16

ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

G.A. Kuz'min and A.Z. Patashinskii

REDUCED DESCRIPTION OF TURBULENT VORTEX STRUCTURES

PREPRINT 91-21

НОВОСИБИРСК

Reduced Description

of Turbulent Vortex Structures

A.Z. Patashinskii

Institute of Nuclear Physics,
630090, Novosibirsk, USSR

G.A. Kuz'min

Institute of Thermophysics,
630090, Novosibirsk, USSR

ABSTRACT

We present a kinematic method of classification of high vorticity regions (vortices) by their integral tensor moments. The simplest moments are shown to represent the high vorticity region either as a vortex ring or an element of a vortex tube. Higher order moments contain more details of the vortex structure and of its deformation. The most noticeable modes are the rotation of the vortex by its own vorticity, deformation of the fluid volume occupied by the vorticity, deformations that twist the vortex. The high order moments are very sensitive to small scale perturbations. We study the space distributions of low order tensor fields for relatively simple vortical structures.

1. INTRODUCTION

Turbulent motion of a fluid is a result of a joint action of nonlinearity and dissipation. To a certain extent, effects of this action are rather contradictory. Both coherence and chaos are revealed in turbulent flow, the former being manifested by existence of organized or coherent structures (Townsend 1976, Cantwell 1981, Hussain 1986), and the latter being represented by their chaotic motion. The large scale coherent structures produce the noticeable part of turbulent energy and of Reynolds stress. Small scale motions are less studied. They are known to be highly intermittent and to occupy regions of small total volume (Monin & Yaglom 1975). The intermittency signifies a marked coherence in the motion at small scales. It was conjectured, that the turbulent flow may be treated as a system of relatively stable vortices. This idea was formulated at the early stages of investigations of turbulence. Recent studies make this idea all the more likely.

At large Reynolds numbers R, vortices of different scales fill in the very wide inertial range. The large number of these modes $N \propto R^{9/4}$ (Monin & Yaglom 1975) considerably complicate the investigation of fully developed turbulence. The problem become more tractable if a modeling of turbulence by a finite number of chaotically moving vortices of various scale is possible (Patashinskii 1991). The problem may be simplified by considering the acts of interactions separately (Kuz'min 1991). After that the statistical study of the full system may be hoped to perform.

Evolution of some particular vortex configurations, which are similar to those occurred in the turbulent flows, were recently investigated on supercomputers (see, for example, Kida & Takaoka 1987, Melander & Hussain 1989, Pumir & Siggia 1989). To describe more complex systems, one needs a simplified description of the inertial range vortices. The tensor space moments, which are described bellow, are intended to give a reduced description of small scale motions. They are useful in problems of recognition and description of complicated motions. Such a problem arises when analyzing the flow field obtained in computer simulations or by a direct multipoint measurement. The methods of recognition and investigation of local flow structure are quite limited. Usually one restricts himself to a visual study of the flow pictures.

We note that the problem under consideration is related to those considered in the information theory, where a general technique of recognition and also of description of possible structures in presence of fluctuations is developed. The basic notion of the theory is the feature space, which

is the space of quantitative characteristics of the structures. For the practical use of the theory, the number of independent characteristics should be as limited as it is necessary to recognize and to describe the flow structure. There exist a resemblance of the present theory to that developed for the local structure of condensed matter (Mitus & Patashinskii 1981,1987). The feature space for vortex structures is the space of tensor moments of the flow field (Kuz'min & Patashinskii 1985, 1986). The similar moments were defined by Melander, Zabusky & Stychek 1986 for two-dimensional flows.

In section 2, we define the feature space for two-dimensional vortices, and describe the simplest vortex structures. The 3-dimensional flow configurations, which are responsible for transfer of the turbulent energy to small scales, are reviewed in section 3. The tensor moments for the 3-dimensional vortices are defined in the section 4. The moments for simple vortex structures are calculated in sec. 5.

2. OUTLINE OF APPROACH
TWO DIMENSIONAL VORTEX STRUCTURES

2.1 Isolated Vortices

There exist a fluid motion, in which the flow along an axis is suppressed or is less essential than the flow along the other two axes. One may mention the large scale geophysical flows, the flow of conducting fluid in a strong magnetic field, the mixing layer, the flows in soap films and so on. In many respects, these flows are similar to strictly two dimensional ones.

The typical structures of a two dimensional flow are the solitary vortices (the regions of high vorticity of any definite sign), the vortex pairs (the bounded states composed of two solitary vortices of the same sign) and the vortex couples (the bounded states of two solitary vortices of different signs). More rarely the more complex bounded states occur. The portion of vortices of a certain type varies in space and time.

The known quantitative characteristics of an isolated vortex are the multipole moments (Batchelor 1967)

$$C = \int \omega\, dA,\quad P_i = e_{ij}\int x_j\,\omega(\vec{x})\, dA,\quad J = -\frac{1}{2}\int x^2\omega(\vec{x})\, dA, \tag{2.1}$$

$$t_{ij} = \int (2x_i x_j - x^2\delta_{ij})\,\omega(\vec{x})\, dA. \tag{2.2}$$

The vortex charge C determines the circulation of velocity around the vortex patch. The vector $\vec{P}$ is the vortex momentum and J is the angular momentum. The tensor t_{ij} describes the difference of the vortex form from the circular one.

