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CHAOTIC QUANTUM SYSTEMS

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ABSTRACT

The overview of recent developments in the theory of quantum chaos is presented with the special emphasis on a number of unsolved problems and current apparent contradictions.

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0. INTRODUCTION

The primary purpose of this lecture is to present an overview of main recent developments in the theory of the so-called chaotic quantum systems or, better to say, of the *quantum chaos*. This is a rather new topic, especially for mathematicians, which arose some time ago from the study of a very peculiar classical phenomenon— the *dynamical chaos* (for review see, e.g., Refs [1, 2]). The latter is fairly well understood by now including many important rigorous mathematical results in the modern ergodic theory [3].

Two properties of the dynamical chaos should be stressed from the beginning. First, such a chaos arises, under certain conditions (see Section 3 below), in a purely dynamical system (hence, the term) without any random parameters, or any noise. To the contrary, the chaotic dynamical system is itself a generator of noise. Second, those systems can be very simple, particularly, a few-dimensional ones (see Section 2 for examples). In a conservative Hamiltonian system $N = 2$ freedoms is already sufficient for chaos. In case of a periodically driven system this small number further reduces to $N = 1$. Both cases can be described also by a two-dimensional mapping (briefly, 2D map).

In case of a dissipative system even 1D map, as a model of such a system, can be chaotic. Yet, we are not going to discuss this case further because the dissipative system is not a purely dynamical one. This is especially clear in quantum mechanics where dissipation is included by the coupling to a heat bath.

All this is completely different from the conceptions of traditional statistical mechanics (TSM) with its thermodynamic limit $N \to \infty$ (for further

discussion see Section 9). Instead, in the new field of dynamical chaos the studies are concentrated on few-dimensional systems.

Now, the crucial physical question is whether this peculiar phenomenon—the dynamical chaos—would survive quantization? This is just the principal problem of quantum chaos. The answer, which will be discussed in detail below, is negative.

On the first glance, this is no surprise since the quantum mechanics is well known to be fundamentally different as compared to the classical mechanics. However, the difficulty, and a very deep one, arises from the fact that the former is commonly believed to be the universal theory, particularly, comprising the latter as the limiting case. Hence, the correspondence principle which requires the transition from quantum to classical mechanics in all cases including the dynamical chaos. Thus, there must exist a sort of quantum chaos!

Of course, one would not expect to find any similarity to classical behavior in essentially quantum region but only sufficiently far in the quasi-classical domain. Usually, it is characterized formally by the condition that Planck's constant $\hbar \to 0$. I prefer to put $\hbar = 1$ (which is a question of the units), and to introduce some quantum parameter q. Generally, it depends on a particular problem, and may be, for instance, the quantum (level) number. The quasi-classical region then corresponds to $q \gg 1$ while in the limit $q \to \infty$ the complete rebirth of classical mechanics must occur somehow. Notice that unlike other theories (of relativity, for example) the quasi-classical transition is rather intricate. Actually, this is the main topic of my lecture (see also Ref. [4]). Thus, the quantum chaos we are going to discuss is essentially a quasi-classical phenomenon in few-dimensional systems.

The number of papers devoted to the studies of quantum chaos and related phenomena is rapidly increasing so that practically impossible to comprise everything in this field. In what follows I have to restrict myself to some selected topics which I know better or which I myself consider as most important. The same is true for references. I apologize beforehand for possible

omissions and inaccuracies. Anyway, Irefer in addition to a number of recent reviews [4 – 9].

My presentation below will be, of course, from a physicist's viewpoint. All I can do for mathematicians is to explain and attract attention to many unsolved problems in this field.

The lecture is organized as follows. In the next Section 1 the principal approach to the problem is explained, and in Section 2 a few simple physical models are described which I am going to use further as illustrations. In Section 3 various defenitions of both classical and quantum chaos are discussed. Sections 4 through 7 are the main part of the lecture where I describe what remains from the classical chaos in quantum mechanics, and how the classical limit reappears from these remnants. Section 8 is devoted to the relation between dynamical quantum chaos and statistical random matrix theory (RMT) while in Section 9 a similar problem with respect to TSM is discussed. I finish with a few exotic examples of the true quantum chaos (Section 10), and with a brief listing some other related problems which I am not yet ready or have no space-time to discuss in detail (Section 11).

1. SOME PHILOSOPHY: PHYSICS AND MATHEMATICS

The well ascertained absence of the classical-like chaos in quantum mechanics apparently contradicts not only with the correspondence principle, as mentioned above, but also with the fundamental statistical nature of quantum mechanics. However, even though the random element in quantum mechanics ("quantum jumps") is inavoidable, indeed, it can be singled out and separated from the proper quantum processes. Namely, the fundamental randomness in quantum mechanics in related only to a very special event—the *quantum measurement*—which, in a sense, is foreign to the proper quantum system itself.

This allows to divide the whole problem of quantum dynamics into two qualitatively different parts:

(i) the proper quantum dynamics as described by the wave function $\psi(t)$, and

(ii) the quantum measurement including the registration of the result, and hence the ψ collapse.

Below I am going to discuss the first part only, and to consider ψ as a specific dynamical variable ignoring the common term for ψ, the probability amplitude. Variable ψ obeys some purely dynamical equation of motion, e.g., the Schrödinger equation.

This part of the problem is essentially mathematical, and it naturally belongs to the general theory of dynamical systems.

The main contribution of physicists to the studies of quantum chaos is in extensive numerical (computer) simulations of quantum dynamics, or numerical experiments as we use to say. But not only that. First of all, numerical experiments are impossible without a theory, if only semiqualitative, and without even rough estimates to guide the study. Mathematicians may consider such physical theories as a collection of hypotheses to prove or disprove them. What is even more important, in my opinion, that those theories require, and are based upon, a set of new notions and concepts which may be also useful in a future rigorous mathematical treatment.

I would like to mention that with all their obvious drawbacks and limitations the numerical experiments have very important advantage (as compared to the laboratory experiments), namely, they provide the complete information about the system under study. In quantum mechanics this advantage becomes crucial because in the laboratory one cannot observe (measure) the quantum system without a radical change of its dynamics.

As to the second part of the problem—the quantum measurement—this is a hard nut for physicists. Currently, there is no common opinion even on the question whether this is a real physical problem or an ill-posed one so that the Copenhagen interpretation of (or convention in) quantum mechanics answers all admissible questions. In any event, there exists as yet no dynamical

description of the quantum measurement including the ψ collapse. An interesting recent discussion of this question in the light of quantum cosmology can be found in Ref. [10]. In my opinion, one could find more "earthy" problems as well.

Below I comment about the quantum measurement on a few occasions, but I will not discuss it in any detail as this certainly goes beyond the frame of my lecture here, and even of this conference as a whole.

Recent breakthrough in the understanding of quantum chaos has been achieved, particularly, due to the above philosophy of separating the dynamical part of quantum mechanics accepted, explicitly or more often implicitly, by most researchers in this field.

2. THE BASIC MODEL AND PHYSICAL EXAMPLES

As an illustration of dynamical chaos, both classical and quantal, I will make use of the following "simple" model. In the classical limit it is described by the so-called *standard map*: $(n, \theta) \to (\bar{n}, \bar{\theta})$ where

$$\bar{n} = n + k \cdot \sin\theta\,; \qquad \bar{\theta} = \theta + T \cdot \bar{n}\,. \tag{2.1}$$

Here n, θ are the action-angle dynamical variables; k, T stand for the strength and period of perturbation. The phase space of this model is an infinite cylinder which can be also "rolled up" into a torus of cirqumference

$$L = \frac{2\pi m}{T} \tag{2.2}$$

with an integer m to avoid discontinuities. Notice that map (2.1) is periodic not only in θ but also in n with period $2\pi/T$. The latter is a nongeneric symmetry of this model. In the studies of general chaotic properties it is a disadvantage. Nevertheless, the model is very popular, apparently because of its formal and technical symplicity combined with the actual richness of

behavior. It can be interpreted as a mechanical system—the rotator driven by a series of short impulses, hence another nickname—"the *kicked rotator*". Essentially, however, it is an abstract mathematical model to try various methods and concepts.

The quantized standard map was first introduced and studied in Ref. [11]. It is described also by a map: $\psi \to \bar{\psi}$ where

$$\bar{\psi} = \hat{R}_T \hat{F}_k \psi \tag{2.3}$$

and where

$$\hat{F}_k = \exp(-ik \cdot \cos \hat{\theta}), \qquad \hat{R}_T = \exp\left(-i \frac{T \hat{n}^2}{2}\right) \tag{2.4}$$

are the operators of a "kick" and of a free rotation, respectively. Momentum operator is given by the usual expression: $\hat{n} = -i\partial/\partial\theta$.

Some time it is more convenient to use the symmetric map

$$\bar{\psi} = \hat{R}_{T/2} \hat{F}_k \hat{R}_{T/2} \psi, \tag{2.5}$$

which differs from Eq. (2.3) by the time shift $T/2$, and which is moreover, time reversible. In the most interesting case of a strong perturbation ($k \gg 1$) the operator $\hat{F}_k$ couples approximately $2k$ unperturbed states.

Notice that in classical limit the motion of model (2.1) depends on a single parameter $K = kT$ but after quantization the two parameters, k and T, can not be combined any longer.

Even though the standard map (2.1)—(2.5) is primarily a simple mathematical model it can serve also to approximately describe some real physical systems or, better to say, some more realistic models of physical systems. One interesting example is the peculiar diffusive photoeffect in Rydberg (highly excited) atoms (see, e.g., Refs [12, 9] for review).

The simplest 1D model is described by the Hamiltonian (in atomic units):

$$H = -\frac{1}{2n^2} + \varepsilon \cdot z(n, \theta) \cos \omega t, \tag{2.6}$$

where z stands for the coordinate along the linearly polarized electric field of strength ε and frequency ω.

Another approach to this problem is constructing a map over a Kepler period of the electron [13]: $(N_\Phi, \theta) \to (\bar{N}_\Phi, \bar{\theta})$ where

$$\bar{N}_\Phi = N_\Phi + k \cdot \sin \theta; \qquad \bar{\theta} = \theta + \frac{\pi}{(2\omega)^{1/2}} (-\bar{N}_\Phi)^{-3/2}. \tag{2.7}$$

Here, $N_\Phi = E/\omega = -1/2\omega n^2$, and perturbation parameter

$$k \approx 2.6 \frac{\varepsilon}{\omega^{5/3}}, \tag{2.8}$$

if the field frequency exceeds that of the electron: $\omega n^3 \geq 1$.

Linearizing the second Eq. (2.7) in N_Φ reduces the Kepler map (2.7) to the standard map with the same k, and with parameter

$$T = 6\pi\omega^2 n^5. \tag{2.9}$$

Thus, the standard map describes the dynamics locally in momentum.

In quantum mechanics, instead of solving Schrödinger's equation with Hamiltonian (2.6) one can directly quantize a simple Kepler map (2.7) to arrive at a quantum map (2.3) with the same perturbation operator $\hat{F}_k$ (2.4) but with a different rotation operator

$$\hat{R}_\nu = \exp(-2i\pi\nu(-2\omega \hat{N}_\Phi^{-1/2})). \tag{2.10}$$

Here parameter $\nu = 1$ (one Kepler's period) for quantum map (2.3), and $\nu = 1/2$ for symmetric map (2.5).

