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COHERENT STRUCTURES FOR THE ISHIMORI EQUATION. II. TIME-DEPENDENT BOUNDARIES

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НОВОСИБИРСК

Coherent Structures for the Ishimori Equation.
ii. Time-dependent Boundaries

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ABSTRACT

The localized solutions for the Ishimori equation are studied. Using the general formulae derived in paper I and the exact solutions of the inverse problem for the modified Kadomtsev-Petviashvili equation we construct the exponentially and rationally localized soliton type solutions of the Ishimori equation with time-dependent boundaries. Explicit examples of such solutions are presented.

1. INTRODUCTION

After the discovery in [1] the exponentially localized solutions of soliton type for the 2+1—dimensional and multidimensional nonlinear equations are studied very intensively [2—11]. The analysis of the structure and properties of the localized exact solutions of the 2+1 and multi-dimensional soliton equations is a very important problem. Most results obtained are concerned to the Davey—Stewartson (DS) or the nonlinear Schrodinger type equations.

Recently we have shown in [11] (hereafter refer as paper I) that the 2+1—dimensional Ishimori equation also possesses the localized solitons of the similar type. This is the equation of the form [11]

$$\vec{S}_t+\frac{1}{2}\vec{S}\times(\vec{S}_{\xi\xi}+\vec{S}_{\eta\eta})+$$

$$+\frac{1}{2}\Big(\int\limits_{-\infty}^{\eta} d\eta'\,\vec{S}(\vec{S}_\xi\times\vec{S}_{\eta'})+2u_2(\xi,t)\Big)\vec{S}_\xi-$$

$$-\frac{1}{2}\Big(\int\limits_{-\infty}^{\xi} d\xi'\,\vec{S}(\vec{S}_{\xi'}\times\vec{S}_\eta)+2u_1(\eta,t)\Big)\vec{S}_\eta=0 \qquad (1.1)$$

where $\vec{S}(x,\ y,\ t)=(S_1,\ S_2,\ S_3)$ is the unit vector $\vec{S}^2=S_1^2+S_2^2+$

$+S_3^2=1$, $\xi=\frac{1}{2}(y+x)$, $\eta=\frac{1}{2}(y-x)$ and $u_1(\eta, t)$, $u_2(\xi, t)$ are arbitrary scalar functions. The functions $u_1(\eta, t)$ and $u_2(\xi, t)$ are the boundary values of the derivatives φ_η and φ_ξ of the auxiliary field φ at $\xi\to-\infty$ and $\eta\to-\infty$ respectivery [11].

In paper I [11] it has been shown that the problem of constructing the localized solitons for equation (1.1) is closely connected with the problem of the explicit solving the linear equation of the type

$$2iX_t(z,t)+X_{zz}+2iu(z,t)X_z=0 \tag{1.2}$$

where $u=u_1(\eta, t)$ (or $u=-u_2(\xi, t)$) is given function.

In I the general formula for the localized solitons for equation (1.1) has been derived. Then in I the case of the stationary boundaries $u_1=u_1(\eta)$, $u_2=u_2(\xi)$ has been studied in detail. The four classes of the localized solutions of the breather type for equation have been constructed explicitly.

In the present paper we study the general case of time-dependent boundaries $u_1(\eta, t)$ and $u_2(\xi, t)$. Using recent exact results for the inverse problem for the linear equation (1.2) (which is associated with the modified Kadomtsev—Petviashvili eguation) [12] we will construct the exact localized soliton type solutions of the Ishimori equation (1.1). The class of exact solutions of the linear equation (1.2) is a very rich one. It includes the decreasing and plane lumps, plane solitons and breathers and so on [12]. As a consequence, the Ishimori equation (1.1) possesses a wide class of the localized soliton type solutions. The rationally localized, rationally-exponentially and fully exponentially localized soliton type solutions are among them. We present several explicit examples of such solutions.

The Ishimori equation (1.1) can be rewritten also as the single equation for the complex variable $q=\frac{S_1+iS_2}{1+S_3}$ which is more complicated than the DS equation. We will compare the localized solitons fo that equation with those for the DS equation.

The paper is organized as follows. In section 2 the principal results of the paper I are presented for convenience. Exact solutions of eguation (1.2) are given is section 3. The localized soliton type solutions of the Ishimori equation with the time-dependent boundaries are constructed in section 4. In section 5 we compare the results obtained with those for the DS equation.

2. GENERAL FORMULA FOR THE EXACT SOLUTIONS OF THE ISHIMORI EQUATION WITH NONTRIVIAL BOUNDARIES

Here for convenience we present the main results of the paper I [11].

The Ishimori equation (1.1) with the nontrivial boundaries $u_1(\eta, t)$ and $u_2(\xi, t)$ is eguivalent to the compatibility condition for the linear system [11]

$$\begin{pmatrix}\partial_\eta & 0\\ 0 & \partial_\xi\end{pmatrix}\Psi+\frac{1}{2}(P+\sigma_3)(\partial_\xi-\partial_\eta)\Psi=0, \tag{2.1}$$

$$\Psi_t-\frac{i}{2}P(\partial_\xi-\partial_\eta)^2\Psi-\frac{i}{4}R(\partial_\xi-\partial_\eta)\Psi+\frac{i}{2\lambda^2}\Psi-$$

$$-\frac{i}{2\lambda}\int_{-\infty}^{+\infty}\frac{dl}{l^2}\gamma\left(\frac{1}{\lambda}-\frac{1}{l}\right)\Psi(l)=0$$

or the linear system

$$\begin{pmatrix}\partial_\eta & 0\\ 0 & \partial_\xi\end{pmatrix}\hat\Psi+\frac{1}{2}(P+\sigma_3)(\partial_\xi-\partial_\eta)\hat\Psi=0, \tag{2.2}$$

$$\hat\Psi_t-\frac{i}{2}P(\partial_\xi-\partial_\eta)^2\hat\Psi-\frac{i}{4}R(\partial_\xi-\partial_\eta)\hat\Psi+$$