Parameters (2.1), (2.2) are the simplest irreducible tensor moments of vorticity distribution. The full set of irreducible moments is written as

$$N^{(n)}_{i_1\ldots i_m}(\vec{x}) = \int r^n\left(r^m\frac{\partial^m \log r}{\partial r_{i_1}\ldots\partial r_{i_m}}\right)\omega(\vec{x}+\vec{r})\, dA \tag{2.3}$$

where $\vec{x}$ is chosen so as the satisfactory description of the vortex structure be achieved by as few tensor moments as

possible. For the solitary vortex and for the vortex pair a vorticity centroid is defined at

$$X_i = \frac{1}{C}\int \omega(\vec{x})\, x_i\, dA = -\frac{1}{C}\, e_{im} P_m .$$

For the vortex couple C=0, and the vector $\vec{P}$ do not determine the vorticity centroid. The plausible expression for the vorticity centroid is

$$X_i = \left(-\frac{1}{2}\, e_{ik} P_k T_{jj} + e_{ij} P_k T_{kj}\right)/P^2, \tag{2.4}$$

where $T_{ij} = \int x_i x_j\,\omega(\vec{x})dA$. In particular, for a couple of point vortices $\omega = \kappa[\delta(\vec{x}) - \delta(\vec{x}')]$ one obtains the natural expression from (2.4)

$$\vec{X} = (\vec{x} + \vec{x}')/2.$$

If, in turn, $\vec{P}$=0, the vorticity centroid has to be determined from tensor moments of higher order.

The moment $N^{(m)}$ is the projection of a vorticity field on the basic set of functions. The radial and the angular dependence of the basic functions are determined accordingly by r^n and by

$$S^{(m)}_{i_1\ldots i_m} = r^m\partial^m \log r/\partial r_{i_1}\ldots\partial r_{i_m} .$$

The tensors $S^{(m)}$ are known to be the circular harmonics of integer degree. The components of the tensors $S^{(m)}$ are equal to $\pm\cos m\phi$, $\pm\sin m\phi$ in polar coordinate system with the origin at the point $\vec{x}$. Therefore, each irreducible tensor $N^{(n)}$ has two independent components. One parameter is a characteristics of orientation of the vortex structure, while the other determines its form. Thus the invariants of the tensor moments can be treated as the formparameters of the vortex structures.

2.2. Organized Vortices of a Given Scale

The vortex structures are naturally described by smoothed fields. The smoothing is to be performed over a scale λ, which is small when compared to the main scale l of the flow, but is large in comparison with the Kolmogorov scale η, $l \gg \lambda \gg \eta$. It is worthy to note, that the similar smoothing is performed when the hydrodynamic equations are derived from the equations for the fluid molecules. As a result of such a smoothing, the microscopic molecular characteristics are replaced by the fluid velocity, density and viscosity. The fluid motions and the microscopic ones are separated by by a wide spectral gap. The large difference between the smallest hydrodynamic scales and the molecular ones implies that the physical point in fluid mechanics is isotropic, that is, the fluid is described only by scalar characteristics (density, enthalpy and so on).

On the contrary, the smoothing of a vortex structure inevitably divides the motions of near scales. So the new physical point whose effective scale is $\lambda \gg \eta$ contains the motions of slightly less scale than λ. These internal motions are described by nonisotropic tensor-type characteristics. One may say, that the Navier- Stokes

equations describe a motion of a scalar fluid, which is composed of physical points without any internal structure. The smoothing over scales $\lambda \gg \eta$ gives a nonisotropic fluid, which has tensor properties at any point.

To extract the vortices of a given scale λ, let us consider the integrals (2.3) over the circular region of radius λ. By moving the center of the circle in space, one may reveal the structure of the flow in the vicinity of any point $\vec{x}$. The moments (2.3) are now the tensor functions of the position $\vec{x}$. The formulae (2.3) may now be considered as a filtration of the vorticity field. The Fourier transforms of the kernels overlap, so the information about the vortex structures of the given scale λ is duplicated in various moments.

We consider the projection of the vorticity field only on the functions with the zeroth radial number n=0 and low azimuthal numbers m. The full set of moments may be used for a detailed investigation of the vortex structures after their type and situation have been established. Let us use the polar coordinates with the origin at the point $\vec{x}$. The moments can be written in the complex form

$$N_m(\vec{x}, \lambda)=\int_{r<\lambda} \omega[\vec{x} + r\vec{e}(\phi)] \exp(im\phi)dA(\vec{r}). \qquad (2.5)$$

The squared modules $|N_m|^2$ are the invariants of the tensors (2.3), so they describe the form of the vortex structures. The phases of complex moments N_m describe the orientation of the vortex.

