\lambda_n+\lambda_m)\rho}\,C_m}{\lambda_n+\lambda_m}\,. \tag{6.16}$$

where $n, m=1, 2, \ldots, N_1+2N_2$ and $\lambda_n=\pm\alpha_n+i\beta_n$. The corresponding general transparent N_1-solition $+$ N_2-breather potential u is given by the formula

$$u(\rho)=2\frac{d}{d\rho}\operatorname{Im}\ln\det(1+iM)=2\frac{d}{d\rho}\operatorname{arctg}\frac{\operatorname{Im}\det(1+iM)}{\operatorname{Re}\det(1+iM)}\,, \tag{6.17}$$

$$u^2(\rho)=\frac{d^2}{d\rho^2}\ln\det(1+M^2)\,.$$

If all $\lambda_n=i\beta_n(\operatorname{Im}\lambda_n=0)$ then the formula (6.17) gives the general N_1-soliton potential. At all $\lambda_n=\pm\alpha_n+i\beta_n$ $(\operatorname{Im}\alpha_n=\operatorname{Im}\beta_n=0)$ we have the general N_2-breather potential. Correspondingly the formula (6.15) gives the N_1-soliton eigenfunction and N_2-breather eigenfunction.

The formulae (6.15) — (6.17) allow us to construct the exact solutions of the problems (6.2), (6.3). For the potential $u_2(\xi)$ and $u_1(\eta)$ given by the formula (6.17) the eigenfunctions X and Y are of the form

$$X_{(N_1,N_2)}(\xi)=(-v_2(\lambda_n,\xi)-iv_1(\lambda_n,\xi))\,e^{-i\int_{-\infty}^{\xi}d\xi' u_2(\xi')}\,, \tag{6.18a}$$

$$Y_{(\tilde{N}_1,\tilde{N}_2)}(\eta)=(v_1(\lambda_n,\eta)+iv_2(\lambda_n,\eta))\,e^{i\int_{-\infty}^{\eta}d\eta' u_1(\eta')}\,, \tag{6.18b}$$

where the function v_1 and v_2 are given by the formula (6.15). Note that the function $X_{(N_1,N_2)}(\xi)$ corresponds to the N_1-soliton $+$ N_2-breather boundary function $u_2(\xi)$ while $Y_{(\tilde{N}_1,\tilde{N}_2)}(\eta)$ correspond to the $\tilde{N}_1$-soliton $+$ $\tilde{N}_2$-breather boundary function $u_1(\eta)$. Now using the formulae (5.14) — (5.19), (6.15) — (6.18), we obtain the exact explicit solutions of the Ishimori-II equation with the general stationary transparent boundaries $u_2(\xi)$ and $u_1(\eta)$.

These solutions are defined by the arbitrary N_1-soliton$+N_2$-breather boundary function $u_2(\xi)$, N_1-soliton$+N_2$-breather boundary function $u_1(\eta)$ and arbitrary constants ρ_{ik}. The matrices a and b involved in the formula (5.17) can be represented in a more compact form. Indeed, taking into account (6.18), one gets

$$a_{lm}=\int_{-\infty}^{\eta} d\eta'\, Y^*(\lambda_m,\eta',t)\,\partial_{\eta'} Y(\lambda_l,\eta',t)=$$

$$=\int_{-\infty}^{\eta} d\eta'\, \tilde{Y}^*(\lambda_m,\eta',t)\,(\tilde{Y}_{\eta'}(\lambda_l,\eta',t)+iu_1(\eta)\,\tilde{Y}(\lambda_l,\eta')=$$

$$=\lambda_l\int_{-\infty}^{\eta} d\eta'\, \tilde{Y}^*(\lambda_m,\eta',t)\,\tilde{X}(\lambda_l,\eta',t) \tag{6.19}$$

and

$$b_{ik}=-\int_{-\infty}^{\xi} d\xi'\, X(\lambda_k,\xi',t)\,\partial_{\xi'} X^*(\lambda_i,\xi',t)=$$

$$=-\int_{-\infty}^{\xi} d\xi'\, \tilde{X}(\lambda_k,\xi',t)\,(\tilde{X}^*_{\xi'}(\lambda_i,\xi',t)+iu_2(\xi')\,\tilde{X}^*(\lambda_i,\xi'\,t))=$$

$$=\lambda_i^*\int_{-\infty}^{\xi} d\xi'\, \tilde{X}(\lambda_k,\xi',t)\,\tilde{Y}^*(\lambda_i,\xi',t)\,. \tag{6.20}$$