$$+\frac{i}{2\lambda^2}\hat\Psi-\frac{i}{2\lambda}\int_{-\infty}^{+\infty}\frac{dl}{l^2}\hat\gamma\left(\frac{1}{\lambda}-\frac{1}{l}\right)\hat\Psi(l)=0$$

where

$$P=\vec S\cdot\vec\sigma\equiv S_1\sigma_1+S_2\sigma_2+S_3\sigma_3,$$

$$R=P_\xi-P_\eta+(P_\xi+P_\eta)P+$$

$$+i(1-P)\int_{-\infty}^{\eta}d\eta'\,\vec S\cdot(\vec S_\xi\times\vec S_{\eta'})+i(1-P)\int_{-\infty}^{\xi}d\xi'\,\vec S\cdot(\vec S_{\xi'}\times\vec S_\eta), \tag{2.3}$$

$$\gamma(k)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\xi e^{ik\xi}u_2(\xi,t),$$

$$\hat{\gamma}(k)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} d\eta e^{-ik\eta}u_1(\eta,t)$$

and

$$\Psi=\begin{pmatrix}\Psi_{11}\\ \Psi_{21}\end{pmatrix},\quad \hat{\Psi}=\begin{pmatrix}\Psi_{12}\\ \Psi_{22}\end{pmatrix}.$$

The Fourier transforms of the inverse problem data are defined as follows

$$\hat{S}(\xi,\eta,t)=-i\int\!\!\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}\frac{d\lambda}{\lambda^2}\lambda T_{21}^{-}(l,\lambda)e^{-\frac{i\eta}{l}-\frac{i\xi}{\lambda}},$$

$$\hat{T}(\xi,\eta,t)=-i\int\!\!\int\limits_{-\infty}^{+\infty}\frac{dl}{l^2}\frac{d\lambda}{\lambda^2}\lambda T_{12}^{+}(l,\lambda)e^{\frac{i\xi}{l}+\frac{i\eta}{\lambda}} \tag{2.4}$$

where

$$T_{12}^{+}(P,\lambda)=\frac{1}{4\pi}\int\!\!\int\limits_{-\infty}^{+\infty}d\xi d\eta\, e^{-\frac{i\xi}{P}-\frac{i\eta}{\lambda}}\left((P+\sigma_3)\left(\partial_\xi-\partial_\eta+\frac{i}{\lambda}\right)\chi^{+}\right)_{12},$$

$$T_{21}^{-}(P,\lambda)=\frac{1}{4\pi}\int\!\!\int\limits_{-\infty}^{+\infty}d\xi d\eta\, e^{\frac{i\eta}{P}+\frac{i\xi}{\lambda}}\left((P+\sigma_3)\left(\partial_\xi-\partial_\eta+\frac{i}{\lambda}\right)\chi^{-}\right)_{21} \tag{2.5}$$

and

$$\chi^{\pm}=\Psi^{\pm}\begin{pmatrix}e^{-\frac{i\xi}{\lambda}} & 0\\ 0 & e^{+\frac{i\eta}{\lambda}}\end{pmatrix}$$

are the solutions of the linear integral equations associated with the first linear problem (2.1).

Using (2.1) and (2.2), one can show that [11] these functions $\hat{S}(\xi,\eta,t)$ and $\hat{T}(\xi,\eta,t)$ obey the linear partial differential eguations

$$\hat{S}_t(\xi,\eta,t)-\frac{i}{2}(\hat{S}_{\xi\xi}+\hat{S}_{\eta\eta})+u_2(\xi,t)\,\hat{S}_\xi-u_1(\eta,t)\,\hat{S}_\eta=0,$$

$$\hat{T}_t(\xi,\eta,t)+\frac{i}{2}(\hat{T}_{\xi\xi}+\hat{T}_{\eta\eta})+u_2(\xi,t)\,\hat{T}_\xi-u_1(\eta,t)\,\hat{T}_\eta=0 \tag{2.6}$$

where the boundary values $u_1(\eta,t)$ and $u_2(\xi,t)$ play a role of variable coefficients.

In the case of the real-valued $\hat{S}(x,y,t)$ one has

$$\hat{S}(\xi,\eta,t)=\hat{T}^{*}(\xi,\eta,t) \tag{2.7}$$

where $*$ denotes the complex conjugation. So in this case one can consider only equation (2.6) for $\hat{S}$.

The linear equation (2.6) is solvable by the method of separation of variables. Its general solution can be represented in the form

$$\hat{S}(\xi,\eta,t)=2\pi\sum_{i,j}\rho_{ij}X_i(\xi,t)\,Y_j(\eta,t) \tag{2.8}$$

where ρ_{ij} are arbitrary constants and $X_i(\xi,t)$, $Y_j(\eta,t)$ are solutions of equations [11]

$$iX_{it}+\frac{1}{2}X_{i\xi\xi}+iu_2(\xi,t)\,X_{i\xi}=0, \tag{2.9}$$

$$iY_{jt}+\frac{1}{2}Y_{j\eta\eta}-iu_1(\eta,t)\,Y_{j\eta}=0. \tag{2.10}$$

The symbol Σ in (2.8) may include also the integration over the continuous indices.

