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THE 2+1-DIMENSIONAL INTEGRABLE GENERALIZATION OF THE SINE-GORDON EQUATION. II. LOCALIZED SOLUTIONS

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НОВОСИБИРСК

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ABSTRACT

The localized solutions for the 2+1-dimensional integrable generalization of the sine-Gordon equation-I (2DISG-I) are studied. General formula for the exact solutions for the 2DISG-I equation with nontrivial time-dependent boundaries is derived. The broad classes of exact solutions of the perturbed string and telegraph equations are obtained via $\bar{\partial}$-dressing method. Exact localized solutions of several types for the 2DISG-I equation are calculated using exact solutions of perturbed string and telegraph equations.

1. INTRODUCTION

Since the discovery in [1] the exponentially localized solutions of soliton type for the 2+1-dimensional and multidimensional nonlinear equations are studied very intensively [2-12]. Spectral theory of such localized solitons (dromions) for the Davey-Stewartson-I (DS-I) equation and their connection with the initial-boundary value problem for the DS-I equation have been studied by different methods in a series of papers [3-7]. Then exponentially localized solutions for the multidimensional nonlinear Schrodinger type equations were obtained with the use of some direct methods [8-10]. In the papers [11-12] the exponentially and rationally localized solutions for the Ishimori equation with nontrivially boundaries were constructed.

The present paper is the second from a series of papers devoted to the study of the coherent structures for the 2+1-dimensional integrable generalization of the sine-Gordon (2DISG) equation. In the part I of the paper it was shown that the 2DISG equation with nontrivial boundaries $u_1(\eta, t)$ and $u_2(\xi, t)$ has the form [14]:

$$\theta_{t\xi\eta} + u_1(\eta, t)\theta_\xi + u_2(\xi, t)\theta_\eta + \frac{1}{4}\theta_\eta\partial_\eta^{-1}(\theta_\xi\theta_\eta)_t +$$

$$+\frac{1}{4}\theta_{\xi}\partial_{\xi}^{-1}(\theta_{\xi}\theta_{\eta})_t = 0 , \qquad (1.1)$$

where $\xi=x+\sigma y$, $\eta = x - \sigma y$; $\sigma^2=1$ corresponds to the case of 2DISG-I equation and $\sigma^2=-1$ -to the case of 2DISG-II equation.

It was demonstrated in the paper-I [14] that the problem of constructing exact solutions for equation (1.1) is closely connected with the problem of the explicit solving the linear equation ofthe type:

$$\Psi_{zt}(z, t) + u(z, t)\Psi(z, t) = 0 , \qquad (1.2)$$

where $u = u_1(\eta, t)$ (or $u = u_2(\xi, t)$) is given function.

In the paper I the case of constant boundaries $u_1(\eta, t) = m_1$ and $u_2(\xi,t)=m_2$ for both types of 2DISG equations (1.1) has been studied in detail. By the dressing method based on the mixed nonlocal $\bar{\partial}$-∂-problem the solutions with functional parameters, line solitons (kinks) and line breathers have been constructed explicitly. The initial value problem for 2DISG equations has been solved.

In the present paper we study the general case of time-dependent boundaries $u_1(\eta, t)$ and $u_2(\xi, t)$ for the 2DISG-I equation. Using the exact solutions of equation (1.2), we construct the exact solutions of 2DISGequation.The localized and also nonlocalized solutions of several types are among these solutions. We present several explicit examples of such solutions.

The paper is organized as follows. In section 2 the principal results of the paper I are presented for convenience. In section 3 the general formula for the exact solutions of the 2DISG-I equation with nontrivial boundaries $u_1(\eta, t)$, $u_2(\xi, t)$ is derived.

Exact solutions $\Psi(z, t)$ with corresponding solvable potentials $u(z, t)$ of equation (1.2) in the case of nonzero

asymptotical values of $u(z,t)$ at infinity (the solutions of perturbed telegraph equation) are obtained in section 4 via nonlocal $\bar{\partial}$-problem. Then in section 5 with the help of these exact solutions of equation (1.2) for corresponding boundaries $u_1(\eta, t)$ and $u_2(\xi, t)$ the localized solutions of soliton and breather type of 2DISG-I equation are constructed.

In section 6 the exact solutions $\Psi(z, t)$ of equation. (1.2) with corresponding solvable potentials $u(z, t)$ in the case of generically zero asymptotical values of $u(z, t)$ at infinity (the solutions of perturbed string equation) via another, different from the used in section 4, nonlocal $\bar{\partial}$-problem are obtained. Then in section 7 with the use of these exact solutions of equation (1.2) for corresponding boundaries $u_1(\eta, t)$ and $u_2(\xi, t)$ the exact but nonlocalized solutions of 2DISG-I equation are constructed.

2. SOME RESULTS FROM THE PAPER I

Here for convenience we present the main results of the paper I. It was shown in the paper I that the 2+1-dimensional integrable generalization of the sine-Gordon equation (2DISG) (1.1) or equivalently the equation

$$\begin{gathered}\theta_{t\xi\eta} + \frac{1}{2}\theta_{\xi}\rho_{\eta} + \frac{1}{2}\theta_{\eta}\rho_{\xi} = 0 , \\ \rho_{\xi\eta} - \frac{1}{2}(\theta_{\xi}\theta_{\eta})_t = 0 ,\end{gathered} \qquad (2.1)$$

is equivalent to the compatibility condition for the following linear system:

$$L_1\Phi = \begin{pmatrix} \partial_{\xi}, & -\frac{i}{2}\theta_{\xi} \\ -\frac{i}{2}\theta_{\eta}, & \partial_{\eta} \end{pmatrix}\Phi = 0,\ L_2\Phi = \begin{pmatrix} \partial^2_{t\eta}+\frac{1}{2}\rho_{\eta}, & -\frac{i}{2}\theta_{\eta}\partial_t \\ -\frac{i}{2}\theta_{\xi}\partial_t, & \partial^2_{t\xi}+\frac{1}{2}\rho_{\xi} \end{pmatrix}\Phi = 0. \qquad (2.2)$$

The operator form of the compatibility condition for the linear system (2.2) looks like [14]:

$$[L_1, L_2] = A_1 L_1 + A_2 L_2 , \tag{2.3}$$

where

$$A_1 = \begin{pmatrix} 0 , & \frac{i}{2}[\theta_{\xi t} + (\theta_\xi + \theta_\eta)\partial_t] \\ \frac{i}{2}[\theta_{\eta t} + (\theta_\xi + \theta_t)\partial_t], & 0 \end{pmatrix} , \quad A_2 = -\frac{i}{2}(\theta_\xi + \theta_\eta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{2.4}$$

Eliminating the variable ρ from the system (2.1), one obtains the single equation (1.1) for θ:

$$\theta_{t\xi\eta} + u_1(\eta, t)\theta_\xi + u_2(\xi,t)\theta_\eta + \frac{1}{4}\theta_\eta \int_{-\infty}^{\eta} d\eta' (\theta_\xi \theta_{\eta'})_t +$$

$$+ \frac{1}{4}\theta_\xi \int_{-\infty}^{\xi} d\xi' (\theta_{\xi'} \theta_\eta)_t = 0 , \tag{2.5}$$

where

$$u_1(\eta,t) \overset{def}{=} \frac{1}{2} \lim_{\xi \to -\infty} \rho_\eta(\xi,\eta,t) ,$$

$$u_2(\xi,t) \overset{def}{=} \frac{1}{2} \lim_{\eta \to -\infty} \rho_\xi(\xi,\eta,t) . \tag{2.6}$$

So the solution of equation (2.5) with the fixed functions $u_1(\eta, t)$ and $u_2(\xi, t)$ gives the solution of the 2DISG equation (2.1) with the boundary values of ρ given by (2.6). The properties of the 2DISG equation, as it was noted in the paper I, essentially depend on the boundaries. In the case

$u_1 = u_2 = 0$ equation (2.5) is the dispersionless one with the linear part $\theta_{t\xi\eta} = 0$. In this case equation (2.5) possesses large symmetry group [14].

Similar to the DS and Ishimori equations the second auxiliary linear problem (2.2b) requires the modification at the presence of nontrivial boundaries $u_1(\eta, t)$ and $u_2(\xi, t)$. This modified linear problem (2.2b) is of the form [14]:

$$L_{2M}\Phi \overset{def}{=} \begin{pmatrix} \partial^2_{t\eta} + u_1(\eta,t) + \frac{1}{4}\partial_\eta^{-1}(\theta_\xi \theta_\eta)_t , & -\frac{i}{2}\theta_\eta \partial_t \\ -\frac{i}{2}\theta_\xi \partial_t , & \partial^2_{t\xi} + u_2(\xi,t) + \frac{1}{4}\partial_\xi^{-1}(\theta_\xi \theta_\eta)_t \end{pmatrix} \cdot \Phi +$$

$$+ \left(\partial_\xi + \partial_\eta - \frac{i}{2}(\theta_\xi + \theta_\eta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) . \tag{2.7}$$

$$\cdot \left(\int_R dl \frac{\hat{u}_1(\lambda-l,t)}{l} (\Phi \cdot \sigma_+)(l) + \int_R dl \frac{\hat{u}_2(\lambda-l,t)}{l} (\Phi \cdot \sigma_-)(l) \right) = 0,$$

where $\sigma_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\hat{u}_1$ and $\hat{u}_2$ are the Fourier-transforms ofthe boundaries $u_1(\eta, t)$and $u_2(\xi, t)$:

$$\hat{u}_1(\lambda-l,t) \overset{def}{=} \frac{i}{2\pi} \int_R d\eta \, u_1(\eta,t) \cdot e^{i\eta(\lambda-l)} ,$$

$$\hat{u}_2(\lambda-l,t) \overset{def}{=} \frac{-i}{2\pi} \int_R d\xi \, u_2(\xi,t) \cdot e^{-i\xi(\lambda-l)} . \tag{2.8}$$

The operator form of the compatibility condition is again (2.3).

The inverse problem data have been introduced in the paper I analogously to the case of Davey-Stewartson (DS) equation [5] by the formulae [14]:

$$S(\lambda,l) = \frac{1}{4\pi}\iint_{R^2} d\xi d\eta(-i\theta_\xi)\cdot\chi^-_{22}(\lambda)\cdot e^{-i\lambda\xi-il\eta} ,$$

$$T(\lambda,l) = \frac{1}{4\pi}\iint_{R^2} d\xi d\eta(-i\theta_\eta)\cdot\chi^-_{11}(\lambda)\cdot e^{i\lambda\eta+il\xi} , \tag{2.9}$$

where

$$\chi^\pm(\xi,\eta,\lambda) = \Phi^\pm(\xi,\eta)\begin{pmatrix} e^{-i\lambda\eta} & 0 \\ 0 & e^{i\lambda\xi}\end{pmatrix} , \tag{2.10}$$

are the solutions of the linear integral equations associated with the linear problem (2.2a).