So one has

$$a_{lm}(\eta)=\lambda_l\, c_{ml}(\eta)\,,\qquad b_{ik}=\lambda_i^*\, C_{ik}(\xi)\,, \tag{6.21}$$

where

$$c_{ik}(\rho)\doteq\int_{-\infty}^{\rho} d\rho'\, X(\lambda_k,\rho')\,Y^*(\lambda_i,\rho')\,. \tag{6.22}$$

Emphasize one more, that $C_{ml}(\eta)$ is calculated via the functions $(Y(\lambda_i,\ \eta),\ X(\lambda_k,\ \eta))=(\Psi_1,\ \Psi_2)$ which correspond to the potential $u=u_1(\eta)$ in (6.9) while $C_{ik}(\xi)$ associated with the potential $u=u_2(\xi)$.

The solitons (s) and breather (b) are quite different transparent

potentials. As a result, we have four different types of the exact solutions of the Ishimori-II equation:

$$\vec{S}_{ss(N,M)}(\xi,\eta,t)\,,\quad \vec{S}_{sb(N,M)}(\xi,\eta,t)\,,\quad \vec{S}_{bs(N,M)}(\xi,\eta,t)\,,\quad \vec{S}_{bb(N,M)}(\xi,\eta,t)\,,$$

which correspond to the choices of X and Y as the soliton's or breather's eigenfunctions. The solutions $\vec{S}_{sb}$ and $\vec{S}_{bs}$ are, obviously, related by the interchange $\xi\leftrightarrow\eta$. So, we have the three essentially different types of the exact solutions

$$\vec{S}_{ss(N,M)}\,,\qquad \vec{S}_{sb(N,M)}\,,\qquad \vec{S}_{bb(N,M)}\,, \tag{6.23}$$

where the intergers N and M correspond the N-soliton (breather) boundary $u_2(\xi)$ and M-soliton (breather) boundary $u_2(\eta)$.

All these solutions can be calculated explicitly. Here we present the simplest examples of such exact solutions. Using the formulae given in the Appendix, one gets the simplest (5.5) localized soliton with $N=M=1$:

$$S_{ss(1,1)1}(\xi,\eta,t)=S_{ss\perp}(\xi,\eta)\cos\left(\frac{(\mu^2+\nu^2)\,t}{2}+\Phi_{ss(1,1)}(\xi,\eta)\right),$$

$$S_{ss(1,1)2}(\xi,\eta,t)=S_{ss\perp}(\xi,\eta)\sin\left(\frac{(\mu^2+\nu^2)\,t}{2}+\Phi_{ss(1,1)}(\xi,\eta)\right),$$

$$S_{ss(1,1)3}(\xi,\eta)=-1+\frac{2\,|X_1|^2|Y_1|^2}{|1-a_{11}b_{11}|^2}=-1+$$

$$+\frac{2\,e^{-2\mu\xi-2\nu\eta}}{\left(\frac{cd}{4\mu\nu}e^{-2\mu\xi-2\nu\eta}+\frac{\mu\nu}{cd}-1\right)^2+\left(\frac{d}{2\nu}e^{-2\nu\eta}+\frac{c}{2\mu}e^{-2\mu\xi}\right)^2}, \tag{6.24}$$

where

$$S_{ss\perp}(\xi,\eta)\doteq\sqrt{S^2_{ss(1,1)1}+S^2_{ss(1,1)2}}=\sqrt{1-S^2_{ss(1,1)3}(\xi,\eta)}=2e^{-\mu\xi-\nu\eta}\times$$

$$\times\frac{\sqrt{\left(\frac{d}{2\nu}e^{-2\nu\eta}-\frac{c}{2\mu}e^{-2\mu\xi}\right)^2+\left(\frac{cd}{4\mu\nu}e^{-2\mu\xi-2\nu\eta}-\frac{\mu\nu}{cd}+1\right)^2}}{\left(\frac{cd}{4\mu\nu}e^{-2\mu\xi-2\nu\eta}+\frac{\mu\nu}{cd}-1\right)^2+\left(\frac{d}{2\nu}e^{-2\nu\eta}+\frac{c}{2\mu}e^{-2\mu\xi}\right)^2},$$