Let us consider the simple examples. The group of point vortices of strength κ_i, $i=1,2,...k$, at the points $\vec{r}_1,...\vec{r}_k$ inside the circle, induces the set of moments

$$N_m(\vec{x})=\sum_{i=1}^{k}\kappa_i \exp(im\phi).$$

For a solitary point vortex

$$N_m = \kappa \exp(im\phi), \text{ so } |N_m|^2 = \kappa^2, \ m=1,2,...,.$$

For $k=2$

$$|N_m|^2 = \kappa_1^2+\kappa_2^2+2\kappa_1\kappa_2 \cos[m(\phi_1-\phi_2)]=$$

$$=(\kappa_1+\kappa_2)^2-4\kappa_1\kappa_2\sin^2[m(\phi_2-\phi_1)/2] \ . \qquad (2.6)$$

If $\vec{x}$ is situated on the line connecting the two vortices, then $\phi_1-\phi_2=\pm\pi$. From (2.6) one sees, that for vortices of the same sign this line is marked by the minimum of the invariants of odd order and by the maximum of the invariants of even order. On the contrary, for vortices of different sins, the line connecting them is marked by the maximum of the invariants of odd order and by the minimum of the invariants of even order. These properties may be useful when searching the vortex structures of the definite kind. The low order invariants are of the primary interest, because they are less sensitive to small scale variations of the vorticity field inside the vortices.

3. REVIEV OF VORTICITY STRUCTURE AT SMALL SCALES

Various terminology is used to name the vortex structures. The term *"coherent structure"* means the vorticity organized into coherent clouds. It is kept in mind that a certain configurations occur more frequently than the other ones, and these structures are the preferred modes of the flow (Hussain 1986). When one says *"organized structure"*, he means that this vortex is reasonably described by a deterministic solution of hydrodynamic equations (Cantwell 1981). Some vortex configurations are neither coherent nor organized structures, but may be favorable to induce the small scale motions. The probability of such *"key structures"* may not be very distinguished, but these structures, if any, are noteworthy.

There exist several scenarios of energy transfer towards small scales in three-dimensional flows. One should mention the generation of small scales due to a cascade of instabilities; the (stochastic or deterministic) stretching of vortex tubes and sheets; the spontaneous generation of a singularity in ideal three-dimensional fluid.

The stability considerations are plausible to explain, why a laminar flow is destroyed at the near-critical Reynolds number. But these considerations are questionable when describing the large scale vortices in fully developed turbulent flows. Some more questions arise when one speaks about generation of small scale structures due to a cascade of instabilities. Of course, almost all realizations of the turbulent flow are believed to be unstable. But this merely signifies that the statistic considerations are necessary to explain the properties of such a realization (see, for example, Lichtenberg & Lieberman 1983).

When analyzing the inviscid vorticity equation

$$\frac{d\vec{\omega}}{dt} = \frac{\partial\vec{\omega}}{\partial t} + (\vec{u}\nabla)\vec{\omega} = (\vec{\omega}\nabla)\vec{u} \, ,$$

the idea of generation of small scale motion due to a stretching of vortex lines arises. The vorticity equation is similar to that for a material fluid element $\delta\vec{l}$ (Batchelor 1967)

$$\frac{d(\delta\vec{l})}{dt} = (\delta\vec{l}\nabla)\vec{u} \, .$$

The mean squared length $\langle(\delta l)^2\rangle$ is believed to grow as a resultof stochastic stretching of $\delta\vec{l}$ by the random velocity field u. The asymptotic form of an infinitesimal fluid volume element will be a small sheet or a small rod, according to the properties of the deformation field

$$D_{ij} = \partial u_i/\partial x_j + \partial u_j/\partial x_i \qquad (3.1)$$

From these considerations it was proposed by Batchelor & Townsend 1949, Corrsin 1962, Tennekes 1968, that the vorticity is concentrated in vortex sheets or in thin vortex tubes. The smallest scale of these vortices is equal to a

viscous scale η and the characteristic curvature radius is of the order of the main scale L. The model of a turbulent flow as a system of chaotically distributed and oriented vortex sheets and tubes explains the high intermittency at small scales, but gives the turbulent spectrum that differs from the Kolmogorov one.

Lundgren 1982 has pointed out, that the vortex sheets have the tendency to be wrapped into spiral structures. He supposed, that just those structures determine the small scale properties of turbulent flows, and derived the Kolmogorov spectrum. Some support of the spiral model give the visual studies of turbulence, performed by Schwartz 1990.

Apart from the mentioned above structures, the vortex rings may occur in turbulent flows (Kutateladze et al 1986). The ring formation as well as their merging and disintegration require the reconnection of the vortex lines. The experimental study of reconnections in vortex rings were maid by Oshima 1977, and some details of the reconnection process were simulated on computers by Kida and Takaoka 1987 and also by Melander and Hussain 1989. In the computer simulations, the partial reconnections of vortex tubes were observed, after that a bridge composed of vorticity is left. In the bridge, an additional reconnection may occur and the vortices of more small scale may be generated.

Other types of interactions between two vortex tubes were also observed. Two antiparallel vortex tubes of equal strength may give a set of vortex rings (see Widnall 1975). Parallel vortex tubes may merge, giving a spiral vortex structure in a transverse section. If the vortex tubes are curved, the merging of their nearest pieces does occurs first. After that the merging process propagates along the tubes (Siggia 1984). If the intensities of the tubes are different in their order of magnitude, the weak tube is wrapped around the strong one (Zabusky & Melander 1989). It was observed in a shear layer that the interactions of this type lead to intensification of longitudinal vortices. The greatest enhancing of vorticity occurs in the stagnation points of the flow, where the tensor (3.1) has its maximum value. The general theory of organized vortex structures in terms of critical points of the flow field was developed by Cantwell 1981.