Notice that in Kepler map's description a new time is discrete (the number of map's iterations), and moreover, its relation to the continuous time t in Hamiltonian (2.6) depends on dynamical variable n or N_Φ:

$$\frac{dt}{d\tau} = 2\pi n^3 = 2\pi\,(-2\omega N)^{-3/2}. \tag{2.11}$$

In quantum mechanics such a change of time variable constitutes the serious problem: how to relae the two solutions, $\psi(t)$ and $\psi(\tau)$? For futher discussion of this problem see Section 5, and Ref. [9]. Besides, map's solution $\psi(N, \tau)$ does not provide the complete quantum description but only some averaged one over the groups of unperturbed states [13].

These difficulties are of a general nature in attempts to make use of the Poincaré map for conservative quantum systems. The straightforward approach would be, first, to solve the Schrödinger equation, and then to construct the quantum map out of $\psi(t)$. Usually, this is a very difficult way. Much simpler one is, first, to derive the classical Poincaré map, and then to quantize it.

However, generally the second way provides only an approximate solution for the original system. The question is how to reconcile the both approaches?

Another physical problem—the Rydberg atom in constant and uniform magnetic field, I will refer to below, is described by the Hamiltonian (for review see Ref. [14]):

$$H = \frac{p_z^2 + p_\rho^2}{2} - \frac{1}{r} + \frac{\omega L_z}{2} + \frac{\omega^2 \rho^2}{8}. \tag{2.12}$$

Here $r^2 = \rho^2 + z^2 = x^2 + y^2 + z^2$; ω is the Larmor frequency in the magnetic field along z axis, and L_z stands for component of the angular momentum (in atomic units).

Unlike the previous model the latter one is conservative (energy preserving). It is simpler for theoretical studies and, hence, more popular

among mathematicians. Physicists prefer time-dependent systems or, to be more precise, the models described by maps which greatly facilitate numerical experiments.

Another important class of conservative models are the billiards, both classical and quantal [16—18, 4]. Especially popular is the billiard model called "stadium" [17]. Interestingly, instead of a quantum ψ wave one may consider classical linear waves, e.g., electromagnetic, sound, elastic etc. In the latter case the billiard is called "cavity". Of course, this problem has been studied since long ago, yet only recently it was related to the brand-new phenomenon of "quantum" chaos [19, 20].

Quantum (wave) billiards are the limiting (and a simpler) case of the general dynamics of linear waves in dispersive media. It seems that the case of a spatially random medium does attract the most attention in this field, particularly, the celebrated phenomenon of the *Anderson localization*. True, this is a statistical rather than dynamical problem. On the other hand, one may consider the random potential as a typical one, and the averaged solution as the representation of typical properties in such systems. Instead, in the spirit of the dynamical chaos, one can extend the problem in question onto a class of regular (but not periodic) potentials (Section 6).

Recently, a deep analogy has been discovered between this rather old problem of wave dynamics in configurational space (in a medium) and of the dynamics in momentum space, particularly, the excitation of a quantum system by driving perturbation [21, 22]. Remarkably, that while the latter problem is described by a time-dependent Hamiltonian the former is a conservative system. This interesting and instructive similarity will be discussed in Section 6.

3. DEFINITION OF DYNAMICAL CHAOS

This is one of the most controversial question even in classical mechanics. There are two main approaches to the problem. The first one is essentially

mathematical [3, 23]. The terms dynamical chaos or randomness are abandoned from the rigorous statements, and left for informal explanations only, usually in quotes *). Instead, a hierarchy of statistical characteristics, such as ergodicity, mixing, K, Markov and Bernoulli properties etc, are introduced. In this hierarchy each property supposed to imply all the preceeding ones.

However, this is not the case in the very important and fairly typical situation when the motion is restricted to a *chaotic component* usually of a very complicated (fractal) structure which occupies only a part of the energy surface in a conservative system or even a submanifold of lesser dimensions (see, e.g., Ref. [26]).

The second approach to the definition of the classical dynamical chaos is essentially physical (see, e.g., Refs [1, 2]). Here the conception of random trajectories of a dynamical system is introduced from the beginning, and it is related to the strong (exponential) *local instability* of motion. This is characterized by a positive Lyapunov's exponent Λ or, more generally, by the Kolmogorov—Sinai (KS) dynamical entropy h.

The main difficulty here is in that the instability itself is not sufficient for chaotic motion. One additional condition is boundedness of the motion to exlude, for example, the hyperbolic motion which hardly can be termed chaotic. Further, the separated unstable periodic trajectories must be also excluded, possibly, by the requirement of some minimal dimensions of a chaotic component. To the best of my knowledge, the complete set of conditions for an arbitrary motion component to be chaotic has not been found as yet, and it constitutes a difficult problem.

The latter definition of classical dynamical chaos is commonly accepted in the physical literature, yet it needs to be modified in the light of our main

*) even in Ref. [24] where a version of rigorous definition of dynamical randomness (chaos) was actuakky given. This is not the case in Chaitin's papers (see, e.g., Ref. [25]) but his is somewhat separated from the rest of ergodic theory.

problem, the quantum chaos. Namely, the *two levels of chaos* can be introduced:

(i) the "first-rate chaos" ($h > 0$) with truly random trajectories, and

(ii) the "second-rate" chaos which is just sufficient to admit a simple statistical description by a sort of kinetic equation.

On the first level the chaos has very strong statistical properties to the extent that the dynamical trajectory completely loses any physical meaning, and only statistical description is possible. Notice, however, that the motion equations can still be efficiently used to completely derive all the statistical properties without any additional statistical hypotheses.

On the second level only some mixing property is required which is roughly equivalent to the continuous component of the spectrum. What is important for quantum chaos that even on the second level of classical chaos the motion phase space has to be continuous to provide traditional statistical discription.

This is not the case in quantum mechanics where the size of an elementary phase-space cell is $\hbar^{1/2}$. Particularly, for the quantum motion bounded in phase space the total number of cells is even finite, and classical mixing is impossible, to say nothing about the exponential instability. Moreover, the energy spectrum in this case is descrete which in the classical mechanics would correspond to the opposite limiting case of regular motion. So, what is then quantum chaos we are going to discuss?

Currently, the most common definition reads (see, e.g., Ref. [6]): *quantum chaos is the quantum dynamics of classically chaotic systems* whatever it could happen to be, I would add.

Logically, this is most simple and clear definition. Yet, it is completely inadequate, in my opinion, just because that chaos may turn out to be a perfectly regular motion, much surpassing that in the classical limit. The point is that the discreteness of quantum spectrum supresses any transitions for a sufficiently weak perturbation, no matter what is the corresponding classical motion [27]. For example, in the standard map this occurs if perturbation

parameter $k \leq 1$ independent of classical parameter $K=kT$ which controls the transition to chaos. This specific quantum stability is also called *perturbative localization*, or *transition localization*.

For this reason Berry proposed [28] to use the term "*quantum chaology*" instead of quantum chaos meaning just the above definition.

The opposite position was taken by Ford [29] who insists that *the quantum chaos is deterministic randomness in quantum mechanics over and above that contained in the wavefunction or the expansion postulate.*

The latter refers to the quantum measurement as discussed in Section 1 above. A few other researchers implicitly accepted the same definition, and successfully constructed the quantum analogue h_Q to classical KS-entropy h. The only problem is that $h_Q = 0$ (see, however, Section 10 and Ref. [97]).

My position is somewhere in between. I would like to define the quantum chaos in such a way to include some essential part of the classical chaos. It would be natural to include the second level of chaos discussed above, that is some mixing property. The difficulty is in that the discrete spectrum prohibits even the mixing in the sense of the ergodic theory. Yet, it turns out that with some, essential though, modification the mixing can be extended on the quantum mechanics (Sections 4, 5).

For this reason, currently, I adhere to the following definition: *the quantum chaos is statistical relaxation in discrete spectrum.* A drawback of this definition is in that such a chaos occurs also in the classical systems of linear waves as already mentioned. New suggestions are welcome!

4. THE TIME SCALES OF QUANTUM DYNAMICS

The main difficulty in the problem of quantum chaos is in that one needs to reconcile the quantum discrete spectrum, which apparently prohibits any dynamical chaos, with the correspondence principle, which does require some chaos, at least, sufficiently far in the quasi-classical region. We resolved this difficulty by introducing characteristic time scales of the quantum motion on which it is close to the classical chaotic dynamics [30].

Actually, the first of those time scales had been discovered and explained by Berman and Zaslavsky already in 1978 [31], and was subsequently confirmed in many numerical experiments (see, e.g., Refs [32]). This *Berman—Zaslavsky time scale* is given, generally, by an estimate

$$t_{BZ} \sim \frac{\ln q}{h}, \tag{4.1}$$

where $q \gg 1$ is some quantum (quasi-classical) parameter, and h stands for the KS-entropy (not Planck's constant $\hbar = 1$!).

In our basic model (2. 1) – (2.5) $h = \Lambda \approx (1/T)\ln(K/2)$ for $K > 4$, and there are two quantum parameters: k and $1/T$. The transition to the classical limit corresponds to $k \to \infty$, $T \to 0$ while the classical parameter $K = kT = \text{const}$. General estimate (4.1) takes now the form [30]

$$t_{BZ} \sim \frac{T\,|\ln T|}{\ln (K/2)}. \tag{4.2}$$

The corresponds to the optimal configuration of the initial Ψ (0), a coherent state.

This physical meaning of this time scale is in a fast (exponential) spreading of the initially narrow wave packet. Thus, the exponential instability is present in quantum mechanics as well but only on a very short time interval (4. 1, 2).

This can be explained in two ways. On the one hand, the initial wave packet cannot be less, in size, than a quantum phase-space cell, that is $\sim \hbar = 1$. On the other hand, in Hamiltonian systems, the local instability leads not only to the expanding in a certain direction but also to the contraction in another direction which rapidly brings the initial wave packet to the size of a quantum cell.

According to the Ehrenfest theorem a wave packet follows the beam of classical trajectories but only as long as it remains narrow, that is only on time scale (4.1). Nevertheless, characteristic time interval t_{BZ}/T grows indefinitely

in quasi-classical region, as $T \to 0$, in accordance with the correspondence principle. However, the transition to the classical chaos is (conceptually) difficult as it includes two limits ($T \to 0$ ($q \to \infty$) and $t \to \infty$) which do not commute. This is a typical situation in the quasi-classical region as was stressed, particularly, by Berry [5].