The derivatives θ_ξ and θ_η are obtained from the following reconstruction formulae [14]:

$$\theta_\xi = \frac{i}{\pi}\iint_{R^2} d\lambda dl\ S(\lambda,l)\cdot e^{il\eta+i\lambda\xi}\cdot\chi^-_{11}(l) ,$$

$$\theta_\eta = \frac{i}{\pi}\iint_{R^2} d\lambda dl\ T(\lambda,l)\cdot e^{-il\xi-i\lambda\eta}\cdot\chi^-_{22}(l) . \tag{2.11}$$

The auxiliary linear spectral problem (2.2a) is the special reduction of the spectral problem for DS equation [5] and therefore the inverse problem data $S(\lambda,l)$ and $T(\lambda,l)$ are not independent, they can be expressed through a single function $\tilde{S}(\lambda,l)$ [14]:

$$S(\lambda,l) = \lambda\cdot\tilde{S}(\lambda,l),\ T(\lambda,l) = -\lambda\cdot\tilde{S}(-l,-\lambda) . \tag{2.12}$$

The reality condition on the field θ gives the restriction on the function $\tilde{S}(\lambda,l)$ [14]:

$$\overline{\tilde{S}(-\lambda,-l)} = \tilde{S}(\lambda,l) . \tag{2.13}$$

The key role in the study of the coherent structures for the 2DISG-I equation is played by linear differential equation for the Fourier-transform $\hat{S}(\xi, \eta, t)$ of the inverse problem data $\tilde{S}(\lambda,l,t)$ (2.13):

$$\hat{S}(\xi,\eta,t) \overset{def}{=} \frac{1}{2\pi}\iint_{R^2} d\lambda dl\ \tilde{S}(\lambda,l,t)\cdot e^{i\lambda\xi+il\eta} . \tag{2.14}$$

As it was shown in the paper I the function $\hat{S}(\xi,\eta,t)$ obeys the following linear partial differential equation [14]:

$$\hat{S}_{t\xi\eta} + u_1(\eta,t)\cdot\hat{S}_\xi + u_2(\xi,t)\cdot\hat{S}_\eta = 0 , \tag{2.15}$$

where the boundaries $u_1(\eta,t)$ and $u_2(\xi,t)$ play a role of the variable coefficients. From (2.13) and (2.14) one can obtain:

$$\hat{S}^*(\xi,\eta,t) = \hat{S}(\xi,\eta,t) . \tag{2.16}$$

Let us note that analogously to the cases of the DS [3, 5] and Ishimori [11, 12] equations equation (2.15) for the Fourier tran sform of the inverse problem data coincides with the linearized 2DISG-I equation.

3. EXACT FORMULAE FOR THE COHERENT STRUCTURES OF THE 2DISG-1 EQUATION

Our aim here is to derive the general formulae for the exact solutions of the 2DISG-I equation via the solutions of equation (2.15), where $S(\xi, \eta, t)$ must be real function. Solutions of eq. (2.15) can be constructed by the method of separation of variables

$$\hat{S}(\xi,\eta,t) = X(\xi,t)\cdot Y(\eta,t) , \qquad (3.1)$$

similar to the DS-I equation [3,5] and Ishimori equation [11, 12]. In our case the functions X and Y obey the equations:

$$X_{t\xi} + u_2(\xi,t)X = \mu X_\xi ,$$
$$Y_{t\eta} + u_1(\eta,t)Y = -\mu Y_\eta . \qquad (3.2)$$

The dependence on separation constant μ is trivial and can be eliminated by the simple gauge transformation of the functions X and Y: $X \to Xe^{\mu t}$, $Y \to Ye^{-\mu t}$. Therefore in the construction of exact soluti-ons of equation (1.1) we can use instead of eqs. (3.2) the equatons:

$$X_{t\xi} + u_2(\xi,t)X = 0 , \qquad (3.3)$$
$$Y_{t\eta} + u_1(\eta,t)Y = 0 . \qquad (3.4)$$

The general factorized solution of eq.(2.15) is of the form:

$$\hat{S}(\xi,\eta,t) = \sum_{i,j}\rho_{ij}X_i(\xi,t)Y_j(\eta,t) \equiv \sum_i X_i(\xi,t)\tilde{Y}_i(\eta,t) , \qquad (3.5)$$

where ρ_{ij} are constants and X_i, Y_j are the solutions of the eqs. (3.3) and (3.4) respectively and for convenience we introduce the notation $\tilde{Y}_i = \sum_j \rho_{ij} Y_j$. The exact solutions X_i and Y_j of the equations (3.3) and (3.4) may be complex, but their mixture $\hat{S}(\xi,\eta,t)$ (3.5) must be real,this impose some constraints on ρ_{ij} which should be taking into account.

Formula (3.5) as follows from (2.14) leads to the degenerate inverse problem data $S(\lambda, l, t)$ and $T(\lambda, l, t)$:

$$S(\lambda,l,t) = \frac{1}{2\pi}\iint_{R^2} d\xi d\eta\; e^{-i\lambda\xi - il\eta}\cdot\lambda\cdot\hat{S}(\xi,\eta,t) =$$

$$= \sum_j S_j(\lambda)\cdot\tilde{S}_j(l) , \qquad (3.6)$$

$$T(\lambda,l,t) = -\frac{1}{2\pi}\iint_{R^2} d\xi d\eta\; e^{il\xi + i\lambda\eta}\cdot\lambda\cdot\hat{S}(\xi,\eta,t) =$$

$$= \sum_j T_j(\lambda)\cdot\tilde{T}(l) , \qquad (3.7)$$

where

$$S_j(\lambda) \overset{def}{=} \frac{1}{\sqrt{2\pi}}\int_R d\xi\; \lambda\cdot e^{-i\lambda\xi}\cdot X_j(\xi,t) =$$

$$= -\frac{i}{\sqrt{2\pi}}\int_R d\xi\; e^{-i\lambda\xi}\cdot X_{j\xi}(\xi,t) , \qquad (3.8)$$

$$\tilde{S}_j(l) \overset{def}{=} \frac{1}{\sqrt{2\pi}}\int_R d\eta\; e^{-il\eta}\cdot\tilde{Y}_j(\eta,t) ;$$

$$T_j(\lambda) \stackrel{def}{=} \frac{i}{\sqrt{2\pi}} \int_R d\eta\, e^{i\lambda\eta} \cdot \lambda \cdot \tilde{Y}_j(\eta,t) = -\frac{1}{\sqrt{2\pi}} \int_R d\eta\, e^{i\lambda\eta} \cdot \tilde{Y}_{j\eta}(\eta,t) ,$$

$$\tilde{T}_j(l) \stackrel{def}{=} \frac{i}{\sqrt{2\pi}} \int_R d\xi\, e^{il\xi} \cdot X_j(\xi,t) . \tag{3.9}$$

Then using the formula for the exact solutions of the inverse problem for the DS equation [5] with degenerate inverse problem data (3.6) and (3.7), one gets for derivatives θ_ξ and θ_η of solution θ of equation (1.1) the following expressions:

$$\theta_\xi = 2 \sum_{j,r,l} X_{j\xi} (1 + \rho\beta\rho^T\alpha)^{-1}_{jr} \cdot \rho_{rl} Y_l ,$$

$$\theta_\eta = 2 \sum_{j,r,l} \rho_{jr} X_r (1 + \alpha\rho\beta\rho^T)^{-1}_{rl} \cdot Y_{l\eta} , \tag{3.10}$$

where

$$\alpha_{rj} \stackrel{def}{=} \int^{\xi} d\xi' X_r(\xi',t) \cdot X_{j\xi'}(\xi',t), \quad \beta_{rj} \stackrel{def}{=} \int^{\eta} d\eta' Y_r(\eta',t) \cdot Y_{j\eta'}(\eta',t) . \tag{3.11}$$

4. EXACT SOLUTIONS OF THE PERTURBED TELEGRAPH EQUATION

So for the study of coherent structures of 2DISG-I equation one needs the exact solutions $X(\xi,t)$ and $Y(\eta,t)$ of equations (3.3), (3.4) for the fixed boundaries $u_2(\xi,t)$ and

$u_1(\eta,t)$. In this section we consider the problem of constructing exact solutions for the equation (as the prototype of eqs.(3.3) and (3.4)):

$$\Psi_{\xi\eta} + u(\xi,\eta)\Psi = 0 , \tag{4.1}$$

where $u(\xi,\eta)$ has generically nonzero asymptotic value $u_\infty = -\varepsilon$ at the infinity:

$$u(\xi,\eta) = \tilde{u}(\xi,\eta) + u_\infty = \tilde{u}(\xi,\eta) - \varepsilon ,$$

$$\tilde{u}(\xi,\eta) \xrightarrow[\xi^2+\eta^2 \to \infty]{} 0 . \tag{4.2}$$

In other words we consider in this section the problem of constructing of the exact solutions of the perturbed telegraph equation:

$$\Psi_{\xi\eta} + \tilde{u}(\xi,\eta)\Psi = \varepsilon\Psi . \tag{4.3}$$

The asymptotic value $u_\infty = -\varepsilon$ of the potential is in our case a parameter.

In order to find exact solutions Ψ with corresponding exactly solvable potentials $\tilde{u}(\xi,\eta)$ of equation (4.3) we explore the $\bar\partial$-dressing method of Zakharov and Manakov [15-19]. The $\bar\partial$-dressing method is based on the use of the nonlocal $\bar\partial$-problem [15-19]:

$$\frac{\partial\chi(\lambda,\bar\lambda)}{\partial\bar\lambda} = (\chi \circ R)(\lambda,\bar\lambda) \stackrel{def}{=} \iint_C \frac{d\lambda' \wedge d\bar\lambda'}{2\pi i} \chi(\lambda',\bar\lambda') R(\lambda',\bar\lambda';\lambda,\bar\lambda). \tag{4.4}$$

The functions χ and R in our case are the scalar complex-valued functions.For the function χ we choose the canonical normalization ($\chi \xrightarrow[\lambda \to \infty]{} 1$). We assume also that the problem (4.4) is uniquelly solvable.

A dependence on the variables ξ and η is introduced via the following dependence of the kernel R of $\bar{\partial}$-problem (4.4) on ξ and η:

$$\frac{\partial R}{\partial \xi} = i\lambda' R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta) - R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta) i\lambda \ , \tag{4.5}$$

$$\frac{\partial R}{\partial \eta} = -\frac{i\varepsilon}{\lambda'} R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta) + R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta) \frac{i\varepsilon}{\lambda} \ ,$$

i.e.

$$R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta) =$$

$$= R_0(\lambda',\bar{\lambda}';\lambda,\bar{\lambda}) \cdot e^{i(\lambda'-\lambda)\xi - i\varepsilon(1/\lambda' - 1/\lambda)\eta} \ . \tag{4.6}$$

With the use of the operators

$$\mathcal{D}_\xi = \partial_\xi + i\lambda \ , \quad \mathcal{D}_\eta = \partial_\eta - i\varepsilon/\lambda \ , \tag{4.7}$$

the equations (4.5) can be rewritten as

$$[\mathcal{D}_\xi, R] = 0 \ , \ [\mathcal{D}_\eta, R] = 0 \ . \tag{4.8}$$

According to the general $\bar{\partial}$-dressing approach [15-18] we must construct the operator L of the form

$$L = \sum_{l,m} u_{lm}(\xi,\eta) \mathcal{D}_\xi^l \mathcal{D}_\eta^m \ , \tag{4.9}$$

where $u_{lm}(\xi,\eta)$ are some functions, which obeys the condition [15-18]: $[\frac{\partial}{\partial \bar{\lambda}}, L] = 0$, (4.10) i.e., which has no singularities on λ. For such operator L the function $L\chi$ obeys the same

$\bar{\partial}$-equation as the function χ. If there are several operators of this type then in virtue of the unique solvability of eq (4.4), one has:

$$L_1 \chi = 0 \ . \tag{4.11}$$

It is not difficult to show that the operator which obeys (4.11) is of the form:

$$(\mathcal{D}_\xi \mathcal{D}_\eta + V\mathcal{D}_\xi + W\mathcal{D}_\eta + u) \cdot \chi = 0 \ . \tag{4.12}$$