$$\Phi_{ss(1,1)}(\xi,\eta)\doteq\operatorname{arctg}\frac{\frac{d}{2\nu}e^{-2\nu\eta}-\frac{C}{2\mu}e^{-2\mu\xi}}{\frac{Cd}{4\mu\nu}e^{-2\nu\eta-2\mu\xi}-\frac{\mu\nu}{Cd}+1}\,.$$

$$\varphi_{ss(1,1)}(\xi,\eta)=-2\,\mathrm{arctg}\frac{\frac{\nu}{2d}e^{-2\mu\xi}+\frac{\mu}{2C}e^{-2\nu\eta}}{\left(\left(\frac{C}{2\mu}\right)^2e^{-4\mu\xi}+1\right)\left(\left(\frac{d}{2\nu}\right)^2e^{-4\nu\eta}+1\right)+\frac{1}{4}e^{-4\mu\xi-4\nu\eta}-\frac{\mu\nu}{Cd}}+$$

$$+4\,\mathrm{arctg}\left(\frac{C}{2\mu}e^{-2\mu\xi}\right)+4\,\mathrm{arctg}\left(\frac{d}{2\nu}e^{-2\nu\eta}\right)+\mathrm{const}$$

where $\mu>0$. $\nu>0$ and C, d are real constants. It is easy see that, this solution is of the breather type in finite t.

The simplest (s,b)-type localized soliton for the Ishimori-II equation is $(N=M=1)$:

$$\rho^T=\begin{pmatrix}\rho_{11}\\ \rho_{12}\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix},\qquad \rho a\rho^+=a_{11},\qquad \rho_{1j}Y_j=Y_1,$$

$$S_{sb(1,1)1}(\xi,\eta,t)=S_{sb\perp}(\xi,\eta,t)\cos\left(\frac{(\alpha^2-\beta^2-\mu^2)\,t}{2}+\Phi_{sb(1,1)}(\xi,\eta,t)\right),$$

$$S_{sb(1,1)2}(\xi,\eta,t)=S_{sb\perp}(\xi,\eta,t)\sin\left(\frac{(\alpha^2-\beta^2-\mu^2)\,t}{2}+\Phi_{sb(1,1)}(\xi,\eta,t)\right),$$

$$S_{ss(1,1)3}(\xi,\eta,t)=-1+\frac{2\,|X_1|^2|Y_1|^2}{|1-a_{11}b_{11}|^2}=-1+$$

$$+\frac{e^{-2\mu\xi+2\alpha\beta t+\Psi}(\mathrm{ch}\,(2\beta\eta+\Psi)-\sin(2\alpha\eta+\varphi-\delta))}{\left|\left(\mathrm{ch}(2\beta\eta+\Psi)+\frac{i\beta}{\alpha}\sin(2\alpha\eta+\varphi-\delta)\right)\left(\frac{d}{2\mu}e^{-2\mu\xi}+i\right)-\frac{i\mu(\alpha+i\beta)}{4\beta d}e^{\Psi+2\alpha\beta t}\right|^2},\quad(6.25)$$

where

$$S_{sb\perp}(\xi,\eta,t)\doteq\sqrt{S^2_{sb(1,1)1}+S^2_{sb(1,1)2}}=\sqrt{1-S^2_{sb(1,1)3}(\xi,\eta,t)},$$

$$\Phi_{sb(1,1)}(\xi,\eta,t)\doteq\mathrm{arctg}\frac{\sin(\alpha\eta+\varphi-\delta)-e^{2\beta\eta+\Psi}\cos\alpha\eta}{\cos(\alpha\eta+\varphi-\delta)-e^{2\beta\eta+\Psi}\sin\alpha\eta}+$$

$$+\mathrm{arctg}\frac{\frac{d}{2\mu}e^{-2\mu\xi}\mathrm{ch}\,(2\beta\eta+\Psi)+\frac{\beta}{\alpha}\sin(2\alpha\eta+\varphi-\delta)-\frac{\mu}{4d}e^{\Psi+2\alpha\beta t}}{\mathrm{ch}\,(2\beta\eta+\Psi)-\frac{d}{2\mu}\frac{\beta}{\alpha}e^{-2\mu\xi}\sin(2\alpha\eta+\varphi-\delta)-\frac{\mu\alpha}{4d\beta}e^{\Psi+2\alpha\beta t}},$$