Still, the first time scale (4.1) is rather short, and the important question is: what happens next? Numerical experiments revealed [11, 30] that some classical-like chaos persists on a much longer time scale t_R which is, generally, of the order

$$\ln t_R \sim \ln q \ . \tag{4.3}$$

For model (2.1)

$$t_R \sim Tk^2 \ . \tag{4.4}$$

On this time scale the diffusion in n proceeds and, moreover, closely follows the classical diffusion in all details, again in agreement with the correspondence principle. Subsequently, these numerical results were confirmed also analytically [33]. We call t_R the *diffusion* or (statistical) *relaxation time scale.*

This similarity to the classical chaos is, however, only partial. Unlike the classical one the quantum diffusion was found to be perfectly stable dynamically. This was proved in striking numerical experiments with the time reversal [34]. In a classical chaotic system the diffusion is immediately recovered due to numerical "errors" (not random!) amplified by the local instability. On the contrary, the quantum "antidiffusion" proceeds until the system passes, to a high accuracy, the initial state, and only then the normal diffusion is restored. It is worthwhile to mention here that there is no time arrow in the dynamical chaos: statistical relaxation proceeds eventually in both directions of time.

Beyond the relaxation time scale, that is for $t \gg t_R$, the quantum diffusion and relaxation typically stops, and a certain steady state is formed which may

or may not be close to the classical statistical equilibrium. The details of quantum relaxation will be discussed in the next Section, and a peculiar quantum steady state is considered in Section 7.

As $k \to \infty$ (kT = const) the time scale $t_R \to \infty$ but this quasi-classical transition is also characterized by the same double limit as for t_{BZ} discussed above.

Thus, various properties of the classical dynamical chaos are present in quantum dynamics but only temporarily, within finite time scales t_{BZ} or t_R. This is a crucial distinction of the quantum ergodic theory from the classical one which is asymptotic in t. It seems that any substantial progress in the mathematical theory requires a generalization of the existing ergodic theory to a finite time. Perhaps, it is better to say that a new nonasymptotic (finite-time) ergodic theory needs to be created.

5. STATISTICAL RELAXATION IN DISCRETE SPECTRUM

Of the two characteristic time scales of quantum motion discussed in the previous Section the relaxation time scale t_R is most important simply because it is much longer than the other one, t_{BZ}. Peculiarity of quantum statistical relaxation is in that it proceeds in spite of the discrete energy spectrum. As is well known, the latter is always the case for the conservative system bounded in phase space. The crucial property is a finite number of quantum states on the energy surface or, better to say, within an *energy shell.* In ihis case [30]

$$t_R \leq \rho \ , \tag{5.1}$$

where ρ is finite energy level density ($\hbar = 1$).

The physical meaning of this estimate is very simple, and it is related to the fundamental uncertainty principle*). For sufficiently short time the discrete spectrum is not resolved, a classical-like diffusion is possible, at most up to $t \sim \rho$. The same is true for the standard map on a torus (2.2) which has also a finite number (L) of now quasi-energy states. Since the quasi-energy is determined mod $2\pi/T$ the level density is

$$\rho = \frac{TL}{2\pi} = m \geq t_R . \tag{5.2}$$

The situation is much less clear for the standard map on a cylinder where the motion can be unbounded in n. In some special cases the quasi-energy spectrum is, indeed, continuous, yet this does not mean a chaotic motion but rather a peculiar quantum resonance (see next Section). A more complicated case of continuous spectrum will be discussed in Section 7.

On the other hand, all the numerical evidence indicates that typically the quasi-energy spectrum is discrete in spite of the infinite number of levels. Formally, the level density ρ is then also infinite. Yet, relaxation time scale t_R is finite. The point is that the quantum motion does not depend on all quasi-energy eigenstates but only on those which are actually present in the initial quantum state $\psi(0)$. We call them *operative eigenstates* (for a given initial conditions). If their density is r a better estimate for t_R is (cf. Eq. (5.1)):

$$t_R \sim \rho_0 . \tag{5.3}$$

For ρ_0 to be finite all eigenfunctions have to be localized that is to decrease sufficiently fast in n. To the best of my knowledge there are as yet no rigorous results on the eigenfunction localization and/or the spectrum even for simple model (2.3).

*) In a different way this first principle was used in Ref. [15] to explain Andetson localization in random potential (see Section 6).

If the localization length is l, the density $\rho_0 \sim Tl/2\pi$ (for sufficiently localized initial state). This latter estimate can be improved by introducing the weighted density (cf. Ref. [35])

$$\rho_0^{-1}(n_0, \omega) = \frac{2\pi}{T} \sum_m |\varphi_m(n_0)|^2 \delta(\omega - \omega_m) . \tag{5.4}$$

Here φ_m are the eigenfunctions with eigenvalues ω_m, and the initial state is assumed in the form: $\psi(n, 0) = \delta_{nn_0}$. Of course, this does not improve the basic estimate (5.3) (see below).

Actually, Eq. (5.3) is an implicit relation because ρ_0 depends, in turn, on dynamics. Consider, first, the unbounded standard map, and introduce the classical diffusion rate in the form

$$D_n \equiv \frac{\langle (\Delta n)^2 \rangle}{\tau} \sim k^2 , \tag{5.5}$$

where $\tau = t/T$ is the number of map's iterations, and the latter estimate holds for $K \gg 1$ (complete classical chaos). Suppose, further, that the width (in n) of the initial state $\Delta n_0 = l_0 \ll l$. Then the final width due to a diffusion during time τ_R is $\Delta n_f \sim (\tau_R D_n)^{1/2} \sim l$. Since $t_R \sim l$, we arrive at the remarkable relation

$$\tau_R \sim l \sim D_n , \tag{5.6}$$

which couples essentially quantum characteristics (τ_R, l) with the classical quantity D_n.

Thus, the quantum diffusion in the unbounded standard map is always localized, and a certain steady state is formed which has no counterpart in classical mechanics (Section 7).

For the bounded standard map the situation is qualitatively different depending on a new parameter

$$\lambda = \frac{l}{L} , \tag{5.7}$$

which we term the *ergodicity parameter*. Indeed, the quantum localization occurs for $\lambda \ll 1$ only. In the opposite limiting case $\lambda \gg 1$ the relaxation time scale, being finite, is nevertheless long enough for the relaxation to the ergodic steady state to occur. In this case the final steady state is close to that in classical mechanics (Section 8). The same is true for conservative systems of two freedoms like billiards or cavities.

Now we turn to a more accurate discription of the quantum relaxation in model (2.3). First, what are the quasi-energy eigenfunctions? We shall discuss this in detail in Section 7. Now it is sufficient to know that the quantum localization is approximately exponential with eigenfunctions

$$\varphi_m(n) \approx \frac{1}{l^{1/2}} \exp\left(-\frac{|m-n|}{l}\right) \tag{5.8}$$

and the steady state

$$g_s(n) = \overline{|\psi_s(n)|^2} \approx \frac{1}{l_s} \exp\left(-\frac{2|n|}{l_s}\right). \tag{5.9}$$

Here the bar means averaging in time, and the initial conditions $g(n,0) = \delta(n)$ so that g is the Green function. Generalization to arbitrary initial conditions is obvious.

Numerically (see, e.g., the second Ref. [30]):

$$l_s \approx 2l \approx D_n. \tag{5.10}$$

This is a more accurate relation between quantum and classical characteristics of motion (cf. Eq. (5.6)). Surprisingly, the localization lengths for eigenfunctions and for the steady state are rather different. This is due to big fluctuations around simple exponential dependence (5.8).

The first attempt to describe the quantum relaxation in standard map was undertaken in Ref. [36]. The idea was very simple: the diffusion rate is proportional to the number of quasi-energy levels which are not yet resolved in time τ. This number decreases, for $\tau \geq \tau_R$, as τ^{-1}, hence

$$D_n(\tau) \sim D_n(0) \frac{\tau_R}{\tau}, \tag{5.11}$$

where $D_n(0)$ is the classical diffusion rate. This result was corrected in Ref. [37] where, in a more sophisticated way, the level repulsion (see Section 8) was taken into account to give for the rate of energy $E = n^2/2$ variation

$$\frac{dE(\tau)}{d\tau} \equiv \dot{E}(\tau) \sim \dot{E}(0) \left(\frac{\tau_R}{\tau}\right)^{1+\beta}, \tag{5.12}$$

where β is the repulsion parameter. Preliminary fitting Eq. (5.12) to some numerical data looked as an agreement with $\beta \approx 0.3$

However recent extensive numerical simulations [38] revealed a different dependence for $\tau \gg \tau_R$ (in our notations)

$$\dot{E}(\tau) \approx c\dot{E}(0) \left(\frac{\tau_R}{\tau}\right)^2 \ln\frac{\tau}{\tau_R} \tag{5.13}$$

supported by a different theory. Numerically (my fit):

$$\tau_R \approx 2l_s; \qquad c \approx 0.2 \tag{5.14}$$

in apparent contradiction with Eq. (5.12).

Still another phenomenological theory was proposed in Ref. [9] and developed in Ref. [62]. It is based on the general diffusion equation (see, e.g., Ref. [1]):

$$\frac{\partial}{\partial\tau} g(n,\tau) = \frac{1}{2} \frac{\partial^2}{\partial n^2} D_n g - \frac{\partial}{\partial n} Ag. \tag{5.15}$$

The second term discribes a "drift"

$$A = \frac{\langle \Delta n \rangle}{\tau} = \frac{dD_n}{dn} + B. \tag{5.16}$$

Introducing this relation into Eq. (5.15), we obtain

$$\frac{\partial g}{\partial \tau} = \frac{1}{2}\frac{\partial}{\partial n} D_n \frac{\partial g}{\partial n} - \frac{\partial}{\partial n} Bg; \tag{5.17}$$

In our problem the last term represents the so-called "backscattering", or reflection of wave propagating in n.

Negligible in the beginning the backscattering eventually suppresses the diffusion and leads to the formation of steady state (5.9).

From Eq. (5.17) the general expression for the steady state $g_s(n)$ is

$$\ln g_s = 2\int \frac{B(n)\, dn}{D_n(n)}. \tag{5.18}$$

For homogeneous diffusion ($D_n = \text{const}$) g_s is given by Eq. (5.9) with $l_s = D_n$, hence

$$B = \mp 1; \qquad n \gtrless 0. \tag{5.19}$$

The analysis of quantum relaxation can be performed using the two first moments of $g(n, \tau)$: $m_1 = \langle n \rangle$ $(n > 0)$ and $m_2 = \langle n^2 \rangle = 2E$. Notice that for initial $g(n, 0) = \delta(n)$ the solution is symmetric with respect to $n = 0$, and we can consider $n > 0$ only. The equations for the moments are derived from Eq. (5.17):

$$\dot{m}_1 = \frac{1}{2} D_n g(0, \tau) + B; \qquad \dot{m}_2 = D_n + 2m_1 B. \tag{5.20}$$

Here $B = -1$ but we keep it in the expressions for further analysis. The second equation shows that one should distinguish the rate of energy variation from the diffusion rate just because of the backscattering.

The quantity $g(0, \tau)$ in the first equation, called *staying probability*, is of independent interest as a characteristic of the relaxation process.