Then using the formula for the exact solutions of the inverse problem for the Ishimori equation [13, 11] with degenerated inverse problem data one gets [11]

$$S_1=\frac{g_{11}g_{12}-g_{21}g_{22}}{g_{11}g_{22}-g_{12}g_{21}},$$

$$S_2=i\frac{g_{11}g_{12}+g_{21}g_{22}}{g_{11}g_{22}-g_{12}g_{21}}, \tag{2.11}$$

$$S_3=-\frac{g_{11}g_{22}+g_{12}g_{21}}{g_{11}g_{22}-g_{12}g_{21}}$$

where

$$g(x,y,t)\overset{def}{=}\chi(x,y,t;\lambda=0)=$$

$$=\begin{pmatrix}1-\langle X,(1-\rho a\rho^{+}b)^{-1}\rho a\rho^{+}X^{*}\rangle, & \langle Y^{*},\rho^{+}(1-b\rho a\rho^{+})^{-1}X^{*}\rangle\\ -\langle X,(1-\rho a\rho^{+}b)^{-1}\rho Y\rangle, & 1+\langle Y^{*},\rho^{+}(1-b\rho a\rho^{+})^{-1}b\rho Y\rangle\end{pmatrix} \tag{2.12}$$

where

$$\langle X,Y\rangle\overset{def}{=}\sum_i X_iY_i,\quad (\rho^{+})_{ij}=\rho_{ji}^{*}.$$

and

$$a_{lm}=\int\limits_{-\infty}^{\eta} d\eta'\, Y_m^*(\eta',t)\,\partial_{\eta'} Y_l(\eta',t)\,,$$

$$b_{ik}=-\int\limits_{-\infty}^{\xi} d\xi'\, X_k(\xi',t)\,\partial_{\xi'} X_i^*(\xi',t)\,. \tag{2.13}$$

The formulae (2.11) — (2.13) are the main result of the paper I [11]. They allow us to construct explicit exact solutions of the Ishimori equation (1.1) with the nontrivial boundaries $u_1(\eta,\ t)$ and $u_2(\xi,\ t)$ using the exact solutions of the linear equations (2.9) and (2.10).

The auxiliary field φ and the topological charge Q are also given by the compact formulae

$$\varphi=2i\ln\det g+2\partial_\xi^{-1}u_2(\xi',t)+2\partial_\eta^{-1}u_1(\eta',t) \tag{2.14}$$

and

$$Q\overset{def}{=}\frac{1}{4\pi}\iint dxdy\,\vec{S}\cdot(\vec{S}_x\times\vec{S}_y)=-\frac{i}{8\pi}\iint d\xi d\eta\,\frac{\partial^2}{\partial\xi\partial\eta}\ln\det g\,. \tag{2.15}$$

In the previous paper I we solved the case of the time-independent boundaries $u_1(\eta)$ and $u_2(\xi)$ and constructed the corresponding exponentially localized solitons for the Ishimori equation (1.1).

The results of the paper [12] allow us now to solve the general case of the time-dependent boundaries.

3. EXACT SOLUTIONS FOR THE MODIFIED KADOMTSEV — PETVIASHVILI EQUATION

For the DS equation the analog of equations (2.9), (2.10) is of the form $iX_t+X_{zz}\pm u(z,\ t)\,X=0$ [3, 5]. So the problem for constructing the localized solitons for the DS equation is closely connected with the spectral theory for the Kadomtsev — Petviashvili equation [3, 5].

In our case the problems (2.9), (2.10) or the problem

$$2i\Psi_t+\Psi_{zz}\pm 2iu(z,t)\,\Psi_z=0 \tag{3.1}$$

is relevant for the application of the inverse spectral transform method for the modified Kadomtsev — Petviashvili (mKP) equation [11, 12]. Namely, the mKP-I equation [14, 12]

$$u_t+u_{zzz}+6u^2u_z-12\partial_z^{-1}u_{yy}+12u_z\,\partial_z^{-1}u_y=0 \tag{3.2}$$

is equivalent to the compatibility condition for the linear system

$$(2i\partial_y+\partial_z^2+2iu\partial_z)\Psi=0\,, \tag{3.3}$$

$$\{\partial_t+4\partial_z^3+12iu\partial_z^2+$$
$$+(6iu_z+12(\partial_z^{-1}u_y)-6u^2)\,\partial_z+\alpha\}\,\Psi=0$$

where α is an arbitrary constant. The mKP equation (3.2) is the $2+1$ — dimensional integrable generalization of the well-known modified Korteweg-de Vries equation.

The mKP equation (3.2) has been solved by the IST method just recently [12]. Using the nonlocal Riemann — Hilbert problem method and the $\bar{\partial}$ — dressing method the wide classes of the exact solutions of the mKP — I equation (3.2) have been constructed [12]. They includes decaying and plane lumps, plane solitons and solutions of the breather type and others.

Here we present those solutions of the mKP equation and their eigenfunctions, more preciessly, of the problem (3.1) which are relevant for the construction of the localized soliton-type solutions of the Ishimori equation (1.1).

The first class of solutions are the real plane lumps. They are of the form [12]

$$u(z,y)=i\frac{\partial}{\partial z}\ln\det((A+B)\,A^{-1}) \tag{3.4}$$

where

$$A_{ik}=\delta_{ik}\left(z-\frac{y}{\lambda_i}+\gamma_i\right)+(1-\delta_{ik})\,\frac{i\lambda_k^2}{\lambda_i-\lambda_k}, \tag{3.5}$$

$$B_{ik}=i\lambda_k$$

and

$$\gamma_k=-\frac{i}{2}\lambda_k+c_k\,,\quad \operatorname{Im}c_k=0\quad (k=1,\ldots,n)$$

where λ_k, c_k are the sets of arbitrary real constants. The corresponding eigenfunctions $\chi_0(z,\ y)=\chi(z,\ y;\ \lambda=0)$ and $\chi(\lambda_k)$ where $\chi(z,\ y;\ \lambda)=\Psi e^{i\left(\frac{z}{\lambda}-\frac{y}{2\lambda^2}\right)}$ are

$$\chi_0(z, y) = 1 + i \sum_{k=1}^{n} \lambda_k \chi(\lambda_k) \tag{3.6}$$

where $\chi(\lambda_k)$ are defined from the algebraic system [12]

$$\chi(\lambda_i)\left(z - \frac{y}{\lambda_i} + \gamma_i\right) + \sum_{k \neq i} \frac{i\lambda_k^2 \chi(\lambda_k)}{\lambda_i - \lambda_k} = 1 \tag{3.7}$$

$$(i, k = 1, \dots, n) .$$

The simplest plane lump is of the form

$$u(z, y) = \frac{\lambda_1}{\left(z - \frac{y}{\lambda_1} + c_1\right)^2 + \frac{\lambda_1^2}{4}} \tag{3.8}$$

and the corresponding $\Psi(\lambda_1)$ is

$$\Psi(\lambda_1) = \frac{e^{i\left(\frac{z}{\lambda_1} - \frac{y}{2\lambda_1^2}\right)}}{z - \frac{y}{\lambda_1} + c_1 - \frac{i\lambda_1}{2}} . \tag{3.9}$$

The solution (3.8) describes the rational nonsingular lump travelling with the velocity $1/\lambda_1$ along axis z. The general multi plane lumps solutions describes the collision the plane lumps (3.8). Their collision is completely trivial: the phase shift is absent.