When the nonlinearity is sufficiently strong, the processes of self-amplification are possible. A known example of such a process is the wave collapse. The similar effects were looked for in vortex flows at high Reynolds numbers. Possibility of a vortex collapse follows from the estimation for the mean squared vorticity (Monin & Yaglom 1975)

$$\langle\omega^2\rangle \propto \varepsilon/\nu,$$

where ε is the mean dissipation of energy density, ν is the viscosity. From the estimation, a singularity in the flow field seems to be possible at $\nu \to 0$. The reasons, that this singularity may occur at a finite time, has also been

adduced (Rose & Sulem 1978, Frisch 1984). If such singularities do occur at a finite time and are essential, a great deal of the small scale phenomena such as intermittency may be associated with a spontaneous singularities in flow fields.

Pure mathematical methods has not been able to give the final conclusions in that area (Frisch 1984). The possibility of a finite time singularity has not been confirmed by direct computer simulations of various flows (Pumir & Siggia 1990). If the vortex collapses were essential in turbulent flows, they had been observed in a large variety of configurations. In particular, they should not require of a fine tuning of the initial configuration.

From the above examples, one may conclude, that the great variety of small scale phenomena can not be attributed to a vortex collapse or to any other single vortex structure. Possibly, the small scale vortex evolution does not follow a typical scenario.

To reveal the relative role of the various small scale phenomena, a great deal of theoretical and experimental investigations have to be done. To describe the vortex configurations, one needs a method that must not be a very special one. Only the most essential characteristics of the vortex structures should be taken into account. Those characteristics are described by the integral moments of low order.

4. TENSOR MOMENTS FOR A 3-DIMENSIONAL REGION OF HIGH VORTICITY

4.1. Low Order Moments

Let us consider the structure of a high vorticity region. Suppose a visual study has indicated a closed volume of high intensity vortex motion. We shall briefly name such a structure as "vortex". Generally, the vorticity distribution inside the vortex is a complicated function of space and time. To classify the vortices, one has to define those parameters of the vorticity distribution, which are relatively stable under the influence of external as well as internal small fine scale motions. We present the classification of the vortices by their integral tensor moments. The following moments are believed to be most important:

a) The integral vorticity

$$\vec{\Omega} = \int \vec{\omega}\, dV. \qquad (4.1)$$

The integration is performed over the volume of the vortex. For an isolated vortex $\vec{\Omega}=0$, otherwise this characteristic represents the vortex as an element of a vortex tube.

b) The vortex momentum (Batchelor 1967)

$$\vec{P} = \frac{1}{2}\int \vec{r}\times\vec{\omega}\, dV \qquad (4.2)$$

represents the vortex as a vortex ring.

c) The angular vortex momentum (Batchelor 1967)

$$\vec{J} = \frac{1}{3}\int \vec{r}\times\vec{r}\times\vec{\omega}\, dV \qquad (4.3)$$

d) The tensor

$$t_{ij} = \int (r_i[\vec{r}\times\vec{\omega}]_j + r_j[\vec{r}\times\vec{\omega}]_i)\, dV \qquad (4.4)$$

describes the deformation of the fluid volume occupied by the vortex, and

e) The moment

$$d_{ij} = \int \{r_i[\vec{r}\times[\vec{r}\times\vec{\omega}]]_j + r_j[\vec{r}\times[\vec{r}\times\vec{\omega}]]_i\}\, dV \qquad (4.5)$$

describes the deformations that twists the isolated vortex (Kuz'min & Patashinskii 1985,1986).

4.2. Toroidal and Poloidal Components of Vorticity Field

We consider the tensor moments in some more details. Because of the condition div $\vec{\omega}=0$, one needs only two scalar functions to describe the vorticity distribution inside the vortex. We use a decomposition of vorticity field in toroidal and poloidal components (see Moffat 1978)

$$\vec{\omega} = \vec{\omega}_t + \vec{\omega}_p$$

where $\vec{\omega}_p = \text{curl curl}(\vec{r}P) = \text{curl}(\vec{r}\times\nabla P)$, $\vec{\omega}_t = \text{curl}(\vec{r}T) = -r\times\nabla\vec{T}$. The poloidal and toroidal potentials P,T are determined by the equations

$$\hat{L}^2P=(\vec{r}.\vec{\omega}),\ \ \hat{L}^2T=(\vec{r}.\text{curl}\ \vec{\omega})$$

where $\hat{L} = ir\times\vec{\nabla}$ is the operator of angular momentum. The eigenvalues of $\hat{L}^2$ are the numbers $l(l+1)$, where $l=0,1,2,\ldots$, and the eigenfunctions of $\hat{L}^2$ are the tensors

$$Y^{(l)}_{i_1\ldots i_l}(\vec{e})=(-1)^l\, r^{l+1}\, \partial^l r^{-1}/\partial r_{i_1}\ldots\partial r_{i_l} \qquad (4.6)$$

where $\vec{e}=\vec{r}/r$. In particular,

$$Y^{(0)}=1; \qquad Y^{(1)}_i=e_i; \qquad Y^{(2)}_{ij}=3e_ie_j-\delta_{ij}; \qquad (4.7)$$

$$Y^{(3)}_{ijk}=15e_ie_je_k-3(e_i\delta_{jk}+e_j\delta_{ik}+e_k\delta_{ij}) \qquad (4.8)$$