In our case Eqs (5.20) describe the evolution of initially spreading Gaussian distribution into the final exponential steady state (5.9). Accidentally, the ratio of moments

$$\frac{m_1^2}{m_2} \equiv \frac{\gamma^2}{2} \approx 0.5 \tag{5.21}$$

remains almost constant which allows for a simple solution

$$-t = \xi + \ln(1 - \xi); \qquad \xi(0) = 0. \tag{5.22}$$

Here the new variable and time are

$$\xi = \frac{2\gamma}{D} E^{1/2}; \qquad t = \frac{\gamma^2}{D_n}\tau. \tag{5.23}$$

Initially, as $\tau \to 0$ Eq. (5.22) describes the classical diffusion ($\xi^2 \approx 2t$, $E \approx D_n\tau/2$) independent of γ. For constant D_n and B the relaxation $\xi \to 1$ is exponential

$$\xi \approx 1 - e^{-t}; \qquad t \to \infty. \tag{5.24}$$

To explain a power law relaxation, observed numerically in Refs [36 – 38], one needs to take account of the explicit time dependence for both $D_n(\tau) = D_n s(\tau)$ and $B(\tau) = -s(\tau)$. Notice that their ratio must be independent of time, at least asymptotically, to provide the exponential steady state (see Eq. (5.18)).

The solution for the moments (see Eq. (5.20)) can be obtained by a change of time:

$$\tau \to \tau' = \int s(\tau)\, d\tau.$$

Then, Eq. (5.22) shows that a power-law tail is only possible for $s(\tau) \sim \tau^{-1}$. This is in accord with the first simple extimate (5.11). Assuming

$$\tau' = \tau_R \ln\left(1 + \frac{\tau}{\tau_R}\right) \tag{5.25}$$

we arrive at the following implicit dependence $E(\tau)$:

$$e^{\xi}(1-\xi)\left(1+\frac{\tau}{\tau_R}\right)^{p} = 1\,; \qquad p = \frac{\gamma^2 \tau_R}{D_n} = 1\,. \tag{5.26}$$

The value of exponent p is obtained from asymptotic relations ($\tau \to \infty$): $\xi(\tau) \sim \tau^{-p} \sim s(\tau) \sim \tau^{-1}$. Hence, the relaxation time scale is

$$\tau_R = \frac{D_n}{\gamma^2} \approx 2D_n\,. \tag{5.27}$$

The value of $\gamma^2 \approx 0.5$ was derived from the best numerical data available [38]. It is only a half of the theoretical value (5.21). Besides, Eq. (5.26) does not contain the logarithmic dependence like Eq. (5.13) [38]. The latter seems better to agree with the numerical data for large τ. The origin of these discrepancies will be discussed in Section 7.

From the first Eq. (5.20) we can derive also implicit dependence of the staying probability on time:

$$g(0,\tau) - \frac{2}{D_n} = \frac{e^{-\xi(\tau)}}{(\tau_R + \tau)\,\xi(\tau)} \to \frac{1}{e\,\tau}\,. \tag{5.28}$$

The latter asymptotic expression is in agreement with numerical data in Ref. [35] as to the time dependence but exceeds the former by a factor of 3. The origin of this discrepancy is unknown as yet.

In many-dimensional systems or for a quasi-periodic driving perturbation the diffusion localization is typically absent beside some special cases (see, e.g. Ref. [39]). For three freedoms or two driving frequencies the localization persists but its length is exponentially large. However, the perturbative localization, mentioned in Section 3, occurs in all cases of discrete spectrum.

On the other hand, even in the lowest dimensions under consideration the so-called delocalization is possible if the motion is allowed to be unbonded. Consider, for example, the standard map on a cylinder with the perturbation $k(n)$ depending on momentum:

$$D(n) = D_0\, n^{2\alpha} \tag{5.29}$$

with some constant α. To solve this problem it is essential to assume that the backscattering remains unchanged, that is $B=-1$ as before, since it does not depend on system's parameters. Then, using Eq. (5.18), we obtain the steady state distribution in the form

$$\ln g_s(n) = \begin{cases} -\dfrac{2n^{1-2\alpha}}{(1-2\alpha)\,D_0}\,; & \alpha \neq \dfrac{1}{2} \\ -\dfrac{2}{D_0}\,; & \alpha = \dfrac{1}{2} \end{cases} \tag{5.30}$$

In agreement with previous results [40] the critical value of the parameter is $\alpha_c = 1/2$. For $\alpha < \alpha_c$ the localization remains exponential while for $\alpha > \alpha_c$ delocalization occurs becausb $g_s(n) \to \text{const} \neq 0$ as $n \to \infty$. In the critical case the steady state distribution is a power law:

$$g_s \sim n^{-2/D_0} \tag{5.31}$$

and the localization takes place for sufficiently small $D_0 < 2$ only, when $g_s(n)$ is normalizable.

Notice that for the localization of energy, that is for the mean energy $\langle E\rangle = \langle n^2\rangle/2$ to be finite in the steady state, a more strong condition is required, namely

$$D_0 < \frac{2}{3}\,. \tag{5.32}$$

This result was recently confirmed numerically in Ref. [41].

In spite of all this theoretical development no rigorous treatment of the quantum relaxation exists so far.

The change of time is a serious problem in quantum mechanics as was explained in Section 2. For a steady-state this problem can be solved [9] as follows. The steady-state distribution is proportional to invariant measure and, hence, to (sojourn) time t. Whence, upon a change of time $t \to \tilde{t}$

$$\frac{\tilde{g}_s(n)}{g_s(n)} = \frac{d\tilde{t}}{dt} \tag{5.33}$$

even though g_s does not depend on time! Now we can change momentum n in such a way to provide $\tilde{\tilde{g}}_s(\tilde{n}) = g_s(n)$. We have

$$\frac{d\tilde{n}}{dn} = \frac{d\tilde{t}}{dt}. \tag{5.34}$$

Particularly, if $\tilde{t} = \tau$, the map time (the number of map's iterations)

$$\frac{\tilde{g}_s}{g_s} = \frac{1}{T} = \frac{d\tilde{n}}{dn}, \tag{5.35}$$

where $T = 2\pi/\Omega(n)$ is map's period. If, moreover, n is action, the map's momentum $\tilde{n} = E/2\pi$ is proportional to the energy (cf. Kepler map (2.7)).

6. QUANTUM CHAOS AND ANDERSON LOCALIZATION

Now we consider in some detail a very fruitful analogy, mentioned already in Section 2, between the quantum localization of dynamical chaos in momentum space and the Anderson localization in configurational space of disordered solids. This analogy had been discovered in Ref. [21] and essentially developed in Ref. [22].

The analogy is based on the equation for eigenfunctions in both problems. Consider quantum map of the type (2.3) with Hamiltonian

$$\hat{H} = T\hat{H}_0(\hat{n}) + \hat{V}(\hat{\theta}). \tag{6.1}$$

For the standard map, for example, $\hat{H}_0 = \hat{n}^2/2$, and $\hat{V} = k\cos\hat{\theta}$. The eigenfunction are determined from the equation

$$e^{iT(\varepsilon - \hat{H}_0)}\varphi_\varepsilon = e^{i\hat{V}}\varphi_\varepsilon, \tag{6.2}$$

which also can be written in momentum representation as

$$\varphi_\varepsilon(n)\, e^{i\chi(n)} - \sum_{m\neq 0} W_m \varphi_\varepsilon(n-m) = W_o \varphi_\varepsilon(n), \tag{6.3}$$

where $\chi(n) = T\varepsilon - TH_0(n)$; ε is quasi-energy, and W_m are the Fourier amplitudes of $\exp(iV(\theta))$. For $V = k\cdot\cos\theta$, for instance, $W_m = i^m J_m(k)$.

Eq. (6.3) is essentially identical to that in a simple Lloyd's model of 1D lattice in solids:

$$v_E(n)\, U(n) - \sum_{m\neq 0} W_m v_E(n-m) = E\, v_E(n), \tag{6.4}$$

where now n is discrete coordinate, and E is energy. The difference is that in Eq (6.3) "potential" ($e^{i\chi}$) and "kinetic" (W_m) energy are complex functions but this can be removed by a transformation of φ.

Consider a case when only two $W_{\pm 1} = -k$ are nonzero, and take

$$V(\theta) = 2\,\mathrm{arctg}\,(E - 2k\cos\theta), \tag{6.5}$$

where E is a parameter.

Then Eq. (6.3) is transformed exactly into Eq. (6.4):

$$v(n)\, tg\frac{\chi(n)}{2} + kv(n+1) + kv(n-1) = Ev(n), \tag{6.6}$$

where $v(n)$ are the Fourier amplitudes of function

$$v(\theta) = \varphi_\varepsilon^{\pm}(\theta)\, e^{\pm iV/2} \cos\frac{V}{2},$$

$$\varphi_\varepsilon^{-}(n) = e^{i\chi(n)}\, \varphi_\varepsilon^{+}(n); \qquad \varphi_\varepsilon^{+}(\theta)\, e^{iV/2} = \varphi_\varepsilon^{-}(\theta)\, e^{-iV/2}. \tag{6.7}$$

These relations determine the correspondence between the two problems which allows to share the concepts, methods and results in both fields.

Particularly, for random and homogeneously distributed phases $\chi(n)$ all the eigenfunctions are known to be localized (see, e.g. Ref. [42]). The localization length (for $l \gg 1$) is given by the expression

$$l = (4k^2 - E^2)^{1/2} = \frac{D_n}{2}, \tag{6.8}$$

where D_n is the diffusion rate in momentum n with perturbation (6.5) (cf. Eq. (5.10)).

In Ref. [21] the discovered analogy was used to prove the diffusion localization in dynamical problems like standard map. But this is not a proof, of course, since dynamical phases $\chi(n) = T(\varepsilon - n^2/2)$ are not random even though they are homogeneously distributed (mod 2π). On the contrary, numerical results on the localization in the standard map show that a random potential in solids is sufficient but not a necessary condition for the Anderson localization. Indeed, now is well established that the localization is possible in the quasi-periodic potential with only two spatial frequencies (see, e.g., Ref. [43]).

Thus, we can speak now about the Anderson localization in a regular potential, but not a periodic one, of course. In this field a number of unsolved problems remain, including the localization in the standard map.

In exactly periodic potential all eigenfunctions are delocalized (the so-called extended Bloch states), and the spectrum is continuous and has the band structure. Interestingly, analogous structure is also possible in the momentum space. For example, in the standard map it corresponds to any rational value of parameter $T/4\pi = p/q$. Then phases $\chi(n) = T\varepsilon - Tn^2/2$ (the

"potential" in solid-state problem) is of period q. This phenomenon had been discovered in Ref. [11] and we call it the *quantum resonance*. The corresponding theory was developed in Ref. [44]. More intricate dynamics for the standard map with special irrational $T/4\pi$ will be discussed in the next Section.

The analogy under consideration, being very fruitful, is nevertheless restricted since it concerns the correspondence between eigenstates only. The properties of motion in two problems, both dynamical and even statistical, are generally different. For example, even though the localization length in both cases is the same (cf. Eqs (6.8) and (5.10)), those for the steady state are already different: $l_s \approx 2l$ in momentum space (5.10), and $l_s = 4l$ in disordered solids (see Ref. [42]).