The plane solitons form the second class of exact solutions of the mKP—I. They are given by the formula [12]

$$u = 2 \frac{\partial}{\partial z} \arg \det A \tag{3.10}$$

where

$$A_{ek} = \delta_{ek} + 2i \frac{R_k \bar{\lambda}_k}{\lambda_e - \bar{\lambda}_k} e^{F(\lambda_e) - F(\bar{\lambda}_k)} \tag{3.11}$$

where

$$F(\lambda_k) = i\left(\frac{z}{\lambda_k} - \frac{y}{2\lambda_k^2} + \gamma_k\right)$$

and $\lambda_1, \dots, \lambda_n$ are arbitrary complex parameters and R_k are arbitrary real parameters. The corresponding eigenfunctions $\chi(\lambda) = \Psi \times \times e^{i\left(\frac{z}{\lambda} - \frac{y}{2\lambda^2}\right)}$ are given by [12]

$$\chi(\lambda = 0) = 1 + 2i \sum_{k=1}^{n} R_k \chi(\lambda_k)\, e^{F(\lambda_k) - F(\bar{\lambda}_k)} \tag{3.12}$$

where $\chi(\lambda_k)$ are defined from the linear algebraic system

$$\chi(\lambda_k) + 2i \sum_{l=1}^{n} \frac{R_l \bar{\lambda}_l}{\lambda_k - \bar{\lambda}_l} e^{F(\lambda_l) - F(\bar{\lambda}_l)} \chi(\lambda_l) = 1$$

$$(k = 1, \dots, n) . \tag{3.13}$$

The simplest one-soliton solution is

$$u = -\frac{4\frac{\lambda_I}{|\lambda|^2}\,\mathrm{Sgn}\,R}{e^f + \left(e^{-f/2} + \frac{\lambda_R}{\lambda_I}(\mathrm{sgn}\,R)\,e^{f/2}\right)^2} \tag{3.14}$$

where

$$f = \frac{2\lambda_I}{|\lambda|^2}\left(z - \frac{\lambda_R y}{|\lambda|^2} + z_0\right), \quad (\lambda = \lambda_R + i\lambda_I) \tag{3.15}$$

and R—some real constant.

The corresponding eigenfunction is

$$\Psi(\lambda) = \frac{e^{i\left(\frac{z}{\lambda} - \frac{y}{2\lambda^2}\right)}}{1 + \frac{R\bar{\lambda}}{\lambda_I} e^f} \tag{3.16}$$

The mKP-I equation possesses also the exact solutions of the breather type [12]. They have rather complicated form and we will not considered them in the present paper.

The mKP-I equation similar to the KP-I equation has a decaying at $x^2 + y^2 \to \infty$ rational solutions [12]. But analogously to the KP case these lumps are not the suitable boundaries u_1 and u_2 for the Ishimori eguation. Indeed in the treatment of the exact solutions of the mKP-I equation as the boundaries the variable y in (3.3) is replaced by the time variable t in (2.9), (2.10) and hence these boundaries should have the character of the traveling alone the axis z solitons.

It is easy to see that the decaying lumps do not satisfy this requirement both for the mKP-I and KP-I equations. But the plane solitons of the mKP-I equation have the required character while for the KP-I equation the plane lumps are absent at all.

Hence there is an essential difference between the DS and Ishimori equations at this point. The latter may have the rationally decaying boundary which give rise to the rationally localized solitons.

4. LOCALIZED SOLITONS OF THE ISHIMORI EQUATION

The use of the formulae presented in the previous section allows us to construct exact solutions of the Ishimori equation via the general formulae (2.11) — (2.13). One should only take in account the difference in sings in (2.9), (2.10) and make the substitution $y\to t$, $z\to\xi$, $u\to u_2(\xi, t)$ for $X_i(\xi, t)$ and $y\to t$, $z\to\eta$, $u\to -u_1(\eta, t)$ for $Y_j(\eta, t)$.

For the boundaries $u_1(\eta, t) = -\beta/\left[\left(\eta-\frac{t}{\beta}+c_1\right)^2+\frac{\beta^2}{4}\right]$ and $u_2(\xi, t)=\alpha/\left[\left(\xi-\frac{t}{\alpha}+c_2\right)^2+\frac{\alpha^2}{4}\right]$, both given by the rational lump (3.8), one gets the rationally localized solition of the Ishimori equation:

$$S_1^{rr}(\xi,\eta,t)=S_\perp^{rr}(\hat\xi,\hat\eta)\cos\left[\frac{1}{\alpha}\hat\xi+\frac{1}{\beta}\hat\eta+\frac{t}{2}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)+\Phi_{rr}(\hat\xi,\hat\eta)\right],$$

$$S_2^{rr}(\xi,\eta,t)=S_\perp^{rr}(\hat\xi,\hat\eta)\sin\left[\frac{1}{\alpha}\hat\xi+\frac{1}{\beta}\hat\eta+\frac{t}{2}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)+\Phi_{rr}(\hat\xi,\hat\eta)\right], \quad (4.1)$$

$$S_3^{rr}(\xi,\eta,t)=-1+$$

$$+\frac{2}{\left[(\hat\xi+c_2)(\hat\eta+c_1)+\frac{\alpha\beta}{4}+\frac{1}{\alpha\beta}\right]^2+\left[\frac{\alpha}{2}(\hat\eta+c_1)-\frac{\beta}{2}(\hat\xi+c_2)\right]^2}$$