$$\varphi_{sb(1,1)}(\xi,\eta)=-8\,\mathrm{arctg}\frac{\beta}{\alpha}+\frac{\sin(2\alpha\eta+\varphi-\delta)}{\mathrm{ch}\,(2\beta\eta+\Psi)}+8\,\mathrm{arctg}\left(\frac{d}{2\mu}e^{-2\mu\xi}\right)-$$

$$-4\,\mathrm{arctg}\frac{\frac{d}{2\mu}e^{-2\mu\xi}\mathrm{ch}\,(2\beta\eta+\Psi)-\frac{\beta}{\alpha}\sin(2\alpha\eta+\varphi-\delta)+\frac{\mu}{4d}e^{\Psi+2\alpha\beta t}}{\frac{\beta}{\alpha}\frac{d}{2\mu}e^{-2\mu\xi}\sin(2\alpha\eta+\varphi-\delta)+\mathrm{ch}\,(2\beta\eta+\Psi)-\frac{\mu\alpha}{4d\beta}e^{\Psi+2\alpha\beta t}},$$

where

$$e^{-\Psi}=\frac{|C|\,\alpha}{2\beta\sqrt{\alpha^2+\beta^2}},\qquad \alpha+i\beta=\sqrt{\alpha^2+\beta^2}\,e^{i\delta},\qquad C=|C|\,e^{-\varphi}.$$

In the formulas for this (s,b) solution $\lambda=i\mu$, $\mu>0$ is the pole of the reflection coefficient for the problem (6.13) which corresponds to soliton, d — real parameter of soliton $\lambda'_{1,2}=\pm\alpha+i\beta$ are the poles of the reflection coefficient for the problem (6.13) which correspond to breather, C — complex parameter of the breather.

At last, the simplest $N=N=1$ localized soliton of the (b,b) type is of the form:

$$\rho=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}0&1\\ 0&0\end{pmatrix},\qquad \rho a\rho^+=\begin{pmatrix}a_{22}&0\\ 0&0\end{pmatrix},$$

$$S_{bb(1,1)1}(\xi,\eta,t)=S_{bb\perp}(\xi,\eta)\cos\left((\alpha^2-\beta^2)\,t+\Phi_{bb(1,1)}(\xi,\eta)\right),$$

$$S_{bb(1,1)2}(\xi,\eta,t)=S_{bb\perp}(\xi,\eta)\sin\left((\alpha^2-\beta^2)\,t+\Phi_{bb(1,1)}(\xi,\eta)\right),$$

$$S_{bb(1,1)3}(\xi,\eta)=-1+\frac{2\,|X_1|^2|Y_2|^2}{|1-a_{22}b_{11}|^2}=-1+$$

$$+\{e^{\Psi_1+\Psi_2}(\mathrm{ch}\,(2\beta\xi+\Psi_2)+\sin(2\alpha\xi+\varphi_2-\delta))\,(\mathrm{ch}\,(2\beta\eta+\Psi_1)+$$

$$+\sin(2\alpha\eta+\varphi_1-\delta))\}/\left|\left(\mathrm{ch}(2\beta\eta+\Psi_1)+\frac{i\beta}{\alpha}\sin(2\alpha\eta+\varphi_1-\delta)\right)\times\right.$$

$$\left.\times\left(\mathrm{ch}(2\beta\xi+\Psi_2)+\frac{i\beta}{\alpha}\sin(2\alpha\xi+\varphi_2-\delta)\right)-\frac{(\alpha^2+\beta^2)}{16\beta^2}e^{\Psi_1+\Psi_2}\right|^2,\quad(6.26)$$

where

$$S_{bb\perp}(\xi,\eta)\doteq\sqrt{S^2_{bb(1,1)1}+S^2_{bb(1,1)2}}=\sqrt{1-S^2_{bb(1,1)3}(\xi,\eta)},$$

$$\Phi_{bb(1,1)}(\xi,\eta)\doteq-\mathrm{arctg}\frac{\cos(\alpha\xi+\varphi_2-\delta)+e^{2\beta\xi+\Psi_2}\sin-\alpha\xi}{\sin(\alpha\xi+\varphi_2-\delta)+e^{2\beta\xi+\Psi_2}\cos-\alpha\xi}-$$

$$-\mathrm{arctg}\frac{\sin(\alpha\eta+\varphi_1-\delta)+e^{2\beta\eta+\Psi_1}\cos\alpha\eta}{\cos(\alpha\eta+\varphi_1-\delta)+e^{2\beta\eta+\Psi_1}\sin\alpha\eta}+$$