The components of the tensors (4.6)-(4.8) are the usual spherical functions. Therefore we have to consider the moments of toroidal and poloidal vorticity

$$\Theta^{(n)}_{i_1 i_2 \ldots i_l}(\vec{x})=\int r^n Y^{(l)}_{i_1\ldots i_l}(\vec{e})[\vec{r}.\text{curl}\,\vec{\omega}(\vec{x}+\vec{r})]\, dV,$$

$$=\int r^n Y^{(l+1)}_{i_1\ldots i_l j}(\vec{e})[\vec{e}.\vec{\omega}(\vec{x}+\vec{r})]_j\, dV, \qquad (4.9)$$

$$\Pi^{(n)}_{i_1 i_2\ldots i_l}(\vec{x})=\int r^n\, Y^{(l)}_{i_1\ldots i_l}(\vec{e})[\vec{e}.\vec{\omega}(\vec{x}+\vec{r})]\, dV \qquad (4.10)$$

where n, l=0,1,2,.... It is convenient to choose the point $\vec{x}$ so that the vorticity field is described correctly by as minimum number of moments as possibly.

4.3. Tensor Moments in Terms of the Vortex Momentum Density

The vectors $\vec{P},\vec{J}$ are, respectively, the resultant force impulse and the resultant angular momentum of the force required to generate the motion from rest (Batchelor 1967).

The similar interpretation is possible for the moments of more high order. We define the vortex momentum density, which is equal to the density of the force impulse, required to generate instantaneously the specified vorticity field from rest (Kuz'min 1983, 1984). We substitute the impulsive force $\vec{f}=\delta(t)\vec{q}$ into the vorticity equation

$$\frac{\partial\vec{\omega}}{\partial t}+(\vec{u}\nabla)\vec{\omega}=(\vec{\omega}\nabla)\vec{u}+\nu\Delta\vec{\omega}+\text{curl}\,\vec{f}\,.$$

Integration of both sides over the infinitesimal time interval $(-\varepsilon,\varepsilon)$, $\varepsilon\to 0$, gives $\vec{\omega}=\text{curl}\,\vec{q}$. Therefore the field $\vec{q}$ differs from velocity $\vec{u}$ by the gradient of a scalar function, $\vec{q}=\vec{u}+\nabla\chi$. We choose the gauge function χ so, that the finite vortex cloud is represented by the finitely distributed vortex momentum density $\vec{q}$. Owing to finiteness of the region, where $\vec{q}$ is non-zero, the moments of the vortex momentum density are well defined. It is easier to imagine, what the fluid motion is represented by the specified moments, when these moments are written in terms of the force impulse $\vec{q}$. So we substitute $\vec{\omega}=\text{curl}\vec{q}$ into (4.9), (4.10) and integrate by parts. Let us consider the moments of poloidal vorticity (4.10) of low orders.

$$\Pi^{(0)}=\int(\vec{r}\vec{\omega})dV=(1/2)\int\partial(r^2\omega_j)/\partial r_j\,dV=0\,,$$

$$\Pi^{(1)}_i=\int r_i(\vec{r}\vec{\omega})dV=\int[\vec{r}\times\vec{q}]_i\,dV=J_i, \qquad (4.11)$$

$$\Pi^{(2)}_{ij}=\int(3r_ir_j-r^2\delta_{ij})(\vec{r}\vec{\omega})\,dV=$$

$$= 3\int\{ r_i[\vec{r}\times\vec{q}]_j + r_j[\vec{r}\times\vec{q}]_i\}\, dV = \frac{3}{4} d_{ij} . \qquad (4.12)$$

The low order moments of toroidal vorticity are

$$\Theta^{(0)} = \int(\vec{r} \text{ curl } \vec{\omega})\, dV = 0$$

$$\Theta^{(1)}_i = \int r_i(\vec{r} \text{ curl } \vec{\omega})dV = \int[\vec{r}\times\vec{\omega}]_i dV = 2\int q_i dV = 2P_i \qquad (4.13)$$

$$\Theta^{(2)}_{ij} = \int(3r_i r_j - r^2\delta_{ij})(\vec{r} \text{ curl } \vec{\omega})dV = 3\int\{r_i[\vec{r}\times\vec{\omega}]_j + r_j[\vec{r}\times\vec{\omega}]_i\}\, dV$$

$$= 3\int[r_i q_j + r_j q_i - \frac{2}{3}(\vec{r}\vec{q})\delta_{ij}]\, dV = 3t_{ij} . \qquad (4.14)$$

From (4.11)-(4.14), the physical sense of the moments become more clear. For example, $\Pi^{(2)}_{ij} = (3/4)d_{ij}$ is the moment of the force impulse, that tends to twist the vortex. The moment $\Theta_{ij} = 3t_{ij}$ describes the quadruple deformation. We see, that the lowest order moments (4.11)-(4.14) are proportional to the moments (4.2)-(4.5). Higher order moments describe more fine details of the vortex.