where

$$\hat\xi=\xi-\frac{t}{\alpha},\quad \hat\eta=\eta-\frac{t}{\beta}$$

and

$$S_\perp^{rr}(\hat\xi,\hat\eta)=$$

$$=2\,\frac{\sqrt{\left[(\hat\xi+c_2)(\hat\eta+c_1)-\frac{\alpha\beta}{4}+\frac{1}{\alpha\beta}\right]^2+\left[\frac{\alpha}{2}(\hat\eta+c_1)+\frac{\beta}{2}(\hat\xi+c_2)\right]^2}}{\left[(\hat\xi+c_2)(\hat\eta+c_1)+\frac{\alpha\beta}{4}+\frac{1}{\alpha\beta}\right]^2+\left[\frac{\alpha}{2}(\hat\eta+c_1)-\frac{\beta}{2}(\hat\xi+c_2)\right]^2},$$

$$\Phi_{rr}(\hat\xi,\hat\eta)=\operatorname{arc\,tg}\frac{\frac{\alpha}{2}(\hat\eta+c_1)+\frac{\beta}{2}(\hat\xi+c_2)}{(\hat\xi+c_2)(\hat\eta+c_1)-\frac{\alpha\beta}{4}+\frac{1}{\alpha\beta}} \quad (4.2)$$

The auxiliary function $\varphi^{rr}(\xi, \eta, t)$ is given by

$$\varphi^{rr}(\xi,\eta,t)=4\operatorname{arc\,tg}\frac{\frac{\alpha}{2}(\hat\eta+c_1)-\frac{\beta}{2}(\hat\xi+c_2)}{(\hat\xi+c_2)(\hat\eta+c_1)+\frac{\alpha\beta}{4}+\frac{1}{\alpha\beta}}+$$

$$+4\operatorname{arc\,tg}\frac{\beta}{2(\hat\eta+c_1)}-4\operatorname{arc\,tg}\frac{\alpha}{2(\hat\xi+c_2)}+4\operatorname{arg\,tg}\frac{2}{\alpha}(\hat\xi+c_2)+$$

$$+4\operatorname{arc\,tg}\frac{2}{\beta}(\hat\eta+c_1)+2\pi(\operatorname{Sgn}\alpha+\operatorname{Sgn}\beta) \quad (4.3)$$

and the density of the topological charge is

$$\partial_\xi\partial_\eta\ln\det g=-2i\partial_\xi\partial_\eta\left\{\operatorname{arc\,tg}\frac{\frac{\alpha}{2}(\hat\eta+c_1)-\frac{\beta}{2}(\hat\xi+c_2)}{(\hat\xi+c_2)(\hat\eta+c_1)+\frac{\alpha\beta}{4}+\frac{1}{\alpha\beta}}+\right.$$

$$\left.+\operatorname{arc\,tg}\frac{\beta}{2(\hat\eta+c_1)}-\operatorname{arc\,tg}\frac{\alpha}{2(\hat\xi+c_2)}\right\}. \quad (4.4)$$

The soliton (4.1) decays as $1/(\xi\eta)$ at $\xi^2+\eta^2\to\infty$ and moves with the velocity $V=(V_\xi, V_\eta)=(\alpha^{-1}, \beta^{-1})$. Emphasize that the rationaly localized soliton (4.1) is the novel phenomena which has been absent in the DS case [1—6].

The next example corresponds to the choise of the boundaries $u_1(\eta, t)$ and $u_2(\xi, t)$ as the plane solitons (3.14):

$$u_1(\eta, t) = \frac{4\frac{\lambda_I}{|\lambda|^2}\operatorname{Sgn} R_1}{\exp\left(\frac{2\lambda_I\hat\eta}{|\lambda|^2}\right) + \left(\exp\left(-\frac{\lambda_I\hat\eta}{|\lambda|^2}\right) + \frac{\lambda_R}{\lambda_I}(\operatorname{Sgn} R_1)\exp\left(\frac{\lambda_I\hat\eta}{|\lambda|^2}\right)\right)^2},$$

$$u_2(\xi, t) = -\frac{4\frac{\mu_I}{|\mu|^2}\operatorname{Sgn} R_2}{\exp\left(\frac{2\mu_I\hat\xi}{|\mu|^2}\right) + \left(\exp\left(-\frac{\mu_I\hat\xi}{|\mu|^2}\right) + \frac{\mu_R}{\mu_I}(\operatorname{Sgn} R_2)\exp\left(\frac{\mu_I\hat\xi}{|\mu|^2}\right)\right)^2}$$

where

$$\hat\eta = \eta - \frac{\lambda_R}{|\lambda|^2}t + \eta_0, \quad \hat\xi = \xi - \frac{\mu_R}{|\mu|^2}t + \xi_0.$$

The corresponding soliton of the Ishimori equation is of form:

$$S_1^{ss}(\xi, \eta, t) = S_\perp^{ss}(\hat\xi, \hat\eta)\cos\left[\frac{\lambda_R}{|\lambda|^2}\hat\eta + \frac{\mu_R}{|\mu|^2}\hat\xi + \frac{t}{2}\left(\frac{1}{|\lambda|^2} + \frac{1}{|\mu|^2}\right) + \Phi_{ss}(\hat\xi, \hat\eta)\right],$$

$$S_2^{ss}(\xi, \eta, t) = S_\perp^{ss}(\hat\xi, \hat\eta)\sin\left[\frac{\lambda_R}{|\lambda|^2}\hat\eta + \frac{\mu_R}{|\mu|^2}\hat\xi + \frac{t}{2}\left(\frac{1}{|\lambda|^2} + \frac{1}{|\mu|^2}\right) + \Phi_{ss}(\hat\xi, \hat\eta)\right],$$

