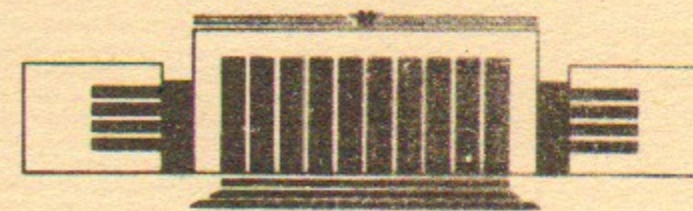8

ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ
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ИЯФ

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FUNCTIONAL INTEGRATION METHOD FOR 1D LOCALIZATION, MULTIPOINT CORRELATORS AND PERSISTENT CURRENT IN MESOSCOPIC RING AT ARBITRARY MAGNETIC FIELDS

НОВОСИБИРСК

Functional integration method for 1D localization, multipoint correlators and persistent current in mesoscopic ring at arbitrary magnetic fields

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Abstract

Starting from the Abrikosov—Ryzhkin formulation of the 1D random potential problem I find closed functional representations for various physical quantities. These functional integrals are calculated exactly without the use of any perturbative expansions. The expressions for the multipoint densities correlators are obtained. Then I evaluate the mean square