5. TURBULENT VORTICES OF A GIVEN SCALE

5.1. Moments of Vortices of the Given Scale

The turbulent flow consists of vortices of different scales. The small scale vortices can be considered as fragments of a large scale one. Consider a spherical volume of radius λ in a turbulent fluid. The integrals (4.9),(4.10) over the volume $r<\lambda$ give the structure of the flow in the

region of scale λ near the point $\vec{x}$. One may scan the flow field by moving the center $\vec{x}$ of the spherical volume. In such a way, the vortex structure may be revealed in the vicinity of any point of the flow.

The tensor moments (4.9),(4.10) are the projections of the vorticity field on the basic set of functions, which is especially simple in the spherical coordinates. The radial dependence of the basic functions is determined by the power factor r^n. The angle dependence is determined by the spherical harmonics (4.6). The irreducible tensors (4.9),(4.10) have $2l+1$ independent components. Three parameters give the orientation of the vortex. The remaining $2(l-1)$ parameters give quantitative characteristics of the vortex which do not depend on its orientation. These characteristics can be considered invariants I_s (s=1,2,...) of irreducible tensors (4.9), (4.10).

Knowing the distribution of invariants inside the volume of the system, it is possible to isolate parts of the fluid occupied by corresponding structures. It is possible to identify the type of structures by comparing the fields $I_s(\vec{x})$ with standard ones, based on specially created vortex-disturbance types (vortex rings and vortex filaments, vortex pairs, etc.). In doing so, it becomes possible to set up a spectroscopy of structures according to their invariants. A comparison of structures according to their moments of low order is the comparison of classes to which these structures belong. It is possible to expect that invariants for small order moments describing the most large-scale deformation of the volume in flow are not too sensitive to fluctuations inside the structures.

Identification of the vortex structures in turbulent flow consists in comparing of its moments and invariants to that of the standard configurations. The moments (4.9), (4.10) for $n = 0$ and for low numbers l are first to be calculated. The full sets (4.9), (4.10) can be used to reveal the fine details of the vortex after its type and situation have been established.

5.2. Low Order Moments for the Elementary Structures

We consider the line vortex configurations in (r_2, r_3) plane which model the vortex structures considered in section 2. The moments (4.9), (4.10) are reduced to line integrals. The coordinate system is shown in Fig. 1.

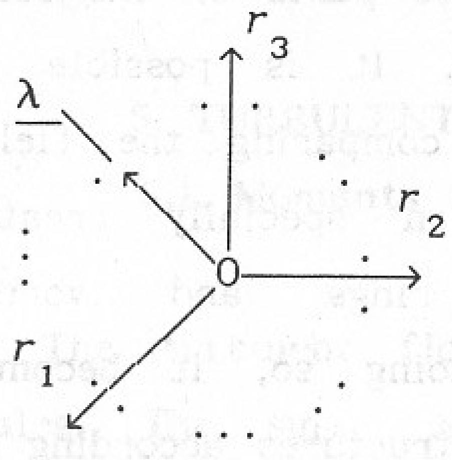


Fig. 1.
Coordinate system

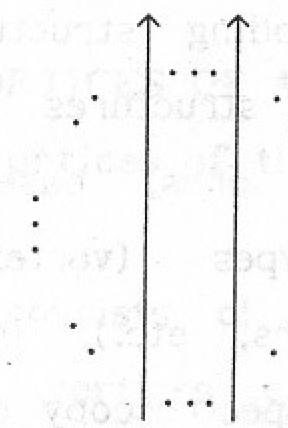


Fig. 2.
Vortex pair

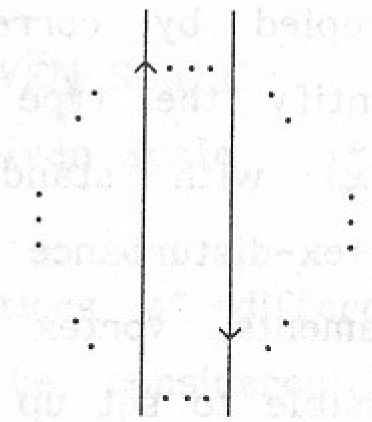


Fig. 3.
Vortex couple

The integration region, which is the sphere of radius λ

is shown schematically by dots in (r_2, r_3) plane. The sketch of the vortices is shown figures 2-5. The first vortex structure is the vortex pair (Fig. 2).

The vorticity distribution is $\omega_1 = \omega_2 = 0$; $\omega_3 = \kappa\delta(r_1)[\delta(r_2 - b) + \delta(r_2 + b)]$. Simple calculations show, that

$$\Pi^{(0)}_i = 4\kappa b\delta_{i3}\left(\sqrt{\lambda^2/b^2 - 1} - \operatorname{arctg}\sqrt{\lambda^2/b^2 - 1}\right); \qquad \Pi^{(0)}_{ij} = 0;$$

$$\Theta^{(0)}_i = 0; \qquad \Theta^{(0)}_{ik} = 12\kappa b\Delta^{(1,2)}_{ik}\operatorname{arctg}\sqrt{\lambda^2/b^2 - 1},$$

where $\Delta^{(l,m)}_{ik} = \delta_{il}\delta_{km} + \delta_{im}\delta_{kl}$ determines the angle dependence of the moment. The vortex couple is composed of two antiparallel line vortices (see Fig. 3) $\omega_1 = \omega_2 = 0$; $\omega_3 = \kappa\delta(r_1) \times [\delta(r_2 - b) - \delta(r_2 + b)]$. The low order moments for the vortex couple are

$$\Pi^{(0)}_i = 0; \; \Pi^{(0)}_{ik} = -12\kappa b\,\Delta^{(2,3)}_{ik}\left[-\sqrt{1 - b^2/\lambda^2} + \log\left(1 + \sqrt{1 - b^2/\lambda^2}\right)\right]$$

$$\Theta^{(0)}_i = 4\kappa b\delta_{i1}\log\left(1 + \sqrt{1 - b^2/\lambda^2}\right); \qquad \Theta^{(0)}_{ik} = 0.$$

The next two structures are the configurations given either by

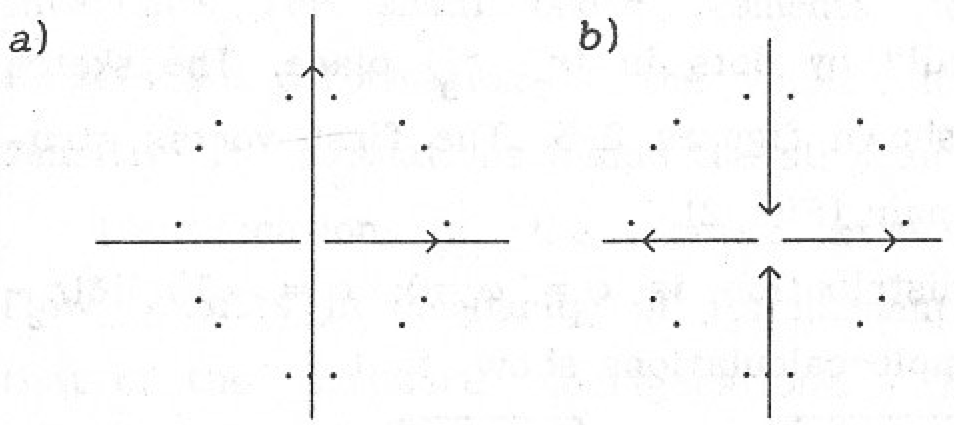
$$\omega_1 = 0, \qquad \omega_2 = \kappa\delta(r_1)\delta(r_3), \qquad \omega_3 = \kappa\delta(r_1)\delta(r_2),$$

or by

$$\omega_1 = 0, \quad \omega_2 = \kappa\,\operatorname{sgn}(r_2)\delta(r_1)\delta(r_3), \quad \omega_3 = \kappa\,\operatorname{sgn}(r_3)\delta(r_1)\delta(r_2)$$

(see fig. 4.(a, b)).

a) b)



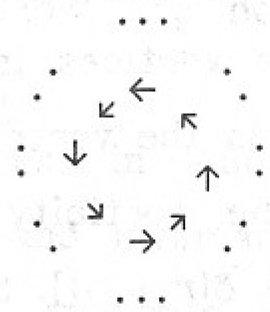


Fig. 4. *Fig. 5.*

a) Intersection b) Cross Vortex ring

For the structure shown in figure 4.a

$$\Pi^{(0)}_1=0,\quad \Pi^{(0)}_2=\Pi^{(0)}_3= 2\kappa\lambda,\quad \Pi^{(0)}_{ik}=0;\quad \Theta^{(0)}_i= \Theta^{(0)}_{ik} =0.$$

For the structure shown in the figure 4.b,

$$\Pi^{(0)}_1= \Theta^{(0)}_i = \Pi^{(0)}_{ik}= \Theta^{(0)}_{ik} =0.$$

For the vortex ring of radius b, placed at the origin of the coordinate system

$$(\vec{e}\vec{\omega})=0;\quad [\vec{e}\times\vec{\omega}]_i= \delta_{i1}\kappa\delta(r-b)\delta(r_3),\qquad \text{so}\quad \Pi^{(n)}_{(i)}=0,$$

$$\Theta^{(0)}_i=- 2\pi\kappa b\delta_{i1},\qquad \Theta^{(0)}_{ik}= 0.$$

Let us consider the two surface vortex configurations. The first one is the shear layer with the vorticity distribution

$$\omega_1=\omega_2=0;\quad \omega_3=\kappa\delta(r_1).$$

The low order moments for the shear layer are

$$\Pi^{(0)}_i=\frac{1}{2}\pi\kappa\lambda^2\delta_{i3};\quad \Pi^{(0)}_{ij}=0;\quad \Theta^{(0)}_i=0;\quad \Theta^{(0)}_{ik}= \frac{3}{2}\pi\kappa\lambda^2\Delta^{(1,2)}_{ik}.$$

The second one is the front of vorticity

$$\omega_1=\omega_2= 0;\ \omega_3= \omega_+\ \text{ for }\ r_2 > 0;\ \omega_3= \omega_-\ \text{ for }\ r_2< 0.$$

For the front of vorticity

$$\Pi^{(0)}_i=\frac{2}{9}\pi\lambda^3(\omega_+ +\omega_-)\delta_{i3};\quad \Pi^{(0)}_{ik}= \frac{1}{4}\lambda^3(\omega_+ +\omega_-)\,\Delta^{(2,3)}_{ik}.$$

The zeroth values of the moments, which are summarized in table 1, indicates, that the corresponding structures are in the origin of the coordinate system. This sign can be used to search and to mark the centers of the structures.

Table 1

Moment	Angle dependence of the moments			
Structure	$\Pi^{(0)}_i$	$\Pi^{(0)}_{ik}$	$\Theta^{(0)}_i$	$\Theta^{(0)}_{ik}$
Pair	δ_{i3}	0	0	$\Delta^{(1,2)}_{ik}$
Couple	0	$\Delta^{(2,3)}_{ik}$	δ_{i1}	0
Intersection	$1-\delta_{i1}$	0	0	0
Cross	0	0	0	0
Ring	0	0	δ_{i1}	0
Shear layer	δ_{i3}	0	0	$\Delta^{(1,2)}_{ik}$
Vortex front	δ_{i3}	$\Delta^{(2,3)}_{ik}$	δ_{i1}	0

The non zeroth moments describe the form as well as the orientation of the structures. The quantitative characteristics of the form are the invariants of the moments which do not depend on the orientation of the axes of the coordinate system. Each vortex structure is characterized by a certain set of the invariants.

REFERENCES

1. *Batchelor G.K. and Townsend A.A.* 1949, Proc. R. Soc. London, Ser.A **199** 238.
2. *Batchelor G.K.* 1967 An Introduction to Fluid Dynamics. (Cambridge University Press).
3. *Cantwell B.J.* 1981 Ann. Rev. Fluid Mech. **13** 457.
4. *Corrsin S.* 1962 Phys. Fluids **5** 1301.
5. *Frisch U.* 1984 The analytic structure of Turbulent Flows. in "Chaos and Statistical Methods" Ed. by Y. Kuramoto, (Springer-Verlag, Berlin, a.o) p. 211-220.
6. *Hussain A.K.M.F.* 1986 J. Fluid Mech. **173** 303.
7. *Kida S. and Takaoka M.*1987 Phys Fluids **30** 2911.
8. *Kuz'min G.A.* 1983 Phys. Lett. **A96** 88.
9. *Kuz'min G.A.* 1984 Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, N4 112. (Engl. Trasl. Journ. Applied Mech. Technical Physics, **4** 735).
10. *Kuz'min G.A. and Patashinskii A.Z.* 1985 Phys. Letters **A**, **113**, 266.
11. *Kuz'min G.A. and Patashinskii A.Z.* 1986 Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No.5 112. (Engl. Trasl. Journ. Applied Mech. Technical Physics, 1986 **5** 735.
12. *Kuz'min G.A.* 1991 Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No.2 44.
13. *Kutateladze S.S., B.P.Mironov et al.* 1986 Arch. Mech., **38** 595.
14. *Lichtenberg A.J. and Lieberman M.A.* 1983 Regular and Stochastic Motion. (Springer-Verlag, New York).
15. *Lundgren T.S.* 1982 Phys. Fluids **25** 2193.
16. *Melander M.V., Zabusky N.J. and Stychek A.S.* 1986 J. Fluid Mech., **167** 95.
17. *Mitus A.S. and Patashinskii A.Z.* 1981 ZhETP **80** 1554.
18. *Mitus A.S. and Patashinskii A.Z.* 1987 preprint 87-124 (Institute of Nuclear Physics, Novosibirk).
19. *Moffat H.K.* 1978 Magnetic Field Generation in Electrically Conducting Fluids. (Cambridge University Press, Cambridge, e.a.)
20. *Monin A.S. and Yaglom A.M.* 1975 Statistical Fluid Mechanics: Mechanics of Turbulence, vol.2, (MIT Press, Cambridge).
21. *Oshima Y. and Asaka S.* 1977 J. Phys. Soc. Japan, **42** 708.
22. *Patashinskii A.Z.* 1991, Preprint, (Institute of Nuclear Physics, Novosibirk).
23. *Pumir A. and Siggia E.D.* 1990 Phys. Fluids **A2** 220.
24. *Rose H.A. and Sulem P.L.* 1978 J. de Physique **39** 441.
25. *Schwarz K.W.* 1990 Phys. Rev. Lett. **64** 415.
26. *Siggia E.D.* 1984 Phys. Fluids **28** 794.
27. *Tennekes H.* 1968 Phys. Fluids **11** 669.
28. *Townsend A.A.* 1976 The structure of Turbulent Shear Flow (Cambridge University Press).
29. *Widnall S.* 1975 Ann. Rev. Fluid Mech. **7** 141.
30. *Zabusky N.J. and Melander M.V.* 1989 Physica **D 37** 555.

G.A. Kuz'min and A.Z. Patashinskii

Reduced Description of Turbulent Vortex Structures

Г.А. Кузьмин, А.З. Паташинский

Редуцированное описание турбулентных вихревых структур

Ответственный за выпуск: С.Г.Попов

Работа поступила 12 марта 1991 г.
Подписано к печати 12.03 1991 г.
Формат бумаги 60×90 1/16
Объем 1,5 п.л., 1,2 уч.-изд.л.
Тираж 200 экз. Бесплатно. Заказ N 21.

Ротапринт ИЯФ СО АН СССР,
г. Новосибирск, 90.