The most striking difference is in the absence of the diffusion stage of motion in 1D solids [45]. This is because the level density of the operative eigenfunctions (see Section 5)

$$\rho_0 \sim \frac{l\,dp}{dE} \sim \frac{l}{u}, \tag{6.9}$$

which is the localization (relaxation) time scale (5.3), is always of the order of time interval for a free spreading of the initial wave packet at a characteristic velocity u. In other words, the localization length l is always of the order of the free path for backscattering. On the contrary, in momentum space, for instance, in the above example (6.5) each scattering (one map's step) couples $\sim k^{1/2}$ unperturbed states, so that $\sim k \gg 1$ scatterings are required to reach the localization $l \sim k$.

Another (qualitative) explanation of this surprising difference is in that the density of quasi-energy levels for driven systems is always higher as compared to that of energy levels. The same is true for a conservative system of two freedoms as compared with the one-freedom motion in solids. Thus, the Anderson localization is the spreading, rather than diffusion, localization.

Interestingly, the asymptotic relaxation ($t \to \infty$) in solids [45] is the same as in the momentum space (5.13). Yet, the decay of the staying probability is different [45] (cf. Eq. (5.28))

$$g_s(0, t) \sim t^{-3}. \tag{6.10}$$

Nevertheless, the analogy in question remains very fruitful and extensively used in the studies of quantum chaos (see, e.g. Ref. [35]).

7. THE QUANTUM STEADY STATE

The quantum diffusion localization generally results in the formation of a peculiar steady state which has no classical counterpart. The statistical relaxation to this steady state is also surprising because the motion spectrum is discrete.

The ultimate origin of this steady state is localization of all the eigenfunctions. In a homogeneous systems like the standard map on a cylinder the localization is asymptotically exponential. This is because Eq. (6.3) for the eigenfunctions is linear whose behavior is described by the Lyapunov exponents in n. This is the most powerful method, borrowed from the solid-state physics, to numerically calculate localization length l [46]. In this method Eq. (6.3) is considered as the "motion" equations for a certain "dynamical" system in n as "time". Formally, it is an infinite-dimensional system but to a good accuracy one can neglect the contribution of Bessel functions $J_m(k)$ with $|m| > N \sim k$. Then the system is described by a canonical $2N$-dimensional map which has N positive Lyapunov exponents. Unlike the usual dynamical problem, where the largest exponent is most important, here the least one $\Lambda_{\min} = 1/l$ determines the asymptotic localization. The role of other exponents is still not clear.

However, a simple exponential dependence (5.8) is only the average behavior superimposed by big fluctuations:

$$\varphi(n) \approx \frac{1}{l^{1/2}} \exp\left(-\frac{|n|}{l} + \xi_n\right). \tag{7.1}$$

By definition $\langle \xi_n \rangle = 0$ while the dispersion not only big but grows with $|\Delta n|$ as [22]

$$\langle (\xi_n - \xi_m)^2 \rangle = D_\xi |n - m|; \qquad D_\xi \approx \frac{1}{l} \approx \frac{2}{D_n}. \tag{7.2}$$

Nevertheless, the accuracy of numerical determination of l can be fairly high:

$$\frac{\Delta l}{l} \approx \left(\frac{l}{n}\right)^{1/2}. \tag{7.3}$$

Fluctuations ξ_n have a big impact on the steady state as was already mentioned in Section 5. Namely, they double the localization length (5.10). This is essentially numerical result, no accurate theory still exists [40]. Also, it is not clear if the steady-state is purely exponential asymptotically or there is a power-law factor like in solids [42].

Initial part ($|n| \sim l$) of the distribution for both eigenstates as well as the steady state must deviate from a simple exponential dependence. Again, big fluctuations impede direct numerical measurements. Instead, two integral characteristics were studied. One is the average energy in the steady state. For exponential localization (5.9)

$$E_s = \frac{\langle n^2 \rangle}{2} = \frac{l_s^2}{4} = \frac{D_n^2}{4} \tag{7.4}$$

and it is in agreement with numerical results within a factor of 2. One difficulty is that E_s depends in a complicated way on the precise value of parameter T owing to the everywhere dense set of quantum resonances mentioned in the previous Section.

Another integral quantity—the entropy H—was introduced in Ref. [47] (see also Ref. [8]) as a different measure of quantum localization. The entropy localization length is defined as

$$l_H = e^H; \qquad H = -\sum_n |\varphi(n)|^2 \ln |\varphi(n)|^2. \tag{7.5}$$

For exponentially localized eigenfunctions

$$l_H = e\, l = \frac{e\, D_n}{2} \approx 1.4 D_n. \tag{7.6}$$

Numerically, $l_H \approx D_n$ that is less, partly due to fluctuations which decrease the entropy and l_H by a factor of 2. Again, deviations from exponential dependence are apparently present but they are not very big.

Just because l_H essentially depends on the main part of the distribution its fluctuations are much bigger as compared to those for l (7.3). Namely [48]:

$$\frac{\Delta l_H}{l_H} \approx 0.5. \tag{7.7}$$

Fluctuations of entropy H were numerically found [48] to be described quite well by a simple expression ($l_H \gg 1$):

$$\frac{dp}{dH} = \frac{a}{\pi \cdot ch\,[a(H - \bar{H})]} \tag{7.8}$$

with $\bar{H} = \ln l_H$, and $a \approx 3$. So far there is no idea as to the explanation of this distribution. For further discussion of the quantum entropy see Section 8.

There is another class of localized eigenfunctions which we call *Mott's states*. They were conjectured by Mott [49] in the context of Anderson localization and further studied in Refs [35, 38, 50, 51]. Mott's state is also called the *double-hump* state for its shape of two exponential peaks separated by distance L (in n). These states exist in pairs of symmetric and

antisymmetric superpositions of the two peaks. The mechanism of their formation can be qualitatively explained as follows. The exponential localization is the effect of resonant backscattering, that is the backscattering on a resonant harmonic of random (or sufficiently irregular) potential. Hence, the exponentially localized states are in a sense unperturbed ones. The perturbation (nonresonant potential) mix them. For close unperturbed states this increases still more the fluctuations. However, for distant states a new, double-hump, structure is formed. The principal parameter is the overlapping integral

$$v = \int_{-\infty}^{\infty} dn\, \varphi_1^0(n)\, \varphi_2^0(n) \approx e^{-L/l} \left(\frac{L}{l} + 1\right), \tag{7.9}$$

which determines the energy splitting in the pair: $\Delta\varepsilon \sim v$.

We studied numerically [52] the structure of Mott's states in the standard map assuming two versions of dependence $\Delta\varepsilon(L)$:

$$l_m \omega = A\, e^{-L/l_m}, \tag{7.10a}$$

$$l_m \omega = A \left(1 + \frac{L}{l_m}\right) e^{-L/l_m}, \tag{7.10b}$$

where $\omega = T\Delta\varepsilon/2\pi$, and A is a constant. The first dependence is usually accepted in literature, the second one is suggested by parameter (7.9). Our preliminary results seem to better confirm the second law with fitting parameters

$$A \approx 0.05; \qquad l_m \approx D_n \approx l_s \approx 2l. \tag{7.11}$$

The fitting to the first dependence gives a close l_m but larger $A \approx 0.15$.

In disordered solids the structure of Mott's states was directly calculated in Ref. [50] via the correlation functions. The result is of the form of Eq. (7.10a) with $A \approx 5$, and $l_m = l = l_s/4$ (cf. Eq. (7.11)).

The importance of Mott's states, for which they actually were sought, is a large matrix element

$$n_{12} = \int dn\, n\varphi_1(n)\, \varphi_2(n) \approx \frac{L}{2}. \tag{7.12}$$

The latter expression holds for $L \gg l_m$. The additional logarithmic dependence in the long-time relaxation (5.13) is explained just by the effect of Mott's states in low-frequency part of the spectrum [38].

The probability for a given unperturbed (exponential) eigenstate to form the Mott pair with $L > L_1$ can be estimated as

$$p_1 \approx 2\alpha \int_{L_1}^{\infty} \omega(L)\, dL = 2\alpha A\, e^{-L_1/l_m}, \tag{7.13a}$$

$$p_1 \approx 2\alpha A \left(2 + \frac{L_1}{l_m}\right) e^{-L_1/l_m} \tag{7.13b}$$

for two dependencies $\omega(L)$ in Eq. (7.10) respectively. In both cases $\alpha \approx 1.5$ according to our numerical experiments. The total probability $p_1(0) \ll 1$, and this explains why milti-hump states are very rare. We have found a few states which could be interpreted as distorted three-hump eigenfunctions.

In disordered solids $p_1(0) > 1$ but this is not necessarily a contradiction because Eqs (7.10, 13) are asymptotic. Nevertheless, it would be interesting to analyze the structure of Mott's states in more detail.

The time-averaged density $g_s(n)$ (5.9) determines a certain invariant measure of the quantum motion which is qualitatively different from the classical measures (microcannical, Gibbs' etc). One important distinction is in that it depends on initial conditions as the quantum steady state is localization of the spreading initial state. Moreover, if the width of initial state exceeds the localization length this dependence becomes even more complicated.

Another difference is in that the relaxation of initial state into the steady state is never as full as in the classical mechanics. For example, average quantities like energy $E_s = \langle n^2 \rangle / 2$ (7.4) oscillate, and can even come back, close to the initial value E_0 since the motion spectrum is discrete.

Does it make any physical sense to speak about statistical relaxation in discrete spectrum? In my opinion, it does. First, such Poincare's recurrences are extremely rare, and their time scale has nothing to do with the characteristic relaxation time scale t_R (4.4). Second, which is even more important, those recurrences are but large fluctuations characteristic for any statistical system.

The same occurs in classical mechanics—for trajectories, and this is the difference. In fact, the quantum density $g(n, \tau)$ plays an intermediate role between classical density (which would never come back for chaotic motion) and a classical chaotic trajectory with its Poincare's recurrences. Namely, the quantum density, which actually describes a single quantum system, represents, nevertheless, a finite statistical ensemble of $M \sim l_s$ systems. Hence, finite fluctuations in the quantum steady state. For example, the energy fluctuations

$$\frac{\Delta E_s}{E_s} \sim \frac{1}{(M)^{1/2}} \sim \frac{1}{k} \tag{7.14}$$

in a reasonable agreement with numerical experiments (see, e.g., Ref. [36]).

One can say also that the mixing, which is responsible for relaxation, is terminated by localization, so that the quantum mixing is only partial or the *finite-time mixin*.

The smooth (up to fluctuations) steady state (5.9) is formed only if localization length $l_s \gg 2\pi/T$, the period of standard map in n. In the opposite limit $l_s \ll 2\pi/T$ the quantum measure $g_s(n)$ reveals the classical resonance structure [40]. Since quantum diffusion requires both $K > 1$ (classical border) and $k > 1$ (quantum border) this regime is only possible near $K = 1$ where the diffusion rate in the chaotic component is very slow:

$$l_s \approx D_n \approx 0.3\,(\Delta K)^3 k^2 \sim \frac{k^2}{\tau_{cr}^3} \tag{7.15}$$

and where the resonance structure is critical with characteristic time scale τ_{cr} [40, 53]. The border between the two regimes is approximately at

$$l_s T \approx \frac{l_s}{k} \approx 1\,. \tag{7.16}$$

At the border $\tau_{cr} \sim k^{1/3}$ as was recently confirmed in Ref. [54].