$$+\mathrm{arctg}\left\{\frac{\beta}{\alpha}(\mathrm{ch}(2\beta\xi+\Psi_2)\sin(2\alpha\eta+\varphi_1-\delta)-\mathrm{ch}(2\beta\eta+\Psi_1)\sin(2\alpha\xi+\varphi_2-\delta))+\right.$$

$$\left.+\frac{\alpha}{8\beta}e^{\Psi_1+\Psi_2}\right\}/\mathrm{ch}\,(2\beta\xi+\Psi_2)\,\mathrm{ch}\,(2\beta\eta+\Psi_1)+$$

$$+\frac{\beta^2}{\alpha^2}\sin(2\alpha\xi+\varphi_2-\delta)\sin(2\alpha\eta+\varphi_1-\delta)+\frac{(\alpha^2-\beta^2)}{16\beta^2}e^{\Psi_1+\Psi_2},$$

$$\varphi_{bb(1,1)}(\xi,\eta)=-8\,\mathrm{arctg}\frac{\beta}{\alpha}\frac{\sin(2\alpha\eta+\varphi_1-\delta)}{\mathrm{ch}\,(2\beta\eta+\Psi_1)}-8\,\mathrm{arctg}\frac{\beta}{\alpha}\frac{\sin(2\alpha\xi+\varphi_2-\delta)}{\mathrm{ch}\,(2\beta\xi+\Psi_2)}+$$

$$+4\operatorname{arctg}\left\{\frac{\beta}{\alpha}(\operatorname{ch}(2\beta\eta+\Psi_1)\sin(2\alpha\xi+\varphi_2-\delta)+\right.$$

$$\left.+\operatorname{ch}(2\alpha\xi+\Psi_2)\sin(2\alpha\eta+\varphi_1-\delta))\right\}/\operatorname{ch}(2\beta\eta+\Psi_1)\operatorname{ch}(2\beta\xi+\Psi_2)-$$

$$-\frac{\beta^2}{\alpha^2}\sin(2\alpha\xi+\varphi_2-\delta)\sin(2\alpha\eta+\varphi_1-\delta)+\frac{(\alpha^2+\beta^2)}{16\beta^2}e^{\Psi_1+\Psi_2},$$

where

$$e^{-\Psi_k}\quad\frac{|C_k|\alpha}{2\beta\sqrt{\alpha^2+\beta^2}},\quad C_k \doteqdot |C_k|e^{i\varphi_k},\quad (k=1,2);\quad \alpha+i\beta=\sqrt{\alpha^2+\beta^2}\,e^{i\delta}.$$

In the formulas for this (b, b) solution $\lambda_1=\alpha+i\beta$, $\alpha_2=-\alpha+i\beta$ are the poles of the reflection coefficient of the problem (6.13) which correspond to breather $u_2(\xi)$. $\lambda_1'=\alpha+i\beta$, $\lambda_1'=-\alpha+i\beta$ are the poles of the reflection coefficient of the problem (6.13) which correspond to breather $u_1(\eta)$. C_1, C_2 are complex parameters of the breathers.

Let us note that the poles of the reflection coefficient which determine the breathers $u_1(\eta)$ and $u_2(\xi)$ are choosen in the formulas (6.26) of (b, b) solution to be the same: $\lambda_1=\lambda_1'=\alpha+i\beta$, $\lambda_2=\lambda_2'=-\alpha+i\beta$.

All these solutions are of the breather type, all of them decrease exponentially at $\xi^2+\eta^2\to\infty$ at all directions, but their forms are quite different. The solution $\bar{S}_{bs}$ is obtained from $\bar{S}_{sb}$ by the interchange $\xi\leftrightarrow\eta$.

These examples demonstrate the richness of the types of the coherent structures for the Ishimori-II equation. In more detail the properties of the localized solitons for the Ishimori-II equation, the case of the time dependent boundaries and the scattering of the localized solitons will be considered elsewhere.

Acknowledgements. We are thankful to A. Degasperis and P. Santini for the useful discussions.

APPENDIX

Here we present for completeness some intermediate formulae which are needed for the calculation of the exact solutions of the Ishimori-II equation.

