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LARGE α^2-CORRECTION TO ORTHOPOSITRONIUM DECAY RATE

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НОВОСИБИРСК

Large α^2–Correction to Orthopositronium Decay Rate

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ABSTRACT

Correction $\sim \alpha^2$ to orthopositronium decay rate is calculated, which appears as square of the one–loop correction to the decay amplitude. Its relative value is equal to $28.8(2)(\alpha/\pi)^2$.

The measured value of orthopositronium decay rate is equal to [1]

$$\Gamma_{exp} = 7.0482(16)\mu s^{-1}.$$

At the same time, its theoretical value (including corrections $\sim (\alpha/\pi)$ and $\sim \alpha^2 \log \alpha$) is firmly established [2, 3, 4, 5, 6, 7]:

$$\Gamma_{th} = m\alpha^6 \frac{2(\pi^2-9)}{9\pi}\left[1 - 10.28\frac{\alpha}{\pi} - \frac{1}{3}\alpha^2 \log\frac{1}{\alpha}\right] = 7.03830\mu s^{-1}.$$

To explain the difference between them by $(\alpha/\pi)^2$–corrections, the factor at $(\alpha/\pi)^2$ should be as large as 250(40), which is very unusual for QED. To remove this discrepancy would be of importance for clear understanding of the relativistic bound–state problem in QED.

However, there exists at least one contribution of the second order with a large coefficient at $(\alpha/\pi)^2$; it arises when we square the one–loop correction to the annihilation amplitude. Naive estimate of this contribution is $26.4(\alpha/\pi)^2$ [7] (here factor 26.4 appears as $(10.28/2)^2$); it will be shown below that this estimate is in fact the lower bound for contribution under consideration. The subject of the present paper is to compute this contribution accurately.

To do this, the *amplitude* of annihilation has been computed, as distinct from all the previous works where the *width* was calculated. After obtaining this amplitude as a function of the polarizations and energies of photons and the polarization of positronium, we can get quadratic one–loop contribution to the width in a trivial way. It is convenient to use three–dimensional transverse polarizations. Since only limited precision is necessary, it is enough to calculate the amplitude at only a few frequences of photons. For checking of the amplitude obtained, it is convoluted with the zero–order amplitude; the result coincides with the first–order correction to the width, as it should be.

Feynman graphs, which contribute to the first order annihilation amplitude, are presented in Fig.1. In the previous works it was shown that graph

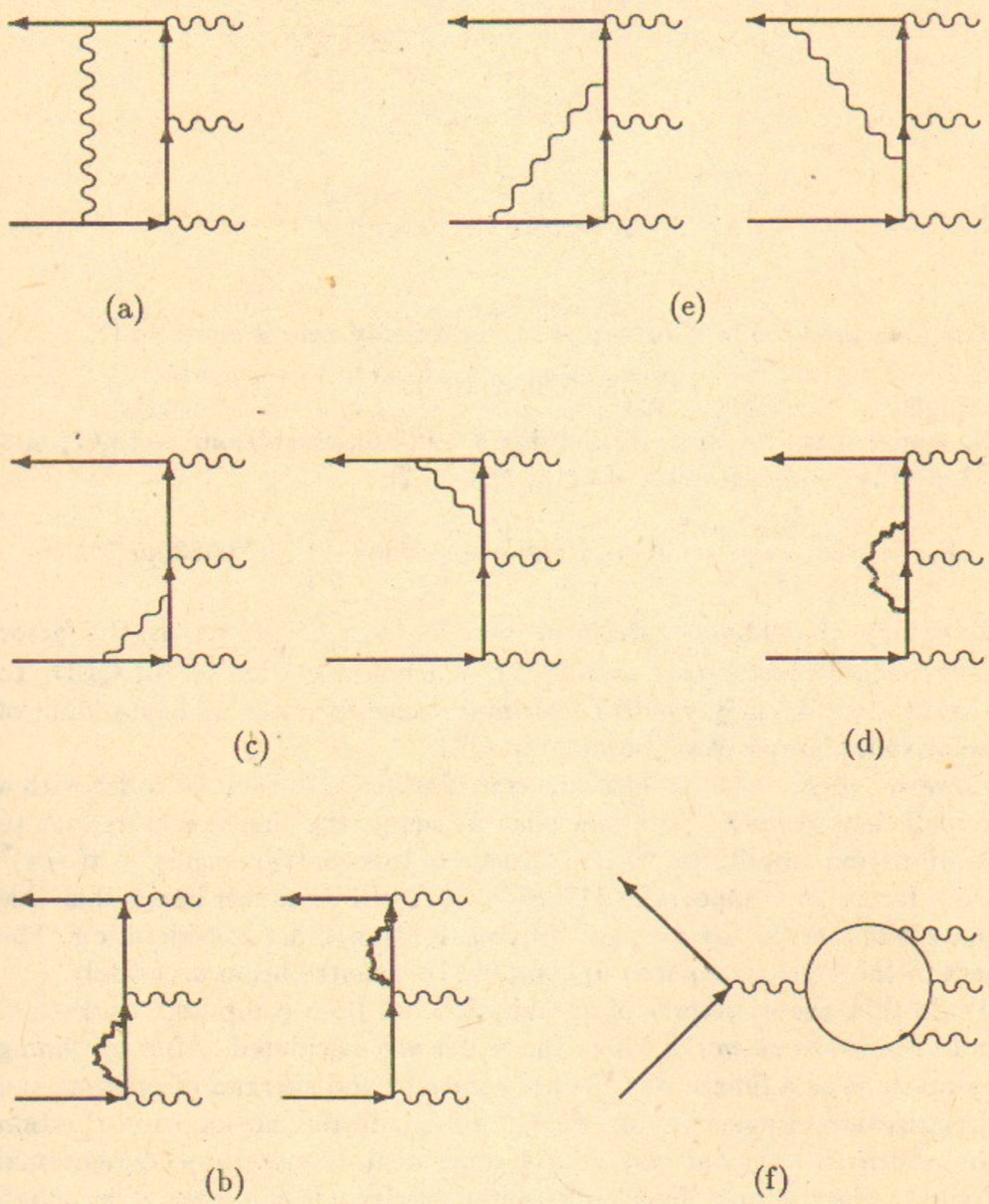

Fig. 1

(a) includes also zeroth order amplitude arising as term $\sim \alpha/v$. It is enough, to the first order in α, to evaluate all the graphs, except (a), for initial particles on mass shell and with zero momenta. Contribution of graph (a) is correctly included if we calculate this graph putting the external legs on mass shell (with non–zero momenta) and then convolute the result with the Coulomb wave function of the initial state, as was shown in [4]; the result of this calculation coincides with the result of [5], where relativistic bound state was described in the zeroth approximation by an exact solution of the Bethe–Salpeter equation with modified kernel [8].

Calculating the amplitude on mass shell we may choose an arbitrary gauge; in the present paper the Feynman gauge was used. In it infrared divergences arise in contributions of graphs *(a),(b),(c),(d)*; however, they easily may be separated analytically [3], and cancel each other in the sum of all the graphs. In [5] the one–loop correction to the width has been calculated in Coulomb gauge, for which infrared divergences do not appear at all in the problem under consideration.

Contribution of graph (b) has been evaluated using the known expression for one–loop mass operator [9]; that of the *(c),(d)* using expression for one–loop vertex as two–dimensional integral over Feynman parameters (numerical integration was carried out by Monte–Carlo procedure). Contribution of graph (e) has been found by taking residues in the zeroth component of loop momentum followed by the numerical integration over remaining space components. Graphs (a) and (f) have been evaluated by two methods: using Feynman parameters, and with direct momenta integration; the obtained numerical results are consistent. The term $\sim \alpha/v$ in the contribution of graph *(a)* has been separated in a way similar to that of [3]. The Pauli–Villars regularization was used when calculating contribution of graph *(f)*. When summation over polarizations of finite photons was carried out covariantly, we rederived contributions to the width, for each of the diagrams separately, obtained in [3].

The numerators of integrands in the Feynman integrals are known to be polynomials in the loop momenta. When the Feynman integrals are calculated in the usual way, these polynomials are found analytically, taking traces of products of γ–matrices and propagators. In this paper, however, analytical expressions for numerators are not used when integrals are evaluated by direct momenta integration. Instead, the Fortran program calculates coefficients at corresponding polynomials starting immediately from the γ–matrix structure of the numerator. While following Monte–Carlo integration, symmetrization of the integrand over substitutions $k_x \to -k_x$, $k_y \to -k_y$, $k_z \to -k_z$ was done to improve the convergence; the numerator polynomial was formed in

such a way that this symmetrization demanded only small extra computer time.

The amplitude obtained as a function of photons energies (their set we denote by ω), their polarizations, and polarizations of positronium(the set of polarizations is denoted by i), may be written in the following way :

$$M(\omega,i) = M_0(\omega,i)[f_0(\omega)(1-g(\alpha/\pi)) + f_1(\omega)(\alpha/\pi)] + (\alpha/\pi)M_2(\omega,i)f_2(\omega)\,, \tag{1}$$

where $f_0(\omega)$, $f_1(\omega)$ and polarization tensors $M_0(\omega,i)$, $M_2(\omega,i)$ are defined so that

$$\sum_i M_0(\omega,i)M_0(\omega,i) = 1, \qquad \sum_i M_2(\omega,i)M_2(\omega,i) = 1,$$
$$\sum_i M_0(\omega,i)M_2(\omega,i) = 0\,,$$
$$\int f_0(\omega)f_1(\omega)d\omega = 0\,, \qquad \text{and } \int f_0^2(\omega)d\omega = 1.$$

Then $g = 5.14$, and

$$\Gamma = \Gamma_0[(1-2g(\alpha/\pi)) + (\alpha/\pi)^2[g^2 + \int f_1^2(\omega)d\omega + \int f_2^2(\omega)d\omega]]\,. \tag{2}$$

Hence we see that $g^2 = 26.4$ is indeed the lower bound for the coefficient at $(\alpha/\pi)^2$ in the width. In fact, the result of the present work is the values for the second and third terms at $(\alpha/\pi)^2$ in (2) . They contribute to width $0.25(5)(\alpha/\pi)^2\Gamma_0$ and $2.15(15)(\alpha/\pi)^2\Gamma_0$ correspondingly. Although precision of calculating of the one–loop amplitude was not large (and corresponds to the error of $0.2(\alpha/\pi)$ in the width), we may employ value of g known with high precision from previous works [3, 5]. Then error in amplitude corresponds in fact to errors in f_1 and f_2. As f_1 and f_2 give to amplitude the contribution essentially smaller than that of g, the error in amplitude squared is as small as $0.2(\alpha/\pi)^2$.

Thus, the final result for considered contribution to the width is 28.8(2) $(\alpha/\pi)^2\Gamma_0$, that differs from naive estimate 26.4 by less then 10%.

We would like to note some fact concerning polarization dependence of the obtained correction to amplitude. This dependence for contributions of graphs *(a)* and *(f)* approximately coincides with the polarization dependence of the basic amplitude M_0 (i. e. these graphs contribute to g and f_1 in (1), but their contributions to f_2 is much less). The polarization structure of contributions of graphs *(b)*, *(c)* and *(d)* differs essentially from M_0, but this difference almost cancels in their sum. As to graph *(e)*, the situation is opposite: although after the three–dimensional transverse summation over polarizations of final photons its contribution to g and f_1 is rather small, f_2 originates mainly from this graph.

Table 1

ω_1	$\omega_a - \frac{\omega_b}{12}$					$\omega_a - \frac{\omega_b}{24}$	
$\omega_1 - \omega_2$	0	$\frac{3}{32}\omega_b$	$\frac{3}{16}\omega_b$	$\frac{9}{32}\omega_b$	$\frac{3}{8}\omega_b$	0	$\frac{\omega_b}{16}$
N	6	7	8	9	10	18	19
$F_1(\omega)$	−0.54	−0.03	0.28	0.42	0.44	−1.47	−0.89
$F_2(\omega)$	3.38	2.12	1.51	1.30	1.04	7.9	4.3

ω_1	$\omega_a - \frac{\omega_b}{4}$	$\omega_a - \frac{\omega_b}{6}$			ω_a				
$\omega_1 - \omega_2$	0	$\frac{\omega_b}{8}$	0	$\frac{\omega_b}{4}$	0	$\frac{1}{8}\omega_b$	$\frac{3}{16}\omega_b$	$\frac{1}{4}\omega_b$	$\frac{3}{8}\omega_b$	$\frac{1}{2}\omega_b$
N	1	2	3	4	5	13	14	15	16	17
$F_1(\omega)$	0.65	0.67	0.43	0.60	0.61	−2.35	−1.13	−0.75	−0.17	0.12
$F_2(\omega)$	0.93	0.88	1.35	1.09	0.89	7.2	4.0	2.85	1.67	1.72

ω_1	$\omega_a - \frac{\omega_b}{48}$						
$\omega_1 - \omega_2$	0	$\frac{\omega_b}{32}$	0	$\frac{1}{32}\omega_b$	$\frac{1}{16}\omega_b$		
N	20	21	11	22	12		
$F_1(\omega)$	−3.3	−2.1	−9.9	−5.9	−3.9		
$F_2(\omega)$	16	9.2	71	26	14		

Taking into account symmetry under the permutation of photons, it is enough to consider the frequences of photons, satisfying the condition $2-\omega_1-\omega_2=\omega_3<\omega_2<\omega_1$ (this region is presented in Fig. 2a; we refer to it as region C). Respectively, the differential width $\frac{d\Gamma}{d\omega_1 d\omega_2}$ is defined in such a way that the integral of it over region C is equal to the full width. This reads

$$\frac{d\Gamma}{d\omega_1 d\omega_2}=\Gamma_0\sum_i M^2(\omega,i)\,,$$

$$\int_C f_0^2 d\omega_1 d\omega_2=1\,.$$

Function $f_1(\omega)f_0(\omega)$ contributes to the differential width in order α, and $f_1^2(\omega)+f_2^2(\omega)$ in order α^2. In the Table 1 the values of functions $F_1(\omega)=$ $=f_1(\omega)f_0(\omega)/6$ and $F_2(\omega)=(f_1^2(\omega)+f_2^2(\omega))/12$ are given at 22 points, which are positioned in (ω_1,ω_2) — plane as represented in Fig.2b, 2c. Here N is the label of each of the points in the Fig.2b,2c. For $F_1(\omega)$ the result obtained is in agreement with the differential width of the first order, quoted in [5]. Below we denote $\omega_a=0.99m$, $\omega_b=0.97m$.

To illustrate obtained results for the amplitude, let us write down its values (for the set of independent polarizations) for few sets of energies ω_1, ω_2, ω_3.

We define the amplitude in the following way.

If we choose the polarizations of the final photons to be three–dimensionally transverse, then the amplitude at any point in the (ω_1,ω_2)–plane represents a set of 24 quantities. For each of the three photons we choose two independent polarization vectors so that one of them lies in the same plane as photons momenta, the other being transverse to this plane. Similarly, the three independent positronium polarizations we choose so that two of them are in the decay plane, the third is transverse to this plane. It may be easily shown that under this choice only 12 of 24 amplitudes differ from zero (those of them, for which *even* number of polarization vectors lies in the decay plane). If we widen energies change region to all sets of ω_1, ω_2, we obtain two independent amplitudes.

Explicitly, let us define $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ to be momenta of the photons 1,2,3, $\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3=0$, $\mathbf{e}_j^a$ to be their polarizations ($a=1,2$ refer to two independent polarizations for each of the photons, $j=1,2,3$ refer to different photons). Let us choose coordinate frame in such a way that $\mathbf{q}_1$ is directed

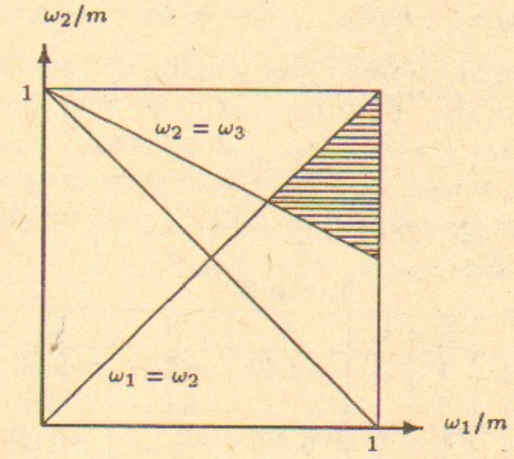

Fig.2a
(dashed region is C: $\omega_3<\omega_2<\omega_1$)

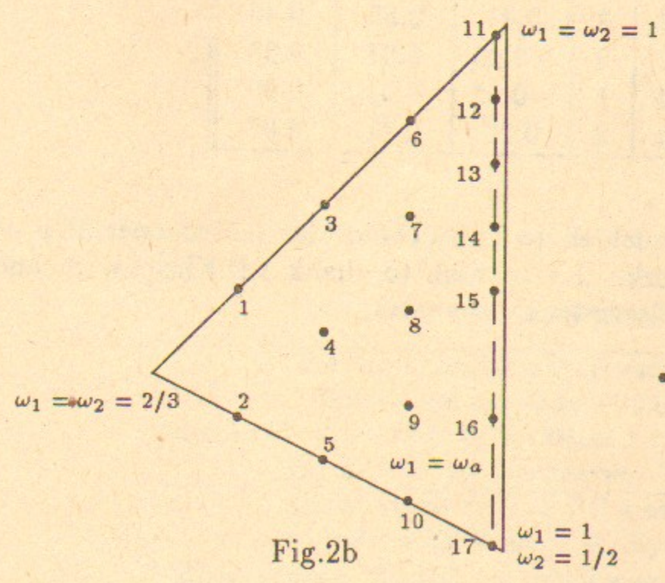

Fig.2b

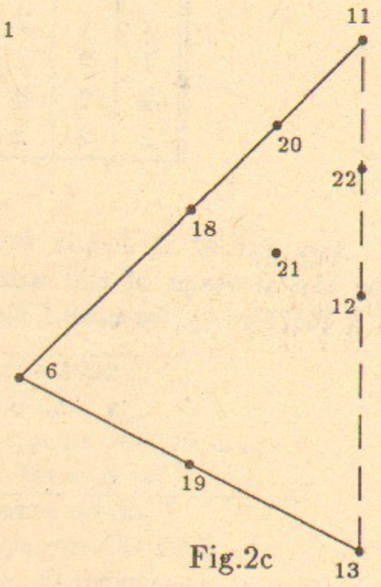

Fig.2c

along the axis x, $\mathbf{q}_2$ lies in (x,y)–plane, and $q_{2y} > 0$. We denote coordinate unit vectors as $\mathbf{n}_x$, $\mathbf{n}_y$, $\mathbf{n}_z$. Then, our choice of independent polarizations reads

$$\mathbf{e}_1^1 = \mathbf{e}_2^1 = \mathbf{e}_3^1 = \mathbf{n}_z\,, \quad \mathbf{e}_1^2 = -\mathbf{n}_y\,,$$

$$\mathbf{e}_2^2 = \mathbf{n}_x q_{2y} - \mathbf{n}_y q_{2x}\,, \quad \mathbf{e}_3^2 = \mathbf{n}_x q_{3y} - \mathbf{n}_y q_{3x}\,.$$

The independent polarizations of positronium, $\mathbf{e}_p^a$, are choosen merely as follows: $\mathbf{e}_p^1 = \mathbf{n}_x$, $\mathbf{e}_p^2 = \mathbf{n}_y$, $\mathbf{e}_p^3 = \mathbf{n}_z$. Let us to write down the values of $M_2(i)$ at three points: 1. $\omega_1 = \omega_2 = \omega_a$, 2. $\omega_1 = \omega_a$, $\omega_2 = \omega_a - \omega_b/4$, 3. $\omega_1 = \omega_a - \omega_b/6$, $\omega_2 = \omega_1 - \omega_b/8$ (in Fig.2b these are points 11,15, 4 respectively); these values are referred as A, B, C, and are given in Table 2.

Table 2

i_1	i_2	i_3	i_p	A	B	C
1	1	1	3	−0.42	−0.32	−0.53
1	1	2	1	0.15	0.01	0.00
1	1	2	2	0.00	−0.07	−0.09
1	2	1	1	−0.40	0.12	0.21
1	2	1	2	−0.24	−0.53	−0.28
2	1	1	1	0.41	0.31	0.48
2	1	1	2	0.41	0.13	0.23
2	2	1	3	0.23	−0.28	−0.07
2	1	2	3	0.41	0.55	0.48
1	2	2	3	−0.06	0.07	0.26
2	2	2	1	−0.15	0.01	0.00
2	2	2	2	0.00	0.31	0.07

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Большой вклад в поправку $\sim \alpha^2$ к ширине ортопозитрония

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