For $l_s T \ll 1$ the localization length $\langle l_s \rangle$ averaged over the resonance structure is

$$\langle l_s \rangle \approx \frac{k}{3^{1/2}} \tag{7.17}$$

and interpolation between the two regimes is approximately destribed by the expression [9]

$$\langle l_s \rangle = \frac{l_s}{2} + \left(\frac{l_s^2}{4} + \frac{k^2}{3}\right)^{1/2}. \tag{7.18}$$

The quantum steady-state is only possible in discrete spectrum. The conditions for the latter in an unbounded quantum map remain unknown. For the standard map on cylinder the spectrum is continuous for the rational values of parameter $T/4\pi = p/q$ due to periodicity of this map in n. In Ref. [55] the continuous spectrum was proved to exist also for very special Liouville's (transcendental) $T/4\pi$ (see below) but if this condition is only a technical limitation remained unclear. This constitutes a very subtle mathematical problem. We shall try to discuss it using a semiempirical theory of the quantum resonance [9] (see Section 6) which leads to the expression

$$\langle n^2 \rangle \approx D_n \tau^2 \cdot \exp\left(-\frac{q}{\pi D_n}\right). \tag{7.19}$$

This is the asymptotic energy growth in quantum resonance with denominator $q \geq D_n$. A detuning $\varepsilon(q) = |T/4\pi - p/q|$ would stop the growth in time $\tau(\varepsilon)$ which we assume to satisfy the condition (see Eq. (2.4))

$$\varepsilon \tau n^2 = \nu\ 1\,. \tag{7.20}$$

Consider now irrational

$$\frac{T}{4\pi} = \frac{1}{m_1 + \dfrac{1}{m_2 + \ldots}} \equiv (m_1, \ldots, m_i, \ldots)\,, \tag{7.21}$$

$$\frac{p_i}{q_i} = (m_1, \ldots, m_i) \to \frac{T}{4\pi}; \qquad q_{i+1} = m_{i+1} q_i + q_{i-1}\,,$$

where p_i/q_i are the convergents of $T/4\pi$. Comparing Eqs (7.19—7.21) we can formulate the following conjecture: there exist infinitely many irrational values of $T/4\pi$ which provide unbouned energy growth and, hence, a continuous spectrum; moreover, $T/4\pi$ can be adjusted in such a way to achieve the desired growth rate. Take the growth law in the form

$$\langle n^2 \rangle = G \tau^{\gamma}. \tag{7.22}$$

Substituting this into Eqs (7.20, 7.21) and excluding τ, we arrive at the relation

$$\varepsilon(q_i) = \frac{\nu}{G}\left(\frac{D_n}{G}\right)^{\frac{1+\gamma}{2-\gamma}} \exp\left(-\frac{q_i}{\pi D_n}\,\frac{1+\gamma}{2-\gamma}\right) \approx \frac{c}{q_i q_{i+1}}\,, \tag{7.23}$$

where the latter expression ($c \sim 1$) follows from the continuous fraction representation (7.21) of $T/4\pi$. This relation determines a map for the construction of desired $T/4\pi$:

$$m_{i+1} \approx \frac{q_{i+1}}{q_i} \approx \frac{cG}{\nu q_i^2}\left(\frac{G}{D_n}\right)^{\frac{1+\gamma}{2-\gamma}} \exp\left(\frac{q_i}{\pi D_n}\,\frac{1+\gamma}{2-\gamma}\right). \tag{7.24}$$

Successive convergents determine the quantum resonances which operate in turn, each one on its own time scale

$$\tau_i = \left(\frac{\nu}{cG} q_i q_{i+1}\right)^{\frac{1}{\gamma+1}} = \left(\frac{G}{D_n}\right)^{\frac{1}{2-\gamma}} \exp\left(\frac{q_i}{\pi D_n (2-\gamma)}\right) . \tag{7.25}$$

Since these time scales rapidly increase the diffusion is inhomogeneous in time, its local rate $\Gamma \equiv d\langle n^2\rangle / d\tau$ oscillating from about zero up to

$$\Gamma_{\max}(\tau_i) = 2D_n \left(\frac{G}{D_n}\right)^{\frac{1}{2-\gamma}} \exp\left(\frac{q_i}{\pi D_n} \frac{\gamma-1}{2-\gamma}\right) . \tag{7.26}$$

The ratio

$$\frac{\Gamma_{\max}(\tau_i)}{\langle\Gamma\rangle} = \frac{2}{\gamma} \geq 1 ; \qquad \langle\Gamma\rangle = \gamma G \tau^{\gamma-1} , \tag{7.27}$$

where $\langle\Gamma\rangle$ is the mean rate from Eq. (7.22).

For maximal $\gamma = 2$ a single resonance operates according to Eq. (7.19). In the whole interval $0 < \gamma \leq 2$ the motion is unbounded, and the spectrum is (singular) continuous with a fractal structure in agreement with the rigorous results in Ref. [55]. Irrationals which are approximated by rationals to exponential accuracy, like those satisfying Eq. (7.23), are called transcendental numbers. A new conjecture is that even among those $T/4\pi$ values there are (infinitely many) such ones that provide the diffusion localization. They correspond, particularly, to $\gamma = 0$ with any finite G. A more general condition is that asymptotically

$$m_{i+1} < \exp(aq_i) ; \qquad a < \frac{1}{2\pi D_n} . \tag{7.28}$$

For a particular value of $T/4\pi$ satisfying this condition the energy $E_s = G/2$ of the quantum steady state is determined by maximal G_i found from Eq. (7.23)

$$G_i^{3/2} = \frac{\nu}{c} D_n^{1/2} m_{i+1} q_i^2 \exp\left(-\frac{q_i}{2\pi D_n}\right) . \tag{7.29}$$

f this $G_{\max} < D_n^2/2$ (7.4) the resonances are irrelevant and usual exponential steady state is formed described by Eqs (5.9) and (7.4). This is just the case for a typical irrational $T/4\pi$ when $G_{\max} \sim D_n^{1/2} << D_n$ if quasi-classical parameter k .>>1 is big enough.

8. QUANTUM CHAOS AND THE RANDOM MATRIX THEORY

The well developed random matrix theory (RMT) (see, e.g. review [56]) is a statistical theory which describes average properties of a "typical" quantum system. At the beginning, the object of this theory was assumed to be a very complicated, particularly, many-dimensional quantum system as the representative of a certain statistical ensemble. With understanding of the phenomenon of dynamical chaos it became clear that the number of system's freedoms is irrelevant. Instead, the number of quantum states, or the quasi-classical parameter, is of importance (for further discussion see Section 9).

Until recently the ergodicity of eigenfunctions, that is the absence of any operators commuting with Hamiltonian, was assumed. Of course, this is not always the case (for a very interesting and instructive review of first attempts to prove the quantum ergodicity, see Ref. [57]). One of a few rigorous results in quantum chaos is an old theorem due to Shnirelman (announced in Ref. [58] with a full proof published only now [59]). Loosely speaking the theorem states that the classical ergodicity implies the ergodicity of most quantum eigenfunctions sufficiently far in the quasi-classical region that is for sufficiently large quantum parameters. The quantum ergodicity was further discussed in Refs [61] and well confirmed in numerical experiments with quantum billiards [18].

Shnirelman's definition of quantum ergodisity is of an integral type:

$$\int dp\, dq\, W_n(p, q)\, f(p, q) \underset{n\to\infty}{\to} \int dp\, dq\, g_\mu(p, q)\, f(p, q) \tag{8.1}$$

for any sufficiently smooth function f of the phase space. Here W_n are Wigner eigenfunctions, and

$$g_\mu = \delta(H(p, q) - E)\, \frac{dE}{dp\, dq} \tag{8.2}$$

is microcanonical (ergodic) measure. The quantity $\rho(E) = dpdq/dE$ is the classical counterpart of the mean level density.

To understand the quantum limitations of egrodicity and the importance of the quasi-classical asymptotics ($n \to \infty$) we consider as an example the Rydberg atom in magnetic field (see Eq. (2.12)).

In Ref. [63] the eigenfunctions of this model were found, for chaotic motion in the classical limit, in the form

$$\psi_i = C \sum_m \frac{\varphi_m}{(\Omega(m))^{1/2}}. \tag{8.3}$$

Here C is normalizing constant, φ_m are some unperturbed eigenfunctions with a fixed quantum number m, and

$$\Omega(m) = 2^{3/2}\left(\omega\left(m + \frac{1}{2}\right) - E\right)^{3/2} \approx \frac{1}{n^3} \tag{8.4}$$

is the electron longitudinal frequency depending on quantum numbers n, m. In the classical limit the ergodic measure

$$g_\mu(m) = \int dn\, \frac{\delta(H(m, n) - E)}{\rho(E)} = \frac{1}{\rho\Omega}, \tag{8.5}$$

where $\Omega = \partial H/\partial n$.

In quantum mechanics this measure in discrete, and to satisfy ergodicity (8.1) the change in Ω must be small, hence, $\omega \to 0$. On the other hand, classical ergodicity (chaos) takes place under condition [14]

$$\varepsilon = \frac{|E|}{\omega^{2/3}} \le 1. \tag{8.6}$$

Therefore, the condition for quantum ergodicity is

$$\omega \ll \varepsilon^3. \tag{8.7}$$

The RMT operates with finite matrices $N_m \times N_m$, so that expansion similar to Eq. (8.3)

$$\psi_i = \sum_j a_{ij}\, \varphi_j \tag{8.8}$$

is always finite, the ergodicity meaning that

$$\langle |a_{ij}|^2 \rangle = \frac{1}{N_m}. \tag{8.9}$$

In other words, all probabilities $|a_{ij}|^2$ are equal at average. This is not the case in a physical system whose energy shell, corresponding to the classical energy surface, is bounded. Hence, the conventional RMT is a *local theory* applicable within a quantum energy shell. We will come back to this important question below.

Statistical properties of quasi-energy eigenstates (for driven systems) were first studied in Refs [64, 65] (see also Ref. [8]) using, as a model, the standard map on a torus (see Eq. (2.2)). Owing to condition (2.2) the parameter $T/4\pi = m/2L$ is rational. But for a finite system, with L states, the spectrum is discrete of course, so that no delicate problems, discussed in Section 7, arise. This system models the quantum dynamics in the energy shell of a two-freedom conservative system.

The ergodicity depends on the parameter mentioned already in Section 5

$$\lambda = \frac{l}{L} = \frac{D_n}{2L} \tag{8.10}$$

and corresponds to large values of the latter. In the quasi-classical region $\lambda \sim \frac{K}{m} k \to \infty$ as $k \to \infty$ ($K = kT$ and $m = LT/2\pi$ remain constant). Thus, sufficiently high quantum states are ergodic in accordance with the Shnirelman theorem.

The structure of ergodic eigenstates well agrees with the prediction of RMT, namely, the fluctuations are nearly Gaussian with the probability density

$$\rho(a) = \frac{\Gamma(N_m/2)}{\pi^{1/2}\Gamma\left(\frac{N_m-1}{2}\right)} (1-a^2)^{\frac{N_m-3}{2}} \approx \left(\frac{N_m}{2\pi}\right)^{1/2} \exp\left(-\frac{N_m a^2}{2}\right). \tag{8.11}$$

Here a, assumed to be real stand for amplitudes in expansion (8.8). Interestingly, a slight difference between the two distributions was clearly observed in Ref. [65] for $N_m = 25$.