Indeed let us consider (4.12) for the series expansion of χ near points $\lambda = 0$ and $\lambda = \infty$: $\chi = \tilde{\chi}_0 + \lambda\chi_1 + \lambda^2\chi_2 + \ldots$, $\chi = \chi_0 + \chi_{-1}/\lambda + \chi_{-1}/\lambda + \ldots$ In the neighbourhood of $\lambda=\infty$, equating to zero the coefficients for degrees of λ ,we obtain:

$$\lambda : i\chi_{0\eta} + V\chi_0 = 0 \ ;$$

$$\lambda^0 : \chi_{0\xi\eta} + V\chi_{0\xi} + i\chi_{-1\eta} + iV\chi_{-1} + W\chi_{0\eta} + \tilde{u}\chi_0 = 0 \ . \tag{4.13}$$

Analogously in the neighbourhood of $\lambda = 0$:

$$\lambda^{-1} : i\tilde{\chi}_{0\xi} + iW\tilde{\chi}_0 = 0 \ ,$$

$$\lambda^0 : \tilde{\chi}_{0\xi\eta} + V\tilde{\chi}_{0\xi} + W\tilde{\chi}_{0\eta} + \tilde{\tilde{u}}\tilde{\chi}_0 - i\varepsilon\chi_{1\xi} - i\varepsilon W\chi_1 = 0 \ , \tag{4.14}$$

where $\tilde{u} = u + \varepsilon$ in accordance with (4.2). From (4.13) and (4.14) one can obtain the condition of potentiality of the operator L. Operator L is potential iff V = W = 0. Putting in (4.13) and (4.14) V = W = 0 one obtains the following condition of potentiality of L:

$$\chi_{0\eta} = \tilde{\chi}_{0\xi} = 0 \ , \tag{4.15}$$

and reconstruction formulae for the potential $\tilde{u}$:

$$\tilde{u} = -i\chi_{-1\eta}/\chi_0 = i\varepsilon\chi_{1\xi}/\tilde{\chi}_0 . \qquad (4.16)$$

According to (4.15) let us choose the following normalizations for χ in the neighbourhood of $\lambda = 0$ and $\lambda = \infty$:

$$\chi \xrightarrow[\lambda \to 0]{} \tilde{\chi}_0 = 1 , \quad \chi \xrightarrow[\lambda \to \infty]{} \chi_0 = 1 . \qquad (4.17)$$

Then the reconstruction formulae (4.16) for $\tilde{u}$ take the form:

$$\tilde{u} = -i\chi_{-1\eta} = i\varepsilon\chi_{1\xi} . \qquad (4.18)$$

The solution of $\bar{\partial}$-problem (4.4)with the canonical normaliza tion (4.17b) is equivalent to the solution of the following singular integral equation:

$$\chi(\xi,\eta;\lambda) = 1 +$$

$$+\iint\limits_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i(\lambda'-\lambda)} \iint\limits_C \frac{d\mu\wedge d\bar{\mu}}{2\pi i} \chi(\mu,\bar{\mu})R_0(\mu,\bar{\mu};\lambda',\bar{\lambda}')\cdot e^{F(\mu)-F(\lambda')}, \qquad (4.19)$$

where $F(\lambda) = i\lambda\xi - i\varepsilon\eta/\lambda$.

The reality of $\tilde{u}(\xi,\eta)$ also gives the restriction on the kernel R of the $\bar{\partial}$-problem. In the limit of the weak fields one finds from (4.18) and (4.19) the following condition of reality of $\tilde{u}$:

$$\overline{R_0(-\bar{\lambda}',-\lambda';-\bar{\lambda},-\lambda)} = - R_0(\lambda',\bar{\lambda}';\lambda,\bar{\lambda}) . \qquad (4.20)$$

From (4.18) and (4.19) we derive the following formula for the reconstruction of the potential $\tilde{u}(\xi,\eta)$:

$$\tilde{u}(\xi,\eta) = i\frac{\partial}{\partial\eta}\iint\limits_C \frac{d\lambda\wedge d\bar{\lambda}}{2\pi i}\iint\limits_C \frac{d\mu\wedge d\bar{\mu}}{2\pi i} R_0(\mu,\bar{\mu};\lambda,\bar{\lambda})\cdot e^{F(\mu)-F(\lambda)} . \qquad (4.21)$$

The condition of potentiality (4.17) by the use of (4.19) takes the form

$$\iint\limits_C \frac{d\lambda\wedge d\bar{\lambda}}{2\pi i\lambda}\iint\limits_C \frac{d\mu\wedge d\bar{\mu}}{2\pi i} \chi(\lambda,\bar{\lambda})\cdot R_0(\mu,\bar{\mu};\lambda,\bar{\lambda})\cdot e^{F(\mu)-F(\lambda)} = 0. \qquad (4.22)$$

The solutions χ of integral equation (4.19) and the dressing formula (4.21) for the potential $\tilde{u}$under the fullfiled conditions of reality (4.20) and potentiality (4.22) give us the method of calculating the broad classes of exact solutions $\Psi(\xi,\eta)=\chi(\xi,\eta)\cdot e^{i(\lambda\xi-\varepsilon\eta/\lambda)}$ of perturbed telegraph equation (4.3) with corresponding solvable potentials $\tilde{u}$.

Following to the formulated $\bar{\partial}$-dressing let us calculate some exact solutions of the perturbed telegraph equation. The conditions of reality (4.20) and potentiality (4.22) are satisfied (as lengthy calculations show), for example, if the kernel R is choosen in the following form:

$$R_0(\mu,\bar{\mu};\lambda,\bar{\lambda}) = i\pi^2\sum_k (S_k^{(1)}(\mu,\lambda)\delta(\mu-i\alpha_k)\delta(\lambda-i\beta_k) +$$

$$+ S_k^{(2)}(\mu,\lambda)\delta(\mu+i\beta_k)\delta(\lambda+i\alpha_k)) , \qquad (4.23)$$

where $\overline{S_k^{(1)}(-\bar{\mu},-\bar{\lambda})}=S_k^{(1)}(\mu,\lambda)$, $\overline{S_k^{(2)}(-\bar{\mu},-\bar{\lambda})}=S_k^{(2)}(\mu,\lambda)$, α_k and β_k are some real constants and

$$\alpha_k S_k^{(1)}(i\alpha_k,i\beta_k) = \beta_k S_k^{(2)}(-i\beta_k,-i\alpha_k) . \qquad (4.24)$$

Substituting the kernel R_0 (4.23) into the integral equation (4.19) one obtains:

$$\chi(\lambda) = 1 +$$

$$+ i\sum_k \left(\frac{S_k^{(1)}\chi(i\alpha_k)\cdot\exp(\Delta F_k)}{i\beta_k - \lambda} + \frac{S_k^{(2)}\chi(-i\beta_k)\cdot\exp(\Delta F_k)}{-i\alpha_k - \lambda} \right), \quad (4.25)$$

where

$$S_k^{(1)} \stackrel{def}{=} S_k^{(1)}(i\alpha_k, i\beta_k),$$

$$S_k^{(2)} \stackrel{def}{=} S_k^{(2)}(-i\beta_k, -i\alpha_k), \quad (4.26)$$

$$\Delta F_k \stackrel{def}{=} F(i\alpha_k) - F(i\beta_k) = -(\alpha_k - \beta_k)\xi - \varepsilon\eta(1/\alpha_k - 1/\beta_k).$$

From (4.25) it follows for χ_{-1}:

$$\chi_{-1} = -i\sum_k (S_k^{(1)}\chi(i\alpha_k)\cdot\exp(\Delta F_k) +$$

$$+ S_k^{(2)}\chi(-i\beta_k)\cdot\exp(\Delta F_k)). \quad (4.27)$$

The system of equations for calculating $\chi(i\alpha_k)$ and $\chi(-i\beta_k)$ as it follows from (4.25)has the form:

$$\chi(i\alpha_k) - \sum_l \frac{S_l^{(1)}\chi(i\alpha_l)\exp(\Delta F_l)}{\beta_l - \alpha_k} +$$

$$+ \sum_l \frac{S_l^{(2)}\chi(-i\beta_l)\exp(\Delta F_l)}{\alpha_l + \alpha_k} = 1, \quad (4.28)$$

$$-\sum_l \frac{S_l^{(1)}\chi(i\alpha_l)\exp(\Delta F_l)}{\beta_l + \beta_k} - \sum_l \frac{S_l^{(2)}\chi(-i\beta_l)\exp(\Delta F_k)}{\beta_k - \alpha_l} + \chi(-i\beta_k) = 1.$$

It is supposed that all the denominators in (4.28) are not equal to zero.

Let us consider the simplest case of the one term in the sum (4.23). In this case the system (4.28) has the solution:

$$\chi(i\alpha) = \chi(-i\beta) = \frac{1}{1 + \dfrac{S^{(1)}(\alpha+\beta)\exp(\Delta F)}{2\beta(\alpha-\beta)}}. \quad (4.29)$$

For $\tilde{u}$ by the use of (4.18), (4.27) and (4.29) one obtains

$$\tilde{u}(\xi,\eta) = \frac{S^{(1)}\varepsilon\cdot\exp(\Delta F)\cdot(\alpha^2-\beta^2)/(\alpha\beta^2)}{\left(1 + \dfrac{S^{(1)}(\alpha+\beta)\exp(\Delta F)}{2\beta(\alpha-\beta)}\right)^2}. \quad (4.30)$$

This expression for $\tilde{u}$ will be nonsingular if we choose parameters $S^{(1)}$, α and β so that

$$0 < \frac{S^{(1)}(\alpha+\beta)}{2\beta(\alpha-\beta)} \stackrel{def}{=} \exp(\alpha-\beta)\xi_0. \quad (4.31)$$

Under the condition (4.31) we have for the $\tilde{u}$in (4.30) the following expression

$$\tilde{u}(\xi,\eta) = -\frac{\varepsilon(\alpha-\beta)^2}{2\alpha\beta\cdot ch^2[\frac{1}{2}(\alpha-\beta)(\xi-\xi_0-\varepsilon\eta/(\alpha\beta))]}. \quad (4.32)$$

Exact solutions $\Psi(\xi,\eta)$ of perturbed telegraph equation (4.1) which correspond to $\tilde{u}$in (4.32) are of the form:

$$\Psi^{(1)}(\xi,\eta)=\chi(\lambda)\cdot e^{i\lambda\xi-i\varepsilon\eta/\lambda}\Big|_{\lambda=i\alpha}=$$

$$=\frac{\exp[-\frac{1}{2}(\alpha+\beta)(\xi+\varepsilon\eta/\alpha\beta)]}{2\exp\frac{(\alpha-\beta)\xi_0}{2}\,\mathrm{ch}\,\frac{(\alpha-\beta)(\xi-\xi_0-\varepsilon\eta/(\alpha\beta))}{2}},$$

$$\Psi^{(2)}(\xi,\eta)=\chi(\lambda)\cdot e^{i\lambda\xi-i\varepsilon\eta/\lambda}\Big|_{\lambda=-i\beta}=$$

$$=\frac{\exp[\frac{1}{2}(\alpha+\beta)(\xi+\varepsilon\eta/\alpha\beta)]}{2\exp\frac{(\alpha-\beta)\xi_0}{2}\,\mathrm{ch}\,\frac{(\alpha-\beta)(\xi-\xi_0-\varepsilon\eta/(\alpha\beta))}{2}}.$$

Unessential constant multiplier $1/[2\exp((\alpha-\beta)\xi_0/2)]$ in these expressions for Ψ can be dropped and we obtain two independent exact solutions $\Psi^{(1)}$and $\Psi^{(2)}$of perturbed telegraph equation (4.1) which correspond to potential $\tilde{u}$ (4.32):

$$\Psi^{(1)}(\xi,\eta)=\frac{\exp[-\frac{1}{2}(\alpha+\beta)(\xi+\varepsilon\eta/(\alpha\beta))]}{\mathrm{ch}[\frac{1}{2}(\alpha-\beta)(\xi-\xi_0-\varepsilon\eta/(\alpha\beta))]},$$

$$\Psi^{(2)}(\xi,\eta)=\frac{\exp[\frac{1}{2}(\alpha+\beta)(\xi+\varepsilon\eta/(\alpha\beta))]}{\mathrm{ch}[\frac{1}{2}(\alpha-\beta)(\xi-\xi_0-\varepsilon\eta/(\alpha\beta))]}. \qquad (4.33)$$

There is therefore two-fold degeneracy of solutions of equation (4.1).