$$S_3^{ss}(\xi, \eta, t) = -1 + \frac{2\exp\left[\frac{\mu_I}{|\mu|^2}\hat\xi + \frac{\lambda_I}{|\lambda|^2}\hat\eta\right]}{A^2 + B^2} \tag{4.5}$$

where

$$S_\perp^{ss}(\hat\xi, \hat\eta) = \frac{2\sqrt{C^2 + D^2}}{A^2 + B^2}\exp\left\{\frac{\lambda_I}{|\lambda|^2}\hat\eta + \frac{\mu_I}{|\mu|^2}\hat\xi\right\}, \tag{4.6}$$

$$\Phi_{ss}(\hat\xi, \hat\eta) = \operatorname{arctg}(D/C)$$

and

$$A = 1 + \frac{1}{4R_1R_2} + \frac{R_1\lambda_R}{\lambda_I}e^{\frac{\lambda_I\hat\eta}{|\lambda|^2}} + \frac{R_2\mu_R}{\mu_I}e^{\frac{\mu_I\hat\xi}{|\mu|^2}} +$$

$$+ \frac{R_1R_2}{\lambda_I\mu_I}(\lambda_R\mu_R + \lambda_I\mu_I)\,e^{\frac{\lambda_I\hat\eta}{|\lambda|^2} + \frac{\mu_I\hat\xi}{|\mu|^2}},$$

$$B = R_2e^{\frac{\mu_I\hat\xi}{|\mu|^2}} - R_1e^{\frac{\lambda_I\hat\eta}{|\lambda|^2}} + R_1R_2\left(\frac{\lambda_R}{\lambda_I} - \frac{\mu_R}{\mu_I}\right)e^{\frac{\lambda_I\hat\eta}{|\lambda|^2} + \frac{\mu_I\hat\xi}{|\mu|^2}},$$

$$C = 1 + \frac{1}{4R_1R_2} + \frac{R_1\lambda_R}{\lambda_I}e^{\frac{\lambda_I\hat\eta}{|\lambda|^2}} + \frac{R_2\mu_R}{\mu_I}e^{\frac{\mu_I\hat\xi}{|\mu|^2}} + \frac{R_1R_2}{\lambda_I\mu_I}(\lambda_R\mu_R - \lambda_I\mu_I)\,e^{\frac{\lambda_I\hat\eta}{|\lambda|^2} + \frac{\mu_I\hat\xi}{|\mu|^2}},$$

$$D = R_1e^{\frac{\lambda_I\hat\eta}{|\lambda|^2}} + R_2e^{\frac{\mu_I\hat\xi}{|\mu|^2}} + R_1R_2\left(\frac{\lambda_R}{\lambda_I} + \frac{\mu_R}{\mu_I}\right)e^{\frac{\lambda_I\hat\eta}{|\lambda|^2} + \frac{\mu_I\hat\xi}{|\mu|^2}}.$$

The soliton (4.5) decays exponentially in all directions on the

plane ξ, η similar to the localized soliton of the DS equation [1–6] and moves with the velocity $V = (V_\xi, V_\eta) = \left(\frac{\mu_R}{|\mu|^2}, \frac{\lambda_R}{|\lambda|^2}\right)$. In the case of time independent boundaries, when λ_R, $\mu_R \to 0$, the soliton (4.5) coincides with corresponding soliton obtained in the paper I.

Our last example here corresponds to the choice of the rational lump (3.8) as the boundary $u_1(\eta, t)$ and of the plane soliton (3.14) as the boundary $u_2(\xi, t)$:

$$u_1(\eta, t) = -\frac{\beta}{(\hat\eta + c)^2 + \frac{\beta^2}{4}},$$

$$u_2(\xi, t) = -\frac{4\frac{\mu_I}{|\mu|^2}\operatorname{Sgn} R}{\exp\left(\frac{2\mu_I\hat\xi}{|\mu|^2}\right) + \left(\exp\left(-\frac{\mu_I\hat\xi}{|\mu|^2}\right) + \frac{\mu_R}{\mu_I}(\operatorname{Sgn} R)\exp\left(\frac{\mu_I\hat\xi}{|\mu|^2}\right)\right)^2}$$

where

$$\hat\eta = \eta - \frac{t}{\beta}, \quad \hat\xi = \xi - \frac{\mu_R}{|\mu|^2}t + \xi_0.$$

In this case one has for the soliton of the Ishimori equation:

$$S_1^{rs}(\xi, \eta, t) = S_\perp^{rs}(\hat\xi, \hat\eta)\cos\left[\frac{\hat\eta}{\beta} + \frac{\mu_R}{|\mu|^2}\hat\xi + \frac{t}{2}\left(\frac{1}{\beta^2} + \frac{1}{|\mu|^2}\right) + \Phi_{rs}(\hat\xi, \hat\eta)\right],$$

$$S_2^{rs}(\xi, \eta, t) = S_\perp^{rs}(\hat\xi, \hat\eta)\sin\left[\frac{\hat\eta}{\beta} + \frac{\mu_R\hat\xi}{|\mu|^2} + \frac{t}{2}\left(\frac{1}{\beta^2} + \frac{1}{|\mu|^2}\right) + \Phi_{rs}(\hat\xi, \hat\eta)\right],$$

$$S_3^{rs}(\xi, \eta, t) = -1 + \frac{2\exp\left\{\frac{2\mu_I\hat\xi}{|\mu|^2}\right\}}{A^2 + B^2}$$

where

$$S_\perp^{rs}(\hat\xi, \hat\eta) = 2\frac{\sqrt{C^2 + D^2}}{A^2 + B^2}\exp\left\{\frac{\mu_I\hat\xi}{|\mu|^2}\right\} \tag{4.8}$$

$$\Phi_{rs}(\hat\xi, \hat\eta) = \operatorname{arc\,tg}\frac{D}{C}$$

and

$$A = (\hat\eta + c)\left(1 + \frac{R\mu_R}{\mu_I}e^{\frac{2\mu_I\hat\xi}{|\mu|^2}}\right) + \frac{\beta R}{2}e^{\frac{2\mu_I\hat\xi}{|\mu|^2}} + \frac{1}{2R\beta},$$