Big spatial fluctuations in a chaotic eigenstate are not completely random but reveal the structure of classical periodic trajectories. This interesting phenomenon had been discovered by Heller in numerical experiments with the quantum stadium billiard [66], and was subsequently confirmed by many others (see, e.g., Ref. [67]), particularly, in quantum maps. The microstructure was observed so far as some enhancements along classical periodic trajectories in both configurational and phase space. Such enhancements were termed "*scars*" by Heller.

A general theory of scars in conservative systems with arbitrary number of freedoms N was developed by Berry [68, 5] (see also Ref. [69]). He made use of the Wigner function W which is the quantum counterpart of the classical fine-grained phase space density. Notice that W is generally not positively definite.

Within a scar W forms complicated diffraction fringes, rapidly oscillating and rather extended along the energy surface. The relative width of the central fringe contracts with the quantum number n as $\sim n^{-1/2}$. In this sence the scars have essentially quantum structure which vanishes in quasi-classical region. Yet, this transition to the classical limit is not a trivial one as the fringe amplitude does not depend on n. To get rid of scars one needs a coarse-grained (averaged) density $\overline{W}$ which is called also the Husimi distribution, and which is positively definite. Then average density of a scar vanishes $\sim n^{-(N-1)}$.

As the scars are maximally localized (essentially within one quantum cell of the phase space) they do not violate Shnirelman's integral ergodicity (8.1). However, it is not completely clear why they are not seen in fluctuations of eigenfunctions.

According to Berry's theory the Wigner chaotic eigenfunction can be approximately represented as a sum over classical periodic trajectories:

$$W(x) \approx \frac{dE}{dx} \delta(E - H(x)) \times$$

$$\times \left[1 + \nu \sum_s \exp\left(-\frac{N-1}{2} \overline{\Lambda}_s T_s\right) \cos\left(S_s + \gamma_s\right) \delta(X_s)\right]. \tag{8.12}$$

Here $x = (p, q)$ is a point in $2N$-dimensional phase space while $X = (P, Q)$ describe $2(N-1)$-dimensional Poincare section transverse to a periodic trajectory at $X = 0$. The periodic trajectory is characterized by action S and quasi-classical phase as well as by instability rate $\overline{\Lambda}$ and period T. Each term in sum (8.12) represents a scar which, by the way, can be of any sign, that is it may produce both a bump or a dip in phase density W. Explicit expression for $\delta(X)$ is given in Refs [68, 5], and ν is some numerical factor.

A difficult mathematical problem in this theory is apparent strong divergence of series (8.12) since the number of periodic trajectories with $T_s < T$ grows as $\exp((N-1)\overline{\Lambda}T)$ (see, e.g., Ref. [70]). One way to approach this problem is as follows [62]. Let us try to consider Eq. (8.12) as an expansion in the basis of certain "coherent" states, the "scars"

$$W_s = \frac{1}{T_s}\,\delta\,(X_s)\,\delta\,(E - H\,(x))\,; \qquad \int W_s\,dx = 1\,, \tag{8.13}$$

which are localized on periodic trajectories. A peculiar property of such coherent states is in that they are stationary that is they don't move in phase space, nor they are spreading. The mechanism of localization is essentially the same as for the diffusion discussed in Section 5, but now it concerns the exponential spreading of a narrow wave packet prior to diffusion. The difference is in the level density which, for a scar, is $\rho_s \sim T_s$. Hence, the time scale for the localization of instability is T_s, and this explains the exponential factor in expression (8.12).

Oscillating $\delta(X)$ tails of unknown length overlap to produce somehow the average ergodic (microcanonical) distribution $\sim\delta(E-H(x))$ (see Eq.(8.12)) as well as the Gaussian fluctuations discussed above. The total number of separated scars is $\sim n^{N-1}$. Since the number of periodic trajectories grows as $\exp((N-1)\overline{\Lambda}T)$ the longest period T_m of basis scars is given by the estimate

$$\overline{\Lambda}T_m \sim \ln n \tag{8.14}$$

and it coincides with the Berman—Zaslavsky time scale (4.1) (with $q \sim n$). This is the time interval for a wave packet spreading over the whole energy surface. The scars with longer periods $T_s \geq T_m$ are not separated from each other, that is even their central fringes do essentially overlap, hence they are crucially modified. As a crude approximation one can simply drop these higher terms which makes series (8.12) convergent. It is not excluded that this approach would provide some physical justification for a formal procedure of smoothing $\delta(E-H)$ [5].

Another characteristic statistical property of chaotic eigenstates is the distribution of their eigenvalues, the energies. Particularly, the spacings s between neighbouring levels are distributed, according to RMT, as

$$p\,(s) \approx A\,s^{\beta} e^{-Bs^2}, \tag{8.15}$$

where A, B are obtained from normalization and condition $\langle s\rangle = 1$.

In the old RMT the level repulsion parameter β could take 3 values only ($\beta = 1; 2; 4$) depending on system's symmetry. In Refs. [64, 8] this property was confirmed for ergodic quasi-energy eigenstates as well.

A new problem is the impact of localization on the statistical properties of chaotic eigenstates. It was first addressed in Ref. [47] to discover a new class of spacing statistics which is now called the *Izrailev distribution*:

$$p\,(s) \approx A\,s^{\beta}\cdot\exp\left(-\frac{\pi^2}{16}\,\beta\,s^2 - \left(B - \frac{\pi\beta}{4}\right)s\right), \tag{8.16}$$

where now β is a continuous parameter in the whole interval (0, 4). This semiempirical relation was found using Dyson's model of charged bars on a ring. In this model the parameter β, which is the inverse bar temperature, can take any value. Yet, for the level repulsion of ergodic eigenstates only 3 values, given above, make sense. Izrailev has found that the intermediate values describe localized eigenstates. The Izrailev distribution is also called *intermediate statistics* as contrasted to the *limiting statistics* (8.15) for ergodic states. This intermediate statistics should be distinguished from another one proposed in Ref. [71] to account for the lack of ergodicity in the classical limit. Earlier a few cases of big deviations of unknown nature from the limiting statistics (mainly in heavy nuclei) were described by a purely empirical *Brody's distribution* ($0 \leq \beta \leq 1$):

$$p\,(s) = A\,s^{\beta} e^{-B\,s^{1+\beta}}. \tag{8.17}$$

The next important step would be to relate parameter β in Eq. (8.16) to the localization length l or rather to the ergodicity parameter $\lambda = l/L$. Instead, Izrailev introduced a new ergodicity parameter

$$\beta_H = \exp\,(H - H_e) \approx \frac{2l_H}{L}. \tag{8.18}$$

Here H, l_H are the entropy of an eigenstate and corresponding length, respectively (7.5), and $H_e \approx \ln(L/2)$ is the entropy of ergodic state which is less than maximal ($\ln L$) owing to fluctuations (8.11). Surprisingly, the new parameter $\beta_H \approx \beta$ proved to be very close to the repulsion parameter β of intermediate statistics (8.16). Why this relation is so simple remains an open question.

Particularly, in case of strong localization ($\beta_H \ll 1$) the spacing distribution (8.16) approaches the Poisson law

$$p(s) = e^{-s}, \tag{8.19}$$

which originally was associated with the completely integrable systems and regular dynamics. Also, this limiting case shows that Eq. (8.16) is an approximation because clearly $p(0) \neq 0$ for sufficiently small β_H. At most, the residual level repulsion could be exponentially small.

In any event, this limit explains the absence of repulsion for Anderson localization in infinite disordered solids. Yet, in a finite sample the repulsion must appear which is also an interesting mathematical problem.

Notice, that Poisson distribution holds only for all levels. For the operative eigenstates (5.3), which determine the quantum dynamics, the repulsion reappears again. This is another difficult problem.

The level repulsion does not change the relaxation time scale (5.3) but may influence the relaxation tail (see, e.g., Eq. (5.12) and Ref. [35]). In this context an interesting question concerns the repulsion among specific Mott's states (7.10). For each pair of such states the repulsion is very strong in the sense that their spacing is bounded from below by overlapping integral (7.9). On the other hand, the total number of Mott's pairs increases as the spacing (ω) decreases owing to the growth of state's size L. Both effects seem to cancel, and the integral repulsion vanishes. Indeed, from Eqs (7.13a) and (7.10a) (both versions (*a*) and (*b*) are asymptotically equivalent), we have

$$p_1 = 2\alpha\, l_m\, \omega\,. \tag{8.20}$$

This is in apparent disagreement with numerical results in Ref. [35] where the level attraction was inferred from the asymptotic behavior of the staying probability (5.28). However, this conclusion is very sensitive to the exact relaxation law. On the other hand, our result is in agreement with another relaxation (5.13) observed in Ref. [38]. To conclude, this question certainly requires further studies.

Empirical dependence $\beta_H(\lambda)$ was found in Refs [47, 72, 73]. Parameter β_H was defined by Eq. (8.18) with the entropy averaged over all eigenstates. The dependence can be approximately described by two expressions

$$\beta_H \approx \begin{cases} \dfrac{4\lambda}{1+4\lambda}; & \lambda \leq 0.5 \\ 1 - \dfrac{1}{4\lambda^{1/2}}; & \lambda \geq 0.1 \end{cases} \tag{8.21}$$

Besides the limit $\lambda \to 0$ there is no explanation of this dependence so far, nor even the physical mechanism underlying Eq. (8.21) has been identified.

For example, we could use a simple Eq. (5.8) for localized eigenstates. On a torus it becomes

$$\varphi_m(n) \approx \left(\frac{2\lambda}{1+\lambda\,\mathrm{sh}\,(1/\lambda)}\right)^{1/2} \mathrm{ch}\left(\frac{|m-n|}{l}\right) \tag{8.22}$$

and Izrailev's ergodicity parameter is

$$\beta_H \approx \begin{cases} 2e\,\lambda\left(1+\dfrac{e^{-1/\lambda}}{\lambda}\right); & \lambda \ll 1 \\ 1 - \dfrac{1}{360\lambda^4}; & \lambda \gg 1 \end{cases} \tag{8.23}$$

that is quite different from Eq. (8.21). Thus, the real dependence $\beta_H(\lambda)$ is related to deviations from simple eigenfunction shape (5.8). One conjecture is that it is the effect of larger Lyapunov's exponents in Eq. (6.3) for eigenfunctions in addition to the main (smallest) exponent $1/l$ which

determines the asymptotic behavior (5.8). The full set of Lyapunov's exponents was studied in Ref. [74].

Remarkably, dependence (8.21) has the nature of a scaling in the sense that β_H and $\beta \approx \beta_H$ depend on the ratio $\lambda = l/L = D_n/2L$ only, whatever the underlying mechanism could be.

The importance of this scaling is in that both quantities, β and λ, are invariant with respect to the rotation of the basis in Hilbert space whereas intermediate quantities, β_H and H, are not.

The statistical counterpart to the theory of quantum localization is not only old Anderson's theory but also a new development in RMT which makes use of the so-called band random matrices (BRM, see, e.g., Refs [75]). These have nonzero random elements within a band of some width $2b$ along the main diagonal only. They are defined in a certain physically significant basis, and also are not invariant under basis rotation.

The unitary matrix in quantized standard map (2.3) is also of a band structure with $b \approx k$ but nonrandom elements. This similarity suggests that appropriate scaling parameter would be [76]

$$\lambda_r = r\frac{b^2}{L}, \tag{8.24}$$

where L is now the matrix size, and r some numerical factor. All matrix elements are assumed to have the same statistical properties. Indeed, the scaling $\beta_H(\lambda_r)$ is similar but not identical to that for the dynamical problem (8.21). In fact, the first dependence is the same for $r \approx 1.5$, and it persists even farther, up to $\lambda_r \approx 3$. The second region ($\beta_H \approx 1$) is apparently different but it has not yet been studied enough. Notice that the origin of the difference can be attributed not only to the distinction between random and regular matrix elements but also to the different boundary conditions for a square matrix and for a torus.

For $\lambda_r \ll 1$ the matrix of eigenfunctions a_{ij} (8.8) is also a band matrix with a_{ij} smoothly decreasing off the diagonal but with a much larger effective width ($\sim b^2$).

In a conservative system parameter b^2 characterizes the width of the energy shell. Hence, the old RMT with its universal limiting statistics describes the local quantum structure only, that is for $L \ll b^2$. The global structure is associated with band matrices. The former approximation is very good, for example, in heavy nuclei ($b^2 \sim 10^6$) but not in heavy atoms ($b^2 \sim 10$ only) [77].

A new type of statistical properties for the quantum chaos has been introduced recently in the second Ref. [43]. It is the statistics of bands (or gaps) in the fractal spectrum of a particle in quasi-periodic critical potential. For a particular model the band "attraction" (or clusterring) was found with the parameter $\beta \approx -3/2$ (cf. Eq. (8.15)) in the limit of small gaps. The attraction parameter characterizes also the fractal dimensions of the spectrum $d_f = -\beta - 1 = 1/2$ in this model. Apparently, the same statistics can be applied to the nonresonant unbounded motion in the standard map (see Eqs (7.22)—(7.24)).

Also, I would like just to mention (and to attract attention to) a very interesting and less known theorem due to Shnirelman [78] (for the proof see Ref. [59]). It is related to the situation intermediate between the complete integrability with independent levels (see Eq. (8.19)) and quantum chaos with level repulsion (8.16). Namely, he assumes, in the classical limit, that the Kolmogorov—Arnold—Moser (KAM) theorem holds. We call this *KAM integrability* [79]. The KAM structure is highly intricate as its chaotic part, being of exponentially small measure, is everywhere dense.

In quantum mechanics the beautiful Shnirelman's theorem, which even doesn't need translation, asserts:

$$\forall N \; \exists C_N > 0\,, \; \forall n > 1 \; \min\left(\lambda_{n+1} - \lambda_n,\, \lambda_n - \lambda_{n-1}\right) < C_N n^{-N}, \tag{8.25}$$

where λ_n^2 are the energy eigenvalues. Thus, asymptotically, as $n \to \infty$, a half of level spacings are exponentially small. A striking difference from both the complete integrability and the quantum chaos!

9. QUANTUM CHAOS AND THERMODYNAMIC LIMIT

Dynamical chaos in classical mechanics seems to be a principally new mechanism underlying statistical laws in physics as compared to the traditional ("old") statistical mechanics (TSM). It is indeed! The only disadvantage is in that the classical chaos does not exist, strictly speaking, as our world is quantal. Now in quantum mechanics the chaos in waning, and becoming a sort of *pseudochaos* which only mimics some properties of the "true" chaos, and moreover, on finite time scales only. Besides, it turns out that such quantum chaos is rather similar in mechanism to TSM [80, 81].

Let us consider these complicated relations in some detail. The paradigm of TSM is many-dimensional *linear* oscillator which can be described by the matrix of coefficients in its quadratic Hamiltonian. So this is a completely integrable system with purely discrete spectrum. But the same is true for a broad class of quantum systems as described by Hamiltonian's or unitary matrices. In both cases the main *dynamical* problem is to diagonalize the matrix, and to find its eigenvalues and eigenvectors. The principal difference between the two problems is in the nature of matrix's size. In TSM it is the number of freedoms N while the quantum-dynamical counterpart is that of states n.

If any of these parameters is big the *statistical* discription becomes meaningful. In TSM it is achieved, in the formal theory, by taking the *thermodynamic limit* $N \to \infty$. Then, the spectrum becomes continuous *if the eigenfunctions* are *delocalized*. This is, indeed, the case, under certain conditions, and not only for the simple linear oscillator but also for a broad class of completely integrable systems (see, e.g., book [3] and references therein). Moreover, in the thermodynamic limit the completely integrable

system (for any finite N) becomes K-system, particularly, its KS-entropy $h > 0$. This is a very strong statistical property.

In quantum mechanics we have the classical limit $n \to \infty$ with its new dynamical chaos. Yet, the main problem in quantum chaos is finite (no matter now large) n. This quasi-classical region is characteristic for the quantum chaos. Actually, the same problem exists in TSM as well. What would be impact of finite N on the statistical properties? From the studies of quantum chaos we know that one still can speak about statistical relaxation in spite of the discrete spectrum. A striking example of such process was observed in old numerical experiments [82] with the completely integrable Toda lattice of 5 freedoms only!

Thus, the new quantum chaos turns out to be very similar in mechanism to the old TSM of completely integrable systems. Remember that TSM equally applicable to both classical and quantum systems under $N \to \infty$. The latter, if bounded, are always integrable in the Hilbert space. Classical systems can exhibit dynamical chaos for any $N \geq 2$. Yet, from the viewpoint of fundamental physics the classical chaos in but a *limiting pattern* to compare with the true (quantum) dynamics (for a discussion from the mathematical point of view see Ref. [83]). As we already know the main distinction of the quantum chaos is restriction of this limiting patterns in time. It reminds the conception of continuous phase space in classical mechanics. Notice that both limits (in time and space) greatly simplify theoretical analysis.

10. TRUE CHAOS IN QUANTUM MECHANICS?

Is the true chaos possible in quantum mechanics? The true means here the chaos like in the classical limit, particularly, on the infinite time interval. Also, remember that we are speaking about the proper quantum dynamics without intermediate measurements (Section 1). The answer is *yes* as was found recently but the examples of such a chaos are rather exotic. The first one was briefly mentioned in Ref. [84]. We consider here another example following

the second Ref. [30] (for a more physical example see Ref. [85] while some general considerations are presented in Ref. [86], and rigorous results see in Ref. [97]).

Consider the flow on an N-dimensional torus specified by the equations

$$\dot{\theta}_i = \nu_i(\theta_k) . \tag{10.1}$$

If $N \geq 3$ the classical chaos is possible with positive Lyapunov exponents that is the solution of the linearized equations is exponentially unstable. Consider now the Hamiltonian system

$$H(n, \theta) = \sum_k n_k \nu_k(\theta_i) \tag{10.2}$$

linear in momenta n_k canonically conjugated to coordinates θ_k. Then, the equations for n_k coincide (in reverse time) with the linearized equations (10.1). Hence, as soon as system (10.1) is chaotic the momenta of system (10.2) grow exponentially.

It is easily verified that the density $g(\theta, t) = |\psi(\theta, t)|^2$ of quantized system (10.2) obeys exactly the same (continuity) equation

$$\frac{\partial g}{\partial t} + \sum_k \frac{\partial}{\partial \theta_k}(g\nu_k) = 0 \tag{10.3}$$

as classical system (10.1) does with the same (particularly, chaotic) solution. The peculiarity of this and similar examples is in that to achieve the true chaos not only the quantum motion must be unbounded and, hence, of a continuous spectrum but the momenta have to grow exponentially in time.

Exotic though, these examples are important for the understanding of the nature of dynamical chaos. Particularly, it is not necessarily related to nonlinear equations. Also, the classical chaotic motion can be equivalently described in terms of the linear Liouville equation. On the other hand, the linear wave equations in classical mechanics exhibit typically the "quantum" chaos only, that is the discrete-spectrum chaos.

The true chaos is also possible in the so-called "semiclassical" systems. In those some freedoms are described classically (often implicitly like for the nonlinear Schrödinger equation, see Ref. [88]) which is, of course, some approximation. For the chaos to occur a single classical freedom is enough. An instructive example of N two-level atoms in a single-mode electromagnetic field is discussed in detail in Ref. [87] (see also Refs [88, 89]). The "semiclassical" approximation is very important for the problem of quantum measurement but this lies beyond the scope of the lecture.

11. CONCLUDING REMARKS

In conclusion I would like, first, to mention briefly a few other interesting developments in the theory of quantum chaos which are not presented in the main Sections above. One is the impact of external noise first studied for the standard map in Ref. [90] (for further developments see Refs [38, 91]). Typically, a sufficiently weak noise does not affect the classical-like diffusion on the relaxation time scale t_R (5.6). Yet, even arbitrarily weak noise destroys localization and provide a finite and permanent diffusion rate D_N where

$$\frac{D_N}{D_n} \sim \begin{cases} D_n \tilde{D}_n ; & D_n \tilde{D}_n \leq 1 \\ 1 ; & D_n \tilde{D}_n \geq 1 \end{cases} \tag{11.1}$$

Here $\tilde{D}_n$ is the diffusion rate under noise only. A sufficiently strong noise restores the permanent classical diffusion (for $1 / D_n \leq \tilde{D}_n \leq D_n$).

A more interesting effect, recently under intensive studies (see, e.g., paper [92] and references therein), is in that the noise of a special type substantially inhibits the quantum transitions preserving the initial state. The importance of this effect is in that it is close to the effect of quantum measurements but unlike the latter admits the dynamical description (cf. Ref. [93] on the quantum Zeno effect).

Another interesting topic is the classical dynamics in discretized phase space. A simple change of the standard map (2.1) to

$$\bar{n} = n + [k \cdot \sin\theta]; \quad \bar{\theta} = \theta + T\bar{n} \tag{11.2}$$

where square brackets denote the integer part, reproduces many quantum effects surprisingly well but, of course, not exactly [30, 9]. Another interesting model — the Arnold cat map — was studied in detail in Refs [94]. A particular quantization is exactly reproduced in the discrete version of this model but it is an exclusion rather than the rule.

The studies of such a discrete dynamics are of special importance for numerical experiments on digital computers where all the quantities (not only the action variables like in Eq. (11.2)) are discrete (integer). So, any dynamics in computer is a "quantal" one. Moreover, as all variables in computer are bounded no exclusions exist for the ban on the true chaos.

This field is rather unexplored, some related mathematical questions were discussed in Refs [95]. Since recently extensive studies in a close field of the so-called cellular automata, which appear to be essentially a particular type of the computer, are in progress (see, e.g., Ref. [96]).

In the very conclusion of this lecture I would like to emphasize again the importance of developing a new ergodic theory which instead of benefiting from the asymptotic approximation ($|t| \to \infty$) could analyze the finite-time statistical properties of dynamical systems.

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CONTENTS

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