The simplest soliton potential for the spectral problem (6.13) corresponds to $N_1=1$, $N_2=0$, $\lambda=i\mu$ $(\mu>0)$ and $c_1=\frac{b(i\mu)}{a'(i\mu)}=$ $=ic(\operatorname{Im}c=0)$ [18, 22, 3]. It is of the form

$$u(\rho)=2\frac{d}{d\rho}\operatorname{arctg}\left(\frac{c}{2\mu}e^{-2\mu\rho}\right)=-\frac{2\mu sgnc}{\operatorname{ch}(2\mu\rho+\rho_0)},\tag{A.1}$$

where $\exp\rho_0=\frac{2\mu}{|c|}$. The corresponding eigenfunction $\tilde{X}$ is

$$\tilde{X}(\rho)=-i\frac{e^{-i\mu\rho+i\frac{\mu^2t}{2}}}{\frac{c}{2\mu}e^{-2\mu\rho}+i}.\tag{A.2}$$

The quantites a and b are

$$a=\frac{\nu}{d}\frac{1}{\frac{d}{2\nu}e^{-2\nu\eta}+i},\quad b=-\frac{\mu}{c}\frac{1}{\frac{c}{2\mu}e^{-2\mu\xi}+i}.\tag{A.3}$$

The functions $X_1(\xi)$ and $Y_1(\eta)$ are

$$X_1(\xi)=\frac{(-i)\,e^{-\mu\xi+\frac{i\mu^2t}{2}}}{\left(\frac{c}{2\mu}\right)e^{-2\mu\xi}-i},\quad Y_1(\eta)=\frac{e^{-\nu\xi+\frac{i\nu^2t}{2}}}{\left(\frac{d}{2\nu}\right)e^{-2\nu\eta}-i}.\tag{A.4}$$

Using the formulae (A.2) — (A.4) and choosing $\rho=1$, one obtains from the formulae (5.17) — (5.18) the solution (6.24).

The simplest breather potential corresponds to $N_1=0$, $N_2=1$, $\lambda_1=\alpha+i\beta$, $\lambda_2=-\alpha+i\beta$, α, $\beta>0$, $c_1=\frac{b(\lambda_1)}{a'(\lambda_1)}=ic$, $c_2=\frac{b(\lambda_2)}{a'(\lambda_2)}=ic^*$ [18, 22, 3]. It looks like

$$u(\rho)=-2\frac{d}{d\rho}\operatorname{arctg}\frac{\beta\sin(2\alpha\rho+\varphi-\delta)}{\alpha\,ch(2\beta\rho+\Psi)},\tag{A.5}$$

where $\operatorname{tg}\delta=\frac{\beta}{\alpha}$, $\exp\Psi=\frac{2\beta\sqrt{\alpha^2+\beta^2}}{\alpha|c|}$, $c=|c|e^{i\varphi}$. The corresponding eigenfunction $\tilde{Y}_1(\rho)$ is of the form

$$\tilde{Y}_1(\rho)=\frac{i\,e^{i\alpha\rho+\beta\rho+\Psi}+e^{-i\alpha\rho-\beta\rho-i\varphi+i\delta}}{2\operatorname{ch}(2\beta\rho+\Psi)-2i\frac{\beta}{\alpha}\sin(2\alpha\rho-\delta+\varphi)}. \tag{A.6}$$

The function $Y_1(\rho)$ looks like

$$Y_1(\rho)=\frac{i\,e^{i\alpha\rho+\beta\rho+\Psi}+e^{-i\alpha\rho-\beta\rho-i\varphi+i\delta}}{2\operatorname{ch}(2\beta\rho+\Psi)+2i\frac{\beta}{\alpha}\sin(2\alpha\rho-\delta+\varphi)}. \tag{A.7}$$

We present here also one general formulae. Using the formula (6.15) for v_1, v_2 and the formula (6.17) for u, it is not difficult to show that the functions X_n and Y_m are representable in the following compact form

$$X_n(\xi,t)=X(\lambda_n,\xi,t)=-\sum_p(1-iM)^{-1}_{np}\,e^{i\lambda_p\xi-\frac{i\lambda_n^2t}{2}}\,e^{-i\int^{\xi}d\xi'\,u_2(\xi')}=$$

$$=-e^{-\frac{i\lambda_n^2t}{2}}\sum_p(1-iM)^{-1}_{np}\,e^{i\lambda_p\xi}\,\frac{(\det(1-iM))^*}{\det(1+iM)}, \tag{A.8}$$

$$Y_m(\eta,t)=Y(\lambda_m,\eta,t)=i\sum_p(1+iM)^{-1}_{np}\,e^{i\lambda_p\eta-\frac{i\lambda_n^2t}{2}}\,e^{-i\int^{\eta}d\eta'\,u_1(\eta')}=$$

$$=i\,e^{-\frac{i\lambda_m^2t}{2}}\sum_p(1+iM)^{-1}_{mp}\,e^{i\lambda_p\eta}\,\frac{\det+(1+iM)}{(\det(1-iM))^*}. \tag{A.9}$$

These formulae essentially simplify all calculations.