$$B = (\hat\eta + c)Re^{\frac{2\mu_I\hat\xi}{|\mu|^2}} - \frac{\beta}{2}\left(1 + \frac{R}{\mu_I}\mu_Re^{\frac{2\mu_I\hat\xi}{|\mu|^2}}\right),$$

$$C=(\hat{\eta}+c)\left(1+\frac{R\mu_R}{\mu_I}e^{\frac{2\mu_I\hat{\xi}}{|\mu|^2}}\right)-\frac{\beta R}{2}e^{\frac{2\mu_I\hat{\xi}}{|\mu|^2}}+\frac{1}{2R\beta},$$

$$D=(\hat{\eta}+c)\,R_1\,e^{\frac{2\mu_I\hat{\xi}}{|\mu|^2}}+\frac{\beta}{2}\left(1+\frac{R\mu_R}{\mu_I}e^{\frac{2\mu_I\hat{\xi}}{|\mu|^2}}\right).$$

The formula (4.7) present the mixed rational-exponential decreasing soliton of the Ishimori equation.

In fact the family of the localized solitons for the Ishimori equation is much richer since the mKP-I equation in addition to the exact solutions mentioned above has other classes of exact solutions which correspond to the multiple-poles plane lumps and plane solitons and so on. The corresponding coherent structures of the Ishimori equation will be considered elsewhere.

5. THE LOCALIZED SOLITONS IN THE TERMS OF THE STEREOGRAPHIC PROJECTION FIELD VARIABLE

$$q=\frac{S_1+iS_2}{1+S_3}.$$

The Ishimori equation (1.1) which is in fact the system of the three (non-independent) nonlinear equations can be rewritten as the simple equation for the stereographic projection $q=\frac{S_1+iS_2}{1+S_3}$. The straightforward but lengthy calculations give:

$$iq_t-\frac{1}{2}(q_{\xi\xi}+q_{\eta\eta})+\frac{\bar{q}}{1+|q|^2}(q_\xi^2+q_\eta^2)-$$

$$-q_\xi\left[\partial_\eta^{-1}\frac{(q_\xi\bar{q}_\eta-q_\eta\bar{q}_\xi)}{(1+|q|^2)^2}+iu_2(\xi,t)\right]+$$

$$+q_\eta\left[\partial_\xi^{-1}\frac{q_\xi\bar{q}_\eta-q_\eta\bar{q}_\xi}{(1+|q|^2)^2}+iu_1(\eta,t)\right]. \qquad (5.1)$$

In the case of the trivial boundaries $u_1=u_2=0$ equation (5.1) has been derived in [15].

It is not difficult to see, that the equation is invariant under the following changes of the dependent variable $q\to-q$, $q\to\bar{q}=\pm 1/q$. So if the q is the solution of the eguation of (5.1), then the $-q$, $\pm q^{-1}$ are also the solutions of (5.1). In terms of the spin variable

$\vec{S}=(S_1, S_2, S_3)$ this means that if the $\vec{S}=(S_1, S_2, S_3)$ is the solution of the Ishimori equation then the $\vec{S}=(-S_1, -S_2, S_3)$, $\vec{S}=(S_1, -S_2, -S_3)$ and $\vec{S}=(-S_1, S_2, -S_3)$ are also the solutions of the Ishimori equation.

The last remark leads us to the conclusion that the two types of solutions of equation (5.1) correspond to the obtained localized solutions (4.1), (4.5) and (4.7) of the Ishimori equation (1.1). The first type is given by the increasing in all directions on the plane ξ, η (polinomially, exponentially, or polynomially-exponentially) solutions of (5.1). The second type is formed by the localized solutions of (5.1), which are decreasing rationally, exponentially or rationally-exponentially in all directions on the plane.

The increasing in all directions of the plane solutions of eg. (5.1) which correspond via stereographic projection $q=\frac{S_1+iS_2}{1+S_3}$ to the localized solutions of Ishimori equation are of the following form:

1. The polynomially increasing solution which corresponds to (4.1) is

$$q(\xi,\eta,t)=\left[\left(\hat{\eta}+c_1+\frac{i\beta}{2}\right)\left(\hat{\xi}+c_2+\frac{i\alpha}{2}\right)+\frac{1}{\alpha\beta}\right]e^{i\left[\frac{\hat{\xi}}{\alpha}+\frac{\hat{\eta}}{\beta}+\frac{t}{2}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\right]} \qquad (5.2)$$

where

$$\hat{\eta}=\eta-\frac{t}{\beta},\quad \hat{\xi}=\xi-\frac{t}{\alpha}.$$

2. The exponentially increasing solution which corresponds to (4.5) is

$$q(\xi,\eta,t)=\left[\left(1+\frac{R_1\lambda}{\lambda_I}e^{\frac{2\lambda_I\hat{\eta}}{|\lambda|^2}}\right)\left(1+\frac{R_2\mu}{\mu_I}e^{\frac{2\mu_I\hat{\xi}}{|\mu|^2}}\right)\right]\times$$

$$\times\exp\left\{-\frac{\lambda_I}{|\lambda|^2}\hat{\eta}-\frac{\mu_I}{|\mu|^2}\hat{\xi}+i\left[\frac{\mu_R}{|\mu|^2}\hat{\xi}+\frac{\lambda_R}{|\lambda|^2}\hat{\eta}+\frac{t}{2}\left(\frac{1}{|\lambda|^2}+\frac{1}{|\mu|^2}\right)\right]\right\} \qquad (5.3)$$

where

$$\hat{\xi}=\xi-\frac{\mu_R}{|\mu|^2}t+\xi_0,\quad \hat{\eta}=\eta-\frac{\lambda_R}{|\lambda|^2}t+\eta_0.$$