Now let us consider the case of another kernel R_0 which

satisfies to the conditions of reality (4.20) and potentiality (4.22). Calculations show that the kernel R_0 can be choosen also in the form of the following sum of the paired terms:

$$R_0(\mu,\bar{\mu};\lambda,\bar{\lambda}) = i\pi^2\sum[S_k(\mu,\lambda)\delta(\mu-\mu_k)\delta(\lambda-\bar{\mu}_k) +$$

$$+\overline{S_k(-\bar{\mu},-\bar{\lambda})}\delta(\mu+\bar{\mu}_k)\delta(\lambda+\mu_k)], \qquad (4.34)$$

where μ_k are some complex constants and

$$\mu_k S_k(\mu_k,\bar{\mu}_k) = \bar{\mu}_k\overline{S_k(\mu_k,\bar{\mu}_k)}. \qquad (4.35)$$

Substituting this kernel R_0 (4.34) into the integral equation (4.19) one obtains:

$$\chi(\lambda) = 1 + i\sum_k\left(\frac{S_k\chi(\mu_k)\exp(\Delta F_k)}{\bar{\mu}_k-\lambda} + \frac{\bar{S}_k\chi(-\bar{\mu}_k)\exp(\Delta F_k)}{-\mu_k-\lambda}\right), \qquad (4.36)$$

where

$$S_k \overset{\mathrm{def}}{=} S_k(\mu_k,\bar{\mu}_k), \quad \mu_k = \mu_{kR} + i\mu_{kI},$$

$$\Delta F_k \overset{\mathrm{def}}{=} F(\mu_k) - F(\bar{\mu}_k) = -2\mu_{kI}(\xi+\varepsilon\eta/|\mu_k|^2. \qquad (4.37)$$

From (4.36) it follows for χ_{-1}:

$$\chi_{-1} = -i\sum[S_k\chi(\mu_k) + \bar{S}_k\chi(-\bar{\mu}_k)]\cdot\exp(\Delta F_k). \qquad (4.38)$$

The systemof equations for calculating $\chi(\mu_k)$ and $\chi(-\bar{\mu}_k)$ as it follows from (4.36) has the form :

$$\chi(\mu_k) - i\sum_l \frac{S_l\chi(\mu_l)\exp(\Delta F_l)}{\bar{\mu}_l - \mu_k} + i\sum_l \frac{\bar{S}_l\chi(-\bar{\mu}_l)\exp(\Delta F_l)}{\mu_l + \mu_k} = 1,$$

$$- i\sum_l \frac{S_l\chi(\mu_l)\exp(\Delta F_l)}{\bar{\mu}_l + \bar{\mu}_k} + \chi(-\bar{\mu}_k) + i\sum_l \frac{\bar{S}_l\chi(-\bar{\mu}_l)\exp(\Delta F_l)}{\mu_l - \bar{\mu}_k} = 1, \tag{4.39}$$

where it is assumed that all denominators in (4.39) are different from zero.

Let us consider the simplest case of the one term in the sum (4.34). In this case the system (4.39) has the solution:

$$\chi(\mu) = \chi(-\bar{\mu}) = \frac{1}{1 + \frac{|S|\cos\delta\cdot\exp(\Delta F)}{2\mu_I}}, \tag{4.40}$$

where $S = |S|\cdot e^{i\delta}$ and according to (4.35) $\mu = \bar{\mu}\cdot e^{-2i\delta}$. For $\tilde{u}$ with the use of (4.18), (4.38) and (4.40) one obtains :

$$\tilde{u} = \frac{4\varepsilon\mu_I|S|\cdot\cos\delta\cdot\exp(\Delta F)}{|\mu|^2\left(1 + \frac{|S|\cdot\cos\delta\cdot\exp(\Delta F)}{2\mu_I}\right)^2}. \tag{4.41}$$

This expression for $\tilde{u}$ will be nonsingular if we choose the parameters S and μ so that

$$0 < \frac{|S|\cdot\cos\delta}{2\mu_I} \stackrel{def}{=} \exp(2\mu_I\xi_0) . \tag{4.42}$$

Under the condition (4.42) we have for the $\tilde{u}$ in (4.41) the following expression:

$$\tilde{u}(\xi,\eta) = \frac{2\varepsilon\mu_I^2}{|\mu|^2\cdot\mathrm{ch}^2[\mu_I(\xi-\xi_0+\varepsilon\eta/|\mu|^2)]} . \tag{4.43}$$

Exact solutions $\Psi(\xi,\eta)$ of perturbed telegraph equation (4.1) which correspond to $\tilde{u}$in (4.43) are of the form:

$$\Psi^{(1)}(\xi,\eta)=\chi(\lambda)\cdot e^{i\lambda\xi-i\varepsilon\eta/\lambda}\Big|_{\lambda=\mu} = \frac{e^{i\mu\xi-i\varepsilon\eta/\mu}}{1 + \frac{|S|\cos\delta\cdot\exp(\Delta F)}{2\mu_I}},$$

$$\Psi^{(2)}(\xi,\eta)=\chi(\lambda)\cdot e^{i\lambda\xi-i\varepsilon\eta/\lambda}\Big|_{\lambda=-\bar{\mu}} = \frac{e^{-i\bar{\mu}\xi+i\varepsilon\eta/\bar{\mu}}}{1 + \frac{|S|\cos\delta\cdot\exp(\Delta F)}{2\mu_I}}.$$

Dropping in this formulae unessential multiplier $1/(2\exp(\mu_I\xi_0))$ one can obtain under the condition (4.42) more convenient expressions for the independent solutions $\Psi^{(1)}$ and $\Psi^{(2)}$ of eq. (4.1), which correspond to the potential $\tilde{u}$(4.43):

$$\Psi^{(1)}(\xi,\eta)=\frac{\exp\ i\mu_R(\xi-\varepsilon\eta/|\mu|^2)}{\mathrm{ch}\ \mu_I(\xi-\xi_0+\varepsilon\eta/|\mu|^2)},\quad \Psi^{(2)}(\xi,\eta) =$$

$$= \frac{\exp(-i\mu_R(\xi-\varepsilon\eta/|\mu|^2))}{\mathrm{ch}\ \mu_I(\xi-\xi_0+\varepsilon\eta/|\mu|^2)} . \tag{4.44}$$

5. EXACT SOLUTIONS OF 2DISG-I EQUATION VIA THE SOLUTIONS OF PERTURBED TELEGRAPH EQUATION

In this section we explore the exact solutions of perturbed telegraph equation (4.1) for obtaining by the general formula (3.10) the exact solutions of 2DISG-I equation. Let us consider at first the boundaries $u_1(\eta,t)$ and $u_2(\xi,t)$ (potentials in eqs. (3.3), (3.4) of the type of(4.32)), i.e. with corresponding changes of variables $(\xi,\eta)\longrightarrow(\xi,t)$ and $(\xi,\eta)\longrightarrow(\eta,t)$ in (4.32):

$$u_1(\eta,t) = -\varepsilon_1 + \frac{\varepsilon_1(\alpha_1-\beta_1)^2}{2\alpha_1\beta_1\cdot \mathrm{ch}^2[\frac{1}{2}(\alpha_1-\beta_1)(\eta-\eta_0-\varepsilon_1 t/(\alpha_1\beta_1))]},$$

$$u_2(\xi,t) = -\varepsilon_2 + \frac{\varepsilon_2(\alpha_2-\beta_2)^2}{2\alpha_2\beta_2\cdot \mathrm{ch}^2[\frac{1}{2}(\alpha_2-\beta_2)(\xi-\xi_0-\varepsilon_2 t/(\alpha_2\beta_2))]}. \qquad (5.1)$$

We choose the corresponding exact solutions $X(\xi,t)$ and $Y(,t)$ of equations (3.3) and (3.4) (according to (4.33)) in the form:

$$X(\xi,t) = \frac{\exp[\frac{1}{2}(\alpha_2+\beta_2)(\xi+\varepsilon_2 t/(\alpha_2\beta_2))]}{\mathrm{ch}[\frac{1}{2}(\alpha_2-\beta_2)(\xi-\xi_0-\varepsilon_2 t/(\alpha_2\beta_2))]},$$

$$Y(\eta,t) = \frac{\exp[\frac{1}{2}(\alpha_1+\beta_1)(\eta+\varepsilon_1 t/(\alpha_1\beta_1))]}{\mathrm{ch}[\frac{1}{2}(\alpha_1-\beta_1)(\eta-\eta_0-\varepsilon_1 t/(\alpha_1\beta_1))]}. \qquad (5.2)$$

Using (3.11) one finds for the matrices α and β (in this case they are scalar functions) the following expressions:

$$\alpha = \frac{1}{2}X^2(\xi,t), \ \beta = \frac{1}{2}Y^2(\eta,t). \qquad (5.3)$$

The matrix ρ is also simply the real number. Inserting these α and β into (3.10), one obtains for the $\theta(\xi,\eta,t)$ the expression:

$$\theta = 4\cdot \mathrm{arctg}\frac{\rho X(\xi,t)Y(\eta,t)}{2} =$$

$$= 4\cdot \mathrm{arctg}\Big\{\rho\cdot\exp[\tfrac{1}{2}(\alpha_2+\beta_2)(\hat\xi+2\varepsilon_2 t/(\alpha_2\beta_2))] \times$$

$$\times \exp[\tfrac{1}{2}(\alpha_1+\beta_1)(\hat\eta+2\varepsilon_1 t/(\alpha_1\beta_1))]\Big\} \Big/ \Big\{2\cdot\mathrm{ch}[\tfrac{1}{2}(\alpha_2-\beta_2)(\hat\xi-\xi_0)] \times$$

$$\times\, \mathrm{ch}[\tfrac{1}{2}(\alpha_1-\beta_1)(\hat\eta-\eta_0)]\Big\}, \qquad (5.4)$$

where for convenience the wave variables $\hat\xi$ and $\hat\eta$ are introduced:

$$\hat\xi = \xi - \varepsilon_2 t/(\alpha_2\beta_2), \ \hat\eta = \eta - \varepsilon_1 t/(\alpha_1\beta_1). \qquad (5.5)$$

Imposing on the parameters ε_k, α_k, $\beta_k (k=1,2)$ in (5.4) the constraint

$$\varepsilon_1(1/\alpha_1 + 1/\beta_1) = -\varepsilon_2(1/\alpha_2 + 1/\beta_2), \qquad (5.6)$$

one obtains from (5.4) the following solution θ:

$$\theta(\xi,\eta,t) =$$

$$= 4\cdot\mathrm{arctg}\frac{\rho\cdot\exp[\frac{1}{2}((\alpha_1+\beta_1)\hat\eta + (\alpha_2+\beta_2)\hat\xi)]}{2\cdot\mathrm{ch}[\frac{1}{2}(\alpha_1-\beta_1)(\hat\eta-\eta_0)]\cdot\mathrm{ch}[\frac{1}{2}(\alpha_2-\beta_2)(\hat\xi-\xi_0)]}. \qquad (5.7)$$

It is evident that this solution of 2DISG-I equation (1.1) under the conditions

$$|\alpha_k - \beta_k| > |\alpha_k + \beta_k|, \quad (k = 1,2) , \qquad (5.8)$$

represents exponentially localized object of dromion type moving on the plane (ξ,η) with the velocity

$$V = (\varepsilon_2/(\alpha_2\beta_2) \; , \; \varepsilon_1/(\alpha_1\beta_1)) . \qquad (5.9)$$

Now let us consider the boundaries $u_1(\eta,t)$ and $u_2(\xi,t)$ of the type of (4.43). With corresponding changes of independent variables one finds from (4.43) the following expressions for $u_1(\eta,t)$ and $u_2(\xi,t)$:

$$u_1(\eta,t) = -\varepsilon_1 + \frac{2\varepsilon_1\mu_{1I}^2}{|\mu_1|^2 \mathrm{ch}^2[\mu_{1I}(\hat{\eta} - \eta_0)]} ,$$

$$u_2(\xi,t) = -\varepsilon_2 + \frac{2\varepsilon_2\mu_{2I}^2}{|\mu_2|^2 \mathrm{ch}^2[\mu_{2I}(\hat{\xi} - \xi_0)]} , \qquad (5.10)$$

where for the convenience the wave variables $\hat{\xi}$ and $\hat{\eta}$ are introduced:

$$\hat{\eta} = \eta + \varepsilon_1 t/|\mu_1|^2 \; , \quad \hat{\xi} = \xi + \varepsilon_2 t/|\mu_2|^2 . \qquad (5.11)$$

The corresponding exact solutions of equations (3.3) and (3.4) in accordance with (4.44) are of the form ,

$$X_1 = X(\xi,t), \; X_2 = \overline{X(\xi,t)}; \; Y_1 = Y(\eta,t), \; Y_2 = \overline{Y(\eta,t)} , \qquad (5.12)$$

where

$$X(\xi,t) = \frac{\exp i\mu_{2R}(\xi - \varepsilon_2 t/|\mu_2|^2)}{\mathrm{ch}\, \mu_{2I}(\hat{\xi} - \xi_0)} ,$$

$$Y(\eta,t) = \frac{\exp i\mu_{1R}(\eta - \varepsilon_1 t/|\mu_1|^2)}{\mathrm{ch}\, \mu_{1I}(\hat{\eta} - \eta_0)} . \qquad (5.13)$$

Using (3.11) and (5.12), (5.13) one finds for the matrices α and β the expressions:

$$\alpha = \frac{1}{2\mathrm{ch}^2 f_2} \begin{pmatrix} \exp(2i\varphi_2), & 1 - i\dfrac{\mu_{2R}}{\mu_{2I}}\mathrm{sh}\, 2f_2 \\ 1 + i\dfrac{\mu_{2R}}{\mu_{2I}}\mathrm{sh}\, 2f_2 & , \exp(-2i\varphi_2) \end{pmatrix} ,$$

$$\beta = \frac{1}{2\mathrm{ch}^2 f_1} \begin{pmatrix} \exp(2i\varphi_1), & 1 - i\dfrac{\mu_{1R}}{\mu_{1I}}\mathrm{sh}\, 2f_1 \\ 1 + i\dfrac{\mu_{1R}}{\mu_{1I}}\mathrm{sh}\, 2f_1 & , \exp(-2i\varphi_1) \end{pmatrix} , \qquad (5.14)$$

where

$$\varphi_1 = \mu_{1R}(\eta - \varepsilon_1 t/|\mu_1|^2) \; , \quad f_1 = \mu_{1I}(\hat{\eta} - \eta_0) ,$$

$$\varphi_2 = \mu_{2R}(\xi - \varepsilon_2 t/|\mu_2|^2), \quad f_2 = \mu_{2I}(\hat{\xi} - \xi_0) . \qquad (5.15)$$

For the construction of the exact solutions θ of the 2DISG-I equation (1.1) one can use several different mixtures (3.5) $S = \sum_{i,j} \rho_{ij} X_i Y_j$ of solutions X_i and Y_j (5.12) of eqs. (3.3) and (3.4).
Let us choose for example the matrix $\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then calculations by the formulae (3.10) with the use of (5.14) give for $\theta(\xi,\eta,t)$ the expression:

$$\theta(\xi,\eta,t) = 4\mathrm{arctg}\frac{\cos(\varphi_1-\varphi_2)}{\mathrm{ch}f_1\cdot\mathrm{ch}f_2 - \frac{\mu_{1R}\mu_{2R}}{\mu_{1I}\mu_{2I}}\mathrm{sh}f_1\cdot\mathrm{sh}f_2} . \quad (5.16)$$

The mixture $\hat{S} = \sum_{i,j}\rho_{ij}X_iY_j$ (3.5) with the matrix $\rho = \begin{pmatrix}10\\01\end{pmatrix}$ leads by the use (3.10) and (5.14) to the following $\theta(\xi,\eta,t)$:

$$\theta(\xi,\eta,t) = 4\mathrm{arctg}\frac{\cos(\varphi_1+\varphi_2)}{\mathrm{ch}f_1\cdot\mathrm{ch}f_2 + \frac{\mu_{1R}\mu_{2R}}{\mu_{1I}\mu_{2I}}\mathrm{sh}f_1\cdot\mathrm{sh}f_2} . \quad (5.17)$$

The other mixture $\hat{S}= \sum_{i,j}\rho_{ij}X_iY_j$(3.5) with the matrix $\rho = \begin{pmatrix}11\\11\end{pmatrix}$ gives by the use of (3.10) and (5.14) the solution θ of the 2DISG-I equation (1.1) of the form:

$$\theta(\xi,\eta,t) = 4\mathrm{arctg}\frac{\cos\varphi_1\cdot\cos\varphi_2}{\mathrm{ch}f_1\cdot\mathrm{ch}f_2} . \quad (5.18)$$

The solutions θ (5.16), (5.17) under the condition $\frac{\mu_{1R}\mu_{2R}}{\mu_{1I}\mu_{2I}} \neq \pm1$ and the solution θ (5.18) represent localized objects of the breather type moving on the plane (ξ,η) with the velocity:

$$V =(-\varepsilon_2/|\mu_2|^2, -\varepsilon_1/|\mu_1|^2) . \quad (5.19)$$

At last, let us consider the boundaries $u_1(\eta,t)$ and

$u_2(\xi,t)$ of different types (4.32) and (4.43). We choose the boundary $u_1(\eta,t)$ of the type (4.43), i.e. (5.10):

$$u_1(\eta,t) = -\varepsilon_1+ \frac{2\varepsilon_1\mu_I^2}{|\mu|^2\cdot\mathrm{ch}^2[\mu_I(\hat{\eta} - \eta_0)]} , \quad (5.20)$$

where $\hat{\eta} = \eta + \varepsilon t/|\mu|^2$. For the corresponding solution Y of eq. (3.4) we take real linear combination of solutions Y_1, Y_2 (5.12):

$$Y(\eta,t) = \frac{\cos\,\mu_R(\eta-\varepsilon_1 t/|\mu|^2)}{\mathrm{ch}\,\mu_I(\hat{\eta}-\eta_0)} . \quad (5.21)$$

Then we choose the boundary $u_2(\xi,t)$ of the type (4.32), i.e. (5.1):

$$u_2(\xi,t) = -\varepsilon_2 + \frac{\varepsilon_2(\alpha-\beta)}{2\alpha\beta\cdot\mathrm{ch}^2[\frac{1}{2}(\alpha-\beta)(\hat{\xi}-\xi_0)]} , \quad (5.22)$$

where $\hat{\xi}= \xi - \varepsilon t/(\alpha\beta)$. The corresponding solution $X(\xi,t)$ of equation (3.3) has the form:

$$X(\xi,t) = \frac{\exp[\frac{1}{2}(\alpha+\beta)(\hat{\xi} + \varepsilon t/(\alpha\beta))]}{\mathrm{ch}[\frac{1}{2}(\alpha-\beta)(\hat{\xi}-\xi_0)]} . \quad (5.23)$$

Then the calculations by the use of (3.10), (5.21), (5.23) give the following solution of 2DISG-I equation (1.1):

$$\theta(\xi,\eta,t) = \tag{5.24}$$

$$= 4\operatorname{arctg}\frac{\rho\cdot\exp[\frac{1}{2}(\alpha+\beta)(\hat{\xi}+\varepsilon_2 t/(\alpha\beta))]\cdot\cos[\mu_R(\eta-\varepsilon_1 t/|\mu|^2)]}{2\operatorname{ch}[\frac{1}{2}(\alpha-\beta)(\hat{\xi}-\xi_0)]\cdot\operatorname{ch}[\mu_I(\hat{\eta}-\eta_0)]} .$$

6. EXACT SOLUTIONS OF THE PERTURBED STRING EQUATION

In this section we consider another class of exact solutions of equations (3.3) and (3.4), which corresponds to boundaries $u_1(\eta,t)$ and $u_2(\xi,t)$ with generically zero asymptotic values at infinity, i.e. we consider (as a prototype of equations (3.3) and (3.4)) the equation of the perturbed string:

$$\Psi_{\xi\eta} + u(\xi,\eta)\Psi = 0 , \tag{6.1}$$

where

$$u(\xi,\eta) \xrightarrow[\xi^2+\eta^2 \longrightarrow \infty]{} 0 .$$

The $\bar{\partial}$-dressing formulated in section 4 now does not work since all derived formulae collapse at the limit $\varepsilon \longrightarrow 0$. The case (6.1) requires a special treatment. We start with the 2×2 matrix $\bar{\partial}$-problem [15-18]:

$$\frac{\partial\chi(\lambda,\bar{\lambda})}{\partial\bar{\lambda}} = (\chi\circ R)(\lambda,\bar{\lambda}) \equiv \iint_C \frac{d\lambda'\wedge d\bar{\lambda}'}{2\pi i}\,\chi(\lambda',\bar{\lambda}')\cdot R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda}). \tag{6.2}$$

The functions χ and R in our present case are the complex-valued matrix functions. We assume that the matrix χ

has the canonical normalization $\chi \xrightarrow[\lambda\to\infty]{} \begin{pmatrix}1&0\\0&1\end{pmatrix}$ and equation (6.2) has a unique solution. The kernel R is choosen to be

$$R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda}) = \begin{pmatrix} 0\ , & R_1(\lambda',\bar{\lambda}';\lambda,\bar{\lambda}) \\ R_2(\lambda',\bar{\lambda}';\lambda,\bar{\lambda}), & 0 \end{pmatrix} . \tag{6.3}$$

A dependence on the variables ξ and η can be introduced via the following dependence of the kernel R of $\bar{\partial}$-problem (6.2) on ξ and η:

$$\frac{\partial R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta)}{\partial\xi} =$$

$$= \begin{pmatrix}0&0\\0&-\lambda'\end{pmatrix} R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta) - R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta)\begin{pmatrix}0&0\\0&-\lambda\end{pmatrix} , \tag{6.4}$$

$$\frac{\partial R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta)}{\partial\eta} =$$

$$= \begin{pmatrix}\lambda'&0\\0&0\end{pmatrix} R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta) - R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta)\begin{pmatrix}\lambda&0\\0&0\end{pmatrix} ,$$

i.e.

$$R = \begin{pmatrix} 0, R_1(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta) \\ R_2(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta), 0 \end{pmatrix} = \begin{pmatrix} 0,\ R_{10}(\lambda',\bar{\lambda}';\lambda,\bar{\lambda})\cdot e^{\lambda'\eta+\lambda\xi} \\ R_{20}(\lambda',\bar{\lambda}';\lambda,\bar{\lambda})\cdot e^{-\lambda'\xi-\lambda\eta},\ 0 \end{pmatrix} . \tag{6.5}$$

In terms of the operators $\mathcal{D}_\xi$ and $\mathcal{D}_\eta$ defined by

$$\mathcal{D}_\xi\chi \overset{def}{=} \chi_\xi + \chi\begin{pmatrix}0&0\\0&-\lambda\end{pmatrix} , \quad \mathcal{D}_\eta\chi \overset{def}{=} \chi_\eta + \chi\begin{pmatrix}\lambda&0\\0&0\end{pmatrix} , \tag{6.6}$$

one can rewrite the equations (6.4) as

$$[\mathcal{D}_\xi, R] = 0 \,, \quad [\mathcal{D}_\eta, R] = 0 \,. \tag{6.7}$$

Our goal is to construct the operator L of the form (4.9) which obeys the condition (4.10). In our case one constructs operator L (4.9) of the form:

$$L\chi = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{D}_\xi + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{D}_\eta \right) \cdot \chi + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \cdot \chi = 0 \,. \tag{6.8}$$

The equations of the system (6.8) are:

$$D_\xi \chi_{11} + q\chi_{21} = 0 \,, \quad D_\xi \chi_{12} + q\chi_{22} = 0 \,,$$
$$D_\eta \chi_{12} + r\chi_{11} = 0 \,, \quad D_\eta \chi_{22} + r\chi_{12} = 0 \,, \tag{6.9}$$

where the action of the operators D_ξ and D_η on the components of χ induces by the formulae (6.6) of the action of $\mathcal{D}_\xi$ and $\mathcal{D}_\eta$ on χ:

$$D_\xi \chi_{11} = \chi_{11\xi} \,, \qquad D_\eta \chi_{11} = \chi_{11\eta} + \lambda\chi_{11} \,,$$
$$D_\xi \chi_{21} = \chi_{21\xi} \,, \qquad D_\eta \chi_{21} = \chi_{21\eta} + \lambda\chi_{21} \,,$$
$$D_\xi \chi_{12} = \chi_{12\xi} - \lambda\chi_{12} \,, \qquad D_\eta \chi_{12} = \chi_{12\eta} \,, \tag{6.10}$$
$$D_\xi \chi_{22} = \chi_{22\xi} - \lambda\chi_{22} \,, \qquad D_\eta \chi_{22} = \chi_{22\eta} \,.$$

Now let us perform the reduction of the $\bar\partial$-problem (6.2) for the DS equation which converts it to the $\bar\partial$-problemfor the perturbed string equation (6.1). The required reduction is q = 1. Indeed, from (6.9) one obtains under q = 1:

$$D_\eta D_\xi \chi_{11} - r\chi_{11} = 0 \,, \quad D_\eta D_\xi \chi_{12} - r\chi_{12} = 0 \,. \tag{6.11}$$

In the terms of components $\Psi_{11} = \chi_{11} \cdot e^{\lambda\eta}$ and $\Psi_{12} = \chi_{12} \cdot e^{-\lambda\xi}$ one has

$$\Psi_{11\xi\eta} - r\Psi_{11} = 0, \Psi_{12\xi\eta} - r\Psi_{12} = 0 \,, \tag{6.12}$$

that is nothing but (6.1) for u = - r. So as the result of reduction q = 1 we obtain from $\bar\partial$-problem (6.2) for DS equation the $\bar\partial$-problem which is applicable to equation (6.1). Spectral problems (6.12) in terms of χ_{11} and χ_{12} have in accordance with (6.11) the forms:

$$\chi_{11\xi\eta} + \lambda\chi_{11\xi} - r\chi_{11} = 0 \,, \quad \chi_{12\xi\eta} - \lambda\chi_{12\eta} - r\chi_{12} = 0. \tag{6.13}$$

In order to obtain the restrictions on the components χ under the reduction q = 1 and the formulae for reconstruction of potenti al r let us consider the system (6.9) for the series expansion of χ near the point $\lambda = \infty$:

$$\chi = \begin{pmatrix} \chi_{11}^{(0)} + \chi_{11}^{(-1)}/\lambda + \ldots, & \chi_{12}^{(0)} + \chi_{12}^{(-1)}/\lambda + \ldots \\ \chi_{21}^{(0)} + \chi_{21}^{(-1)}/\lambda + \ldots, & \chi_{22}^{(0)} + \chi_{22}^{(-1)}/\lambda + \ldots \end{pmatrix} . \tag{6.14}$$

Equating the coefficients of (6.9) with χ from (6.14) for certain degrees of λ to zero we obtain the relations:

$$\chi_{12}^{(0)} = \chi_{21}^{(0)} = 0 \,, \quad \chi_{22}^{(0)} = \chi_{12}^{(-1)} = \chi_{11}^{(0)} = 1, \; \chi_{21}^{(-1)} = -\chi_{11\xi}^{(-1)} , \tag{6.15}$$
$$\chi_{22}^{(-1)} = \chi_{12}^{(-2)} , \quad r = -\chi_{22\eta}^{(-1)} = -\chi_{21}^{(-1)} = \chi_{11\xi}^{(-1)} .$$

From (6.2) one has the system of equations of our $\bar\partial$-problem for the component s χ_{11} and χ_{12}:

$$\frac{\partial\chi_{11}(\lambda,\bar\lambda)}{\partial\bar\lambda} = (\chi_{12}\circ R_2)(\lambda,\bar\lambda), \quad \frac{\partial\chi_{12}(\lambda,\bar\lambda)}{\partial\bar\lambda} = (\chi_{11}\circ R_1)(\lambda,\bar\lambda), \qquad (6.16)$$

where for χ_{11} and χ_{12}, according (6.15), one can choose the following series expansions at the point $\lambda = \infty$:

$$\chi_{11} = 1 + \chi_{11}^{(-1)}/\lambda + \ldots, \quad \chi_{12} = 1/\lambda + \chi_{12}^{(-2)} + \ldots . \qquad (6.17)$$

Let us denote $\chi_2 \stackrel{def}{=} \lambda\chi_{12}$, $\chi_1 \stackrel{def}{=} \chi_{11}$ and redefine the kernels R_1 and R_2 as follows: $\lambda R_1(\lambda',\bar\lambda';\lambda,\bar\lambda) \Rightarrow R_1(\lambda',\bar\lambda';\lambda,\bar\lambda)$, $R_2(\lambda',\bar\lambda';\lambda,\bar\lambda)/\lambda' \Rightarrow R_2(\lambda',\bar\lambda';\lambda,\bar\lambda)$.

In the terms of χ_1 and χ_2 and redefined kernels R_1 and R_2 one obtains from (6.16) the desired system of equations of $\bar\partial$-problem for the construction of exact solutions of equation (6.1)

$$\frac{\partial\chi_1(\lambda,\bar\lambda)}{\partial\bar\lambda} = \iint_C \frac{d\lambda'\wedge d\bar\lambda'}{2\pi i}\chi_2(\lambda',\bar\lambda')\cdot R_{20}(\lambda',\bar\lambda';\lambda,\bar\lambda)\cdot e^{-\lambda'\xi-\lambda\eta},$$

$$\frac{\partial\chi_2(\lambda,\bar\lambda)}{\partial\bar\lambda} = \iint_C \frac{d\lambda'\wedge d\bar\lambda'}{2\pi i}\chi_1(\lambda',\bar\lambda')\cdot R_{10}(\lambda',\bar\lambda';\lambda,\bar\lambda)\cdot e^{\lambda'\eta+\lambda\xi}, \qquad (6.18)$$

where near the point $\lambda = \infty$:

$$\chi_1 = 1 + \chi_1^{(-1)}/\lambda + \ldots, \quad \chi_2 = 1 + \chi_2^{(-1)}/\lambda + \ldots, \qquad (6.19)$$

The formulae for the reconstruction of the potential $u = -r$

and the constraint from the reduction $q = 1$ in terms of χ and χ_2 have according to (6.15) the forms:

$$u = -r = -\chi_{1\xi}^{(-1)} = \chi_{2\eta}^{(-1)}, \qquad (6.20)$$

$$\chi_2^{(0)} = 1 . \qquad (6.21)$$

With the use of (6.19) the system of (6.18) of our $\bar\partial$-problem is equivalent to the system of singular integral equations:

$$\chi(\lambda,\bar\lambda) = 1 + \iint_C \frac{d\mu\wedge d\bar\mu}{2\pi i(\mu-\lambda)}\iint_C \frac{d\lambda'\wedge d\bar\lambda'}{2\pi i}\chi_2(\lambda',\bar\lambda')R_{20}(\lambda',\bar\lambda';\mu,\bar\mu)\cdot e^{-\lambda'\xi-\mu\eta}, \qquad (6.22)$$

$$\chi_2(\lambda,\bar\lambda) = 1 + \iint_C \frac{d\mu\wedge d\bar\mu}{2\pi i(\mu-\lambda)}\iint_C \frac{d\lambda'\wedge d\bar\lambda'}{2\pi i}\chi_1(\lambda',\bar\lambda')R_{10}(\lambda',\bar\lambda';\mu,\bar\mu)\cdot e^{\lambda'\eta+\mu\xi} .$$

Using the condition (6.20) $\chi_{1\xi}^{(-1)} = -\chi_{2\eta}^{(-1)}$ in the limit of weak fields ($\chi_1 \sim 1$, $\chi_2 \sim 1$) one obtains from (6.22) the relation between the kernels R_{10} and R_{20}:

$$\lambda' R_{20}(\lambda',\bar\lambda';\lambda,\bar\lambda) = -\lambda R_{10}(-\lambda,-\bar\lambda;-\lambda',-\bar\lambda') . \qquad (6.23)$$

The reality of $u = -r$ imposes on the kernels R_{10} and R_{20} the other restrictions. Under $u = \bar u$ in the limit of the weak fields one has from (6.22):

$$\overline{R_{10}(\bar{\lambda}',\lambda';\bar{\lambda},\lambda)}=R_{10}(\lambda',\bar{\lambda}';\lambda,\bar{\lambda}) ,$$

$$\overline{R_{20}(\bar{\lambda}',\lambda';\bar{\lambda},\lambda)}=R_{20}(\lambda',\bar{\lambda}';\lambda,\bar{\lambda}) . \qquad (6.24)$$

The solution χ_1, χ_2 of the system of integral equations (6.22) and the dressing formulae (6.20) for the potential u = -r under fulfiled conditions of the reduction q = 1 and reality $\bar{u}$ = u (6.20), (6.21) or (6.23), (6.24) give us the method of calculation of the broad classes of exact solutions $\Psi = \chi_1 \cdot e^{\lambda\eta}$ of perturbed string equation (6.1) together with corresponding solvable potentials u.

By the use of formulated in this section $\bar{\partial}$-dressing let us calculate some exact solutions of the perturbed string equation (6.1). The conditions (6.23) and (6.24) of the reduction q=1 and the reality u=$\bar{u}$ are satisfied if the kernels R_{10} and R_{20} are choosen, for instance, in the form of the following sum of delta-functional terms:

$$R_{10}(\mu,\bar{\mu};\lambda,\bar{\lambda}) = 2\pi^2\sum_k S_k\lambda\cdot\delta(\mu-\mu_k)\delta(\lambda-\lambda_k) ,$$
$$(6.25)$$
$$R_{20}(\mu,\bar{\mu};\lambda,\bar{\lambda}) = 2\pi^2\sum_k S_k\lambda\cdot\delta(\mu+\lambda_k)\delta(\lambda+\mu_k) ,$$

where $S_k = \bar{S}_k$. Substituting these kernels R_{10} and R_{20} into the system (6.22), one obtains:

$$\chi_1(\lambda) - \sum_k \frac{2S_k\mu_k\chi_2(-\lambda_k)\cdot \exp F_k}{\mu_k + \lambda} = 1 ,$$
$$(6.26)$$
$$\chi_2(\lambda) - \sum_k \frac{2S_k\lambda_k\chi_1(\mu_k)\cdot \exp F_k}{\lambda_k - \lambda} = 1 ,$$

where $F_k \overset{def}{=} \mu_k\eta + \lambda_k\xi$. From (6.26) it follows for $\chi_1^{(-1)}$:

$$\chi_1^{(-1)} = 2\sum_k S_k\mu_k\chi_2(-\lambda_k)\cdot \exp F_k . \qquad (6.27)$$

The system of equations for calculating $\chi_1(\mu_k)$ and $\chi_2(-\lambda_k)$ has due to the equations (6.26) the form:

$$\chi_1(\mu_k) - \sum_l \frac{2S_l\mu_l\chi_2(-\lambda_l)\cdot \exp F_l}{\mu_l + \mu_k} = 1 ,$$
$$(6.28)$$
$$\chi_2(-\lambda_k) - \sum_l \frac{2S_l\lambda_l\chi_1(\mu_l)\cdot \exp F_l}{\lambda_l + \lambda_k} = 1 .$$

It is assumed that in (6.28) all denominators are not equal to zero.

Now let us calculate the simplest exact solutions of equation (6.1) For the kernels R_{10} and R_{20} (6.25) with one term in the sums the system (6.28) has the solution:

$$\chi_1(\mu_0) = \chi_2(-\lambda_0) = \frac{1}{1 - S_0\cdot \exp F_0} , \qquad (6.29)$$

where $F_0 = \mu_0\eta + \lambda_0\xi$. For u with the use of (6.20), (6.27) and (6.29) one obtains:

$$u = -r = \frac{2S_0\lambda_0\mu_0\cdot \exp F_0}{(1 - S_0\cdot \exp F_0)^2} = \frac{-\lambda_0\mu_0}{2\mathrm{ch}^2[\frac{1}{2}(\lambda_0\xi+\mu_0\eta+C_0)]} , (6.30)$$

where for providing the nonsingularity of u we imposed on the parameter S_0 the condition:

$$0 < -S_0 \overset{def}{=} \exp C_0 \, . \qquad (6.31)$$

Two independent exact solutions Ψ of the problem (6.1) which correspond to potential u (6.30) have, due to the (6.29), the form:

$$\Psi^{(1)}(\xi,\eta) \overset{def}{=} 2\cdot e^{\lambda\eta}\cdot \chi_1(\lambda)\Big|_{\lambda=\mu_0} =$$

$$= \frac{2\exp(\mu_0\eta)}{1 - S_0\cdot \exp F_0} = \frac{\exp\frac{1}{2}(\mu_0\eta-\lambda_0\xi-C_0)}{ch\,\frac{1}{2}(\lambda_0\xi+\mu_0\eta+C_0)} \, , \qquad (6.32)$$

$$\Psi^{(2)}(\xi,\eta) \overset{def}{=} 2\cdot e^{-\lambda\xi}\cdot \chi_2(\lambda)\Big|_{\lambda=-\lambda_0} =$$

$$= \frac{2\exp(\lambda_0\xi)}{1 - S_0\cdot \exp F_0} = \frac{\exp\frac{1}{2}(\lambda_0\xi-\mu_0\eta-C_0)}{ch\,\frac{1}{2}(\lambda_0\xi+\mu_0\eta+C_0)} \, .$$

The more complicated solutions of equation (6.1) correspond to kernels R_{10} and R_{20} in the form of the following sums of the paired delta-functional terms:

$$R_{10}(\mu,\bar\mu;\lambda,\bar\lambda) = 2\pi^2\sum_k [S_k(\mu,\lambda)\lambda\cdot\delta(\mu-\mu_k)\delta(\lambda-\lambda_k) +$$

$$+ \overline{S_k(\bar\mu,\bar\lambda)\lambda}\cdot\delta(\mu-\bar\mu_k)\delta(\lambda-\bar\lambda_k) \, ,$$

$$R_{20}(\mu,\bar\mu;\lambda,\bar\lambda) = 2\pi^2\sum_k [S_k(-\lambda,-\mu)\lambda\delta(\mu+\lambda_k)\delta(\lambda+\mu_k) +$$

$$+ \overline{S_k(-\bar\lambda,-\bar\mu)}\lambda\delta(\mu+\bar\lambda_k)\delta(\lambda+\bar\mu_k). \qquad (6.33)$$

For the kernels (6.33) the conditions (6.23), (6.24) of the reduction $q = 1$ and reality $u = \bar u$, as can be easily seen, are satisfied. Substituting these kernels R_{10} and R_{20} into the system of integral equations (6.22), one obtains:

$$\chi_1(\lambda) - \sum_k \frac{2S_k\mu_k\chi_2(-\lambda_k)\cdot \exp F_k}{\mu + \lambda} - \sum_k \frac{2\bar S_k\bar\mu_k\chi_2(-\bar\lambda_k)\cdot \exp\bar F_k}{\bar\mu + \lambda} = 1 \, ,$$

$$(6.34)$$

$$\chi_2(\lambda) - \sum_k \frac{2S_k\lambda_k\chi_1(\mu_k)\cdot \exp F_k}{\lambda_k - \lambda} - \sum_k \frac{2\bar S_k\bar\lambda_k\chi_1(\bar\mu_k)\cdot \exp\bar F_k}{\bar\lambda_k - \lambda} = 1 \, ,$$

where $S_k \overset{def}{=} S_k(\mu_k,\lambda_k), F_k \overset{def}{=} \mu_k\eta + \lambda_k\xi$. From (6.34) it follows for $\chi_1^{(-1)}$:

$$\chi_1^{(-1)} = 2\sum_k [S_k\mu_k\chi_2(-\lambda_k)\cdot \exp F_k + \bar S_k\bar\mu_k\chi_2(-\bar\lambda_k)\cdot \exp\bar F_k] \, . \qquad (6.35)$$

The system of equations for calculating $\chi_1(\mu_k)$, $\chi_1(\bar\mu_k)$ and $\chi_2(-\lambda_k)$, $\chi_2(-\bar\lambda_k)$ according to (6.34) has the form:

$$\chi_1(\mu_k) - \sum_l \frac{2S_l\mu_l\chi_2(-\lambda_l)\cdot \exp F_l}{\mu_l + \mu_k} - \sum_l \frac{2\bar S_l\bar\mu_l\chi_2(-\bar\lambda_l)\cdot \exp\bar F_l}{\bar\mu_l + \mu_k} = 1 \, ,$$

$$\chi_1(\bar\mu_k) - \sum_l \frac{2S_l\mu_l\chi_2(-\lambda_l)\cdot \exp F_l}{\mu_l + \bar\mu_k} - \sum_l \frac{2\bar S_l\bar\mu_l\chi_2(-\bar\lambda_l)\cdot \exp\bar F_l}{\bar\mu_l + \bar\mu_k} = 1 \, ,$$

$$\chi_2(-\lambda_k) - \sum_l \frac{2S_l\lambda_l\chi_1(\mu_l)\cdot \exp F_l}{\lambda_l + \lambda_k} - \sum_l \frac{2\bar{S}_l\bar{\lambda}_l\chi_1(\bar{\mu}_l)\cdot \exp \bar{F}_l}{\bar{\lambda}_l + \lambda_k} = 1 ,$$

$$\chi_2(-\bar{\lambda}_k) - \sum_l \frac{2S_l\lambda_l\chi_1(\mu_l)\cdot \exp F_l}{\lambda_l + \bar{\lambda}_k} - \sum_l \frac{2\bar{S}_l\bar{\lambda}_l\chi_1(\bar{\mu}_l)\cdot \exp \bar{F}_l}{\bar{\lambda}_l + \bar{\lambda}_k} = 1 . \quad (6.36)$$

It is supposed that in (6.36) all denominators are not equalto zero.

Let us calculate the exact solution of equation (6.1) which corresponds to the kernels R_{10} and R_{20} of the type (6.33) with one term in the sums. The system (6.36) has in this case the solution:

$$\chi_1(\mu_0) = \overline{\chi_1(\bar{\mu}_0)} = \frac{1 - i\bar{P}\cdot\mu_{0I}/\mu_{0R}}{1 - P - \bar{P} - Q} ,$$

$$\chi_2(-\lambda_0) = \overline{\chi_2(-\bar{\lambda}_0)} = \frac{1 - i\bar{P}\cdot\lambda_{0I}/\lambda_{0R}}{1 - P - \bar{P} - Q} , \quad (6.37)$$

where

$$P = S_0\cdot \exp F_0 , \quad Q = |P|^2\cdot\lambda_{0I}\mu_{0I}/(\lambda_{0R}\mu_{0R}) ,$$

$$F_0 = \lambda_0\xi + \mu_0\eta . \quad (6.38)$$

By the formula of reconstruction (6.20) using (6.35) and (6.37) one finds for the potential $u(\xi.\eta)$ in (6.1) the expression:

$$u(\xi,\eta) = -2 \times$$

$$\times \frac{\lambda_0\mu_0(P+Q)(1-\bar{P})+\bar{\lambda}_0\bar{\mu}_0(\bar{P}+Q)(1-P)+(\lambda_0\bar{\mu}_0+\bar{\lambda}_0\mu_0)(|P|^2+Q)}{(1 - P - \bar{P} - Q)^2} . \quad (6.39)$$

For providing the nonsingularity of the potential u (6.39) let us impose on the parameters λ_0, μ_0 and S_0 the conditions:

$$-\frac{\lambda_{0I}\mu_{0I}}{\lambda_{0R}\mu_{0R}} = 1 , \quad \text{i.e. } \lambda_0\bar{\mu}_0 + \bar{\lambda}_0\mu_0 = 0 , \quad (6.40)$$

$$-S_0 > 0 .$$

Under the conditions (6.40) one obtains for u from (6.39) the expression:

$$u = -\frac{2\lambda_0\mu_0 P}{(1 - P)^2} - \frac{2\bar{\lambda}_0\bar{\mu}_0\bar{P}}{(1 - \bar{P})^2} . \quad (6.41)$$

The exact solutions Ψ of (6.1) which correspond to u in (6.41) as follows from (6.12), (6.37) and (6.40) are:

$$\Psi^{(1)}(\xi,\eta) = \chi_1(\lambda)\cdot e^{\lambda\eta}\Big|_{\lambda=\mu} =$$

$$= \overline{\Psi^{(2)}(\xi,\eta)} = \frac{(1-i\bar{P}\cdot\mu_{0I}/\mu_{0R})\cdot\exp(\mu_0\eta)}{|1 - P|^2} , \quad (6.42)$$

In conclusion of this section let us make some remarks about another approaches to the spectral problem (6.1) which is known as equation of perturbed string.

Nizhnik developed [20, 21] approach to this spectral problem which is based on the classical scattering theory for the hyperbolic equations. In the frameworks of this approach [20, 21] general formulae were obtained for the transparent potentials $u(\xi,\eta)$ and corresponding to this potentials wave functions $\Psi(\xi,\eta)$. The Cauchy problem in the class of rapidly decreasing at infinity initial data was also solved.

In the paper of Boiti et al [22] another approach to the spectral problem (6.1) was developed. This approach is based on the use of nonlocal Riemann-Hilbert problem. In the paper [22] the general formulae of inverse problem for calculating potentials u and corresponding wave functions Ψ were obtained, the time evolution of the inverse problem data was found. The possibility of solution of the Cauchy problem in the class of rapidly decreasing at infinity initial data has been discussed.

7. EXACT SOLUTIONS OF 2DISG-I EQUATION VIA THE SOLUTIONS OF PERTURBED STRING EQUATION

In this section we explore the exact solutions of perturbed string equation (6.1) for obtaining via the general formulae (3.10) the exact solutions of 2DISG-I equation (1.1).Let us choose at first the boundaries $u_1(\eta,t)$ and $u_2(\xi,t)$ (potentials in (3.3), (3.4)) of the type of (6.30), i.e. with corresponding changes of the variables $(\xi,\eta) \longrightarrow (\xi,t)$ and $(\xi,\eta) \longrightarrow (\eta,t)$ in (6.30):

$$u_1(\eta,t) = -\frac{\lambda_1\mu_1}{2\mathrm{ch}^2\frac{1}{2}(\lambda_1\eta+\mu_1 t+C_1)},$$

$$u_2(\xi,t) = -\frac{\lambda_2\mu_2}{2\mathrm{ch}^2\frac{1}{2}(\lambda_2\xi+\mu_2 t+C_2)}. \tag{7.1}$$

The corresponding to these boundaries the exact solutions $X(\xi,t)$ and $Y(\eta,t)$ of equations (3.3) and (3.4) we choose, according to (6.32), in the form:

$$X(\xi,t) = \frac{\exp\frac{1}{2}(\lambda_2\xi-\mu_2 t)}{\mathrm{ch}\frac{1}{2}(\lambda_2\xi+\mu_2 t+C_2)},$$

$$Y(\eta,t) = \frac{\exp\frac{1}{2}(\lambda_1\eta-\mu_1 t)}{\mathrm{ch}\frac{1}{2}(\lambda_1\eta+\mu_1 t+C_1)}. \tag{7.2}$$

Using (3.10) and (7.2), after simple calculations we obtain corresponding exact solution $\theta(\xi,\eta,t)$ of the 2DISG-I equation:

$$\theta(\xi,\eta,t) = 4\mathrm{arctg}\frac{\rho\cdot\exp[\frac{1}{2}(\lambda_1\hat{\eta}+\lambda_2\hat{\xi})-\mu_1 t-\mu_2 t]}{2\mathrm{ch}[\frac{1}{2}\lambda_1(\hat{\eta}-\eta_0)]\cdot\mathrm{ch}[\frac{1}{2}\lambda_2(\hat{\xi}-\xi_0)]}, \tag{7.3}$$

where for convenience the wave variables

$$\hat{\eta} = \eta + \mu_1 t/\lambda_1, \quad \hat{\xi} = \xi + \mu_2 t/\lambda_2, \tag{7.4}$$

are introduced. Imposing on the parameters μ_1, μ_2 the condition:

$$\mu_1 + \mu_2 = 0, \tag{7.5}$$

one obtains from (7.3)

$$\theta(\xi,\eta,t) = 4\ \mathrm{arctg}\frac{\rho\cdot\exp[\frac{1}{2}(\lambda_1\hat{\eta}+\lambda_2\hat{\xi})]}{2\mathrm{ch}[\frac{1}{2}\lambda_1(\hat{\eta}-\eta_0)]\cdot\mathrm{ch}[\frac{1}{2}\lambda_2(\hat{\xi}-\xi_0)]}. \tag{7.6}$$

This solution describes nonlocalized object which moves on the plane (ξ,η) with velocity:

$$V = (\mu_2/\lambda_2 = -\mu_1/\lambda_2,\ \mu_1/\lambda_1). \tag{7.7}$$

Now we consider the boundaries $u_1(\eta,t)$ and $u_2(\xi,t)$ of the type of (6.41). With corresponding changes of variables one finds from (6.41) the following expressions for u (η,t) and $u_2(\xi,t)$:

$$u_1(\eta,t) = -\frac{2\lambda_1\mu_1 P_1}{(1-P_1)^2} - \frac{2\bar{\lambda}_1\bar{\mu}_1\bar{P}_1}{(1-\bar{P}_1)^2}, \qquad (7.8)$$

$$u_2(\xi,t) = -\frac{2\lambda_2\mu_2 P_2}{(1-P\ )^2} - \frac{2\bar{\lambda}_2\bar{\mu}_2\bar{P}_2}{(1-\bar{P}\ )^2},$$

where

$$P_1 = S_1\cdot\exp(\lambda_1\eta+\mu_1 t)\ ,\ P_2 = S_2\cdot\exp(\lambda_2\xi+\mu_2 t)\ . \qquad (7.9)$$

The corresponding to these boundaries exact solutions of equations (3.3) and (3.4) we choose in the form

$$X_1 = X(\xi,t),\ X_2 = \overline{X(\xi,t)},\ Y_1 = Y(\eta,t)\ ,\ Y_2 = \overline{Y(\eta,t)}\ , \qquad (7.10)$$

where

$$X(\xi,t) = \frac{(1-i\bar{P}_2\mu_{2I}/\mu_{2R})\cdot\exp(\mu_2 t)}{|1-P_2|^2},$$

$$Y(\eta,t) = \frac{(1-i\bar{P}_1\mu_{1I}/\mu_{1R})\cdot\exp(\mu_1 t)}{|1-P\ |^2}. \qquad (7.11)$$

Parameters λ_k, μ_k, S_k $(k=1,2)$ in (7.8)-(7.11) satisfy according to (6.40) to the conditions:

$$-\frac{\lambda_{kI}\mu_{kI}}{\lambda_{kR}\mu_{kR}} = 1\ ,\ \text{i. e.}\ \ \lambda_k\bar{\mu}_k + \bar{\lambda}_k\mu_k = 0\ ,$$

$$-S_k > 0\ ,\ (k=1,2)\ . \qquad (7.12)$$

Using (3.10) and (7.10), one obtains for the matrices α and β the expressions:

$$\alpha = \frac{1}{2}\begin{pmatrix} X^2, & |X|^2 - \dfrac{i\mu_{2R}\cdot\exp(2\lambda_{2R}t)}{\mu_{2I}|1-P_2|^2} \\ |X|^2 + \dfrac{i\mu_{2R}\cdot\exp(2\lambda_{2R}t)}{\mu_{2I}|1-P_2|^2}, & \bar{X}^2 \end{pmatrix},$$

$$(7.13)$$

$$\beta = \frac{1}{2}\begin{pmatrix} Y^2, & |Y|^2 - \dfrac{i\mu_{1R}\cdot\exp(2\lambda_{1R}t)}{\mu_{1I}|1-P_1|^2} \\ |Y|^2 + \dfrac{i\mu_{1R}\cdot\exp(2\lambda_{1R}t)}{\mu_{1I}|1-P_2|^2}, & \bar{Y}^2 \end{pmatrix}.$$

For the construction of the exact solutions θ of 2DISG-I equation (1.1) one can use several different mixtures (3.4) $\hat{S}(\xi,\eta,t) = \sum_{i,j}\rho_{ij}X_iY_j$ of solutions X_i,Y_j (7.10) of equations (3.3) and (3.4). Let us take for example the matrix $\rho = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Then calculations by thew formula (3.10) with the use of (7.13) give for $\theta(\xi,\eta,t)$ the expression:

$$\theta(\xi,\eta,t) =$$

$$=4\text{arctg}\frac{\left((1+i\dfrac{\mu_{2I}P_2}{\mu_{2R}})(1-i\dfrac{\mu_{1I}\bar{P}_1}{\mu_{1R}})\cdot\exp[(\mu_1+\bar{\mu}_2)t] + \text{c.c.}\right)}{\left(|1-P_1|^2|1-P_2|^2 - \dfrac{\mu_{1R}\mu_{2R}}{\mu_{1I}\mu_{2I}}\cdot\exp[2(\mu_{1R}+\mu_{2R})t]\right)}. \qquad (7.14)$$

We impose on the parameters μ_1, μ_2 the restriction

$$\mu_1 + \bar{\mu}_2 = 0 \, , \tag{7.15}$$

then

$$-\frac{\mu_{1R}\mu_{2R}}{\mu_{1I}\mu_{2I}} = \frac{\mu_{1R}^2}{\mu_{1I}^2} \equiv \delta^2 > 0$$

The formula (7.14) for θ takes the form:

$$\theta(\xi,\eta,t) =$$

$$= 4\mathrm{arctg}\frac{\left\{(1+i\mu_{2I}P_2/\mu_{2R})(1-i\mu_{1I}P_1/\mu_{1R}) + c.c.\right\}}{|1-P_1|^2|1-P_2|^2 + \delta^2} \, . \tag{7.16}$$

This solution describes nonlocalized object which moves on the plane (ξ,η) with velocity

$$V = (-\mu_{2R}/\lambda_{2R} \, , \, -\mu_{1R}/\lambda_{1R}) = (\mu_{1R}/\lambda_{2R} \, , \, -\mu_{1R}/\lambda_{1R}). \tag{7.17}$$

Another mixture (3.4) $\hat{S} = \sum_{i,j}\rho_{ij}X_iY_j$ of solutions (7.10) of equations (3.3) and (3.4) with the matrix $\rho = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ leads after some calculations by the use of (3.10), (7.10)-(7.13) and (7.15) to the following solution $\theta(\xi,\eta,t)$ of 2DISG-I equation (1.1):

$$\theta(\xi,\eta,t) =$$

$$=4\mathrm{arctg}\frac{\left(\frac{2\mu_{1I}(P_1+\bar{P}_1)}{\mu_{1R}} - \frac{2\mu_{2I}(P_2+\bar{P}_2)}{\mu_{2R}} + \frac{i\mu_{1I}\mu_{2I}(\bar{P}_1P_2 - P_1\bar{P}_2)}{\mu_{1R}\mu_{2R}}\right)}{|1 - P_1|^2|1 - P_2|^2 + \delta^2} \, . \tag{7.20}$$

This solution describes nonlocalized object which moves on the plane (ξ,η) with velocity (7.17). So, exact solutions (7.6), (7.16) and (7.20) θ of the 2DISG-I equation with generically zero values at infinity boundaries $u_1(\eta,t)$ and $u_2(\xi,t)$ are not localized.

In conclusion of this section let us consider the curious example of the exact solution of 2DISG-I equation which can be obtained using the following exact "hand" solution of eq. (6.1). It is not difficult to check that equation (6.1) with potental

$$u(\xi,\eta) = \frac{6(\xi-V\eta)^2 - 2\gamma^2}{[(\xi-V\eta)^2 + \gamma^2]} \, , \tag{7.21}$$

has the solution Ψ of the form:

$$\Psi(\xi,\eta) = \frac{1}{(\xi - V\eta)^2 + \gamma^2} \, , \tag{7.22}$$

where γ and V are some real parameters.

Taking the boundaries $u_1(\eta,t)$ and $u_2(\xi,t)$ (with corresponding changes variables) of the type (7.21):

$$u_1(\eta,t) = \frac{(\eta-V_1t)^2-2\gamma_1^2}{[(\eta-V_1t)^2+\gamma_1^2]^2} \, , \; u_2(\xi,t) = \frac{(\xi-V_2t)^2-2\gamma_2^2}{[(\xi-V_2t)^2+\gamma_2^2]^2}, \tag{7.23}$$

and corresponding exact solutions X and Y of eqs. (3.3), (3.4) in the form (7.22):

$$X(\xi,t) = \frac{1}{(\xi-V_2t)^2 + \gamma_2^2} \, , \quad Y(\eta,t) = \frac{1}{(\eta-V_1t)^2 + \gamma_1^2} \, , \tag{7.24}$$

with the use of the formulae (3.10) and (7.24)one obtains the following exact localized solution $\theta(\xi,\eta,t)$ of the 2DISG-I equation:

$$\theta(\xi,\eta,t) = 4\operatorname{arctg}\frac{\rho}{2[(\eta-V_1t)^2+\gamma_1^2]^2\cdot[(\xi-V_2t)^2+\gamma_2^2]^2}.$$

This solution describes moving on the plane (ξ,η) with velocity (V_2, V_1) localized object.

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