REFERENCES

1. *V.E. Zakharov, S.V. Manakov, S.P. Novikov, L.P. Pitaevski.* Theory of Solitons. The Inverse Problem Method, Nauka, Moscow, 1980. Consultant Bureau, 1984.
2. *M.J. Ablowitz, H. Segur.* Solitons and Inverse Scattering Transform, SIAM, Philadelphia, 1981.
3. *R.K. Dodd, J.C. Eilbeck, J.D. Gibbon, H.C. Morris.* Solitons and Nonlinear Waves, Academic Press, New York, 1982.
4. *Y. Satsuma*, J. Phys. Soc. Japan, 40 (1976) 286.
5. *L.A. Bordag, A.R. Its, V.B. Matveev, S.V. Manakov, V.E. Zakharov.* Phys. Lett., 63A (1979) 205.
6. *H. Cornille.* J. Math. Phys., 20 (1979) 144.
7. *M. Boiti, J.J.-P.Leon, L. Martina, F. Pempinelli.* Phys. Lett., 132A (1988) 432.
8. *M. Boiti, J.J.-P.Leon, F. Pempinelli.* Phys. Lett., 141A (1989) 101.
9. *M. Boiti, J.J.-P.Leon, L. Martina, F. Pempinelli.* Preprints Montpellier, PM/88-40, 1988; PM/88-44, 1988; PM/89-17, 1989; PM/90-04, 1990.
10. *A.S. Fokas, P.M. Santini.* Phys. Rev. Lett., 63 (1989) 1329.
11. *A.S. Fokas, P.M. Santini.* Coherent Structures and Boundary Value Problem for the Davey—Stewartson Equation. Preprint INS 121, Clarkson University, 1989; Physica D.
12. *P.M. Santini, A.S. Fokas.* The Inital-Boundary Value Problem for the Davey—Stewartson Equation... Preprint 684, Universita di Roma, 1898.
13. *P.M. Santini.* Physica D, 41 (1990) 26.
14. *Y. Ishimori.* Progr. Theor. Phys., 72 (1984) 33.
15. *B.G. Konopelchenko, B.T. Matkarimov.* Phys. Lett., 135A (1989) 183.
16. *B.G. Konopelchenko, B.T. Matkarimov.* J. Math. Phys., 1990; Preprint Saclay SPhT/88/229, 1988.
17. *R. Beals, R.R. Coifman.* Inverse Problems, 5 (1989) 81.
18. M. Wadati. J. Phys. Soc., Japan, 32 (1972) 168.
19. *B.G. Konopelchenko, B.T. Matkarimov.* Stud. Appl. Math., 82 (1990) 319.
20. *B.G. Konopelchenko.* Phys. Lett., 92A (1982) 323.
21. *B.G. Konopelchenko, V.G. Dubrovsky.* Phys. Lett., 102A (1984) 15.
22. *M. Wadati.* J. Phys. Soc., Japan, 34 (1973) 1289.
23. *M. Wadati, K. Ohkuma.* Journ. Phys. Soc., Japan, 51 (1982) 2029.

V.G. Dubrovsky, B.G. Konopelchenko

Coherent Structures for the Ishimori Equation. I. Localized Solitons with the Stationary Boundaries

В.Г. Дубровский, Б.Г. Конопельченко

Когерентные структуры для Ишимори. I. Локализованные солитоны со стационарными границами

Ответственный за выпуск С.Г.Попов

Работа поступила 21 мая 1990 г.
Подписано в печать 25.06 1990 г. МН 02338
Формат бумаги 60×90 1/16 Объем 2,9 печ.л., 2,4 уч.-изд.л.
Тираж 250 экз. Бесплатно. Заказ № 76

Набрано в автоматизированной системе на базе фотонаборного автомата ФА1000 и ЭВМ «Электроника» и отпечатано на ротапринте Института ядерной физики СО АН СССР,
Новосибирск, 630090, пр. академика Лаврентьева, 11.