3. The polinomially-exponentially increasing solution which corresponds to (4.7) is

$$q(\xi,\eta,t)=\left[\left(\hat{\eta}+c+\frac{i\beta}{2}\right)\left(1+\frac{R\mu}{\mu_I}e^{\frac{\mu_I\hat{\xi}}{|\mu|^2}}\right)+\frac{1}{2R\beta}\right]\times$$

$$\times \exp\left\{-\frac{\mu_I}{|\mu|^2}\hat{\xi}+i\left[\frac{\hat{\eta}}{\beta}+\frac{\mu_R}{|\mu|^2}\hat{\xi}+\frac{t}{2}\left(\frac{1}{|\mu|^2}+\frac{1}{\beta^2}\right)\right]\right\} \qquad (5.4)$$

where

$$\hat{\eta}=\eta-\frac{t}{\beta}, \quad \hat{\xi}=\xi-\frac{\mu_R}{|\mu|^2}t.$$

The localized solutions of eq. (5.1) which correspond to the localized solutions of Ishimori equation via the formula $\tilde{q}=q^{-1}=\frac{1+S_3}{S_1+iS_2}=\frac{S_1-iS_2}{1-S_3}$ are of the form:

1. The rationally localized soliton of (5.1) which corresponds to (4.1) is

$$\tilde{q}(\xi,\eta,t)=\frac{\exp\left\{(-i)\left[\frac{\hat{\xi}}{\alpha}+\frac{\hat{\eta}}{\beta}+\frac{t}{2}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\right]\right\}}{\left(\hat{\eta}+c_1+\frac{i\beta}{2}\right)\left(\hat{\xi}+c_2+\frac{i\alpha}{2}\right)+\frac{1}{\alpha\beta}}. \qquad (5.5)$$

2. The exponentially localized soliton of (5.1) which corresponds to (4.5) is

$$\tilde{q}(\xi,\eta,t)=$$

$$=\frac{\exp\left\{\frac{\lambda_I\hat{\eta}}{|\lambda|^2}+\frac{\mu_I\hat{\xi}}{|\mu|^2}\right\}}{\left(1+\frac{R_1\lambda}{\lambda_I}e^{\frac{2\lambda_I\hat{\eta}}{|\lambda|^2}}\right)\left(1+\frac{R_2\mu}{\mu_I}e^{\frac{2\mu_I\hat{\xi}}{|\mu|^2}}\right)+\frac{1}{4R_1R_2}}\times$$

$$\times\exp\left\{(-i)\left[\frac{\mu_R}{|\mu|^2}\hat{\xi}+\frac{\lambda_R}{|\lambda|^2}\hat{\eta}+\frac{t}{2}\left(\frac{1}{|\lambda|^2}+\frac{1}{|\mu|^2}\right)\right]\right\}. \qquad (5.6)$$

3. The rationally-exponentially localized soliton of (5.1) which corresponds to (4.7) is

$$\tilde{q}(\xi,\eta,t)=\frac{\exp\left(\frac{\mu_I\hat{\xi}}{|\mu|^2}\right)}{\left(\hat{\eta}+c+\frac{i\beta}{2}\right)\left(1+\frac{R\mu}{\mu_I}exp\left(\frac{\mu_I\hat{\xi}}{|\mu|^2}\right)\right)+\frac{1}{2R\beta}}\times$$

$$\times\exp\left\{(-i)\left[\frac{\hat{\eta}}{\beta}+\frac{\mu_R}{|\mu|^2}\hat{\xi}+\frac{t}{2}\left(\frac{1}{|\mu|^2}+\frac{1}{\beta^2}\right)\right]\right\}. \qquad (5.7)$$

In conclusion remind that the DS equation is of the form [1—6]

$$ip_t+p_{\xi\xi}+p_{\eta\eta}-$$

$$-p\left[\frac{1}{2}\int_{-\infty}^{\xi}d\xi'\,|p|^2_\eta-u_1(\eta,t)+\frac{1}{2}\int_{-\infty}^{\eta}d\eta'\,|p|^2_\xi-u_2(\xi,t)\right]=0. \qquad (5.8)$$

The simplest exponentially localized soliton of the DS equation is [1—6]:

$$p(\xi,\eta,t)=4\rho\sqrt{\lambda_R\mu_R}\times$$

$$\times\frac{\exp\{-\lambda_R(\hat{\eta}-\bar{\eta})-\mu_R(\hat{\xi}-\bar{\xi})+i[-\lambda_I\hat{\eta}+\mu_I\hat{\xi}+(|\mu|^2+|\lambda|^2)t+\arg(lm)]\}}{(1+e^{-2\lambda_R(\hat{\eta}-\bar{\eta})})(1+e^{-2\mu_R(\hat{\xi}-\bar{\xi})})+|\rho|^2} \qquad (5.9)$$

where

$$\hat{\eta}\overset{def}{=}\eta+2\lambda_I t, \quad \hat{\xi}\overset{def}{=}\xi+2\mu_I t,$$

$$\bar{\eta}=\frac{1}{\lambda_R}\ln\frac{|l|}{\sqrt{2\lambda_R}}, \quad \bar{\xi}=\frac{1}{\mu_R}\ln\frac{|m|}{\sqrt{2\mu_R}}.$$

The solution (5.9) corresponds to the boundaries $u_1(\eta,t)$ and $u_2(\xi,t)$ of the form

$$u_1(\eta,t)=\frac{2\lambda_R^2}{\mathrm{ch}^2\lambda_R(\hat{\eta}-\bar{\eta})}, \quad u_2(\xi,t)=\frac{2\mu_R^2}{\mathrm{ch}^2\mu_R(\hat{\xi}-\bar{\xi})}. \qquad (5.10)$$

Comparing (5.9) and (5.6) we see that they are very similar in form. Thus the Ishimori equation (1.1) on the one hand is more complicated in its form than DS equation but on the other hand it is more rich as far as concerning the localized coherent structures.

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Coherent Structures for the Ishimori Equation.
II. Time-dependent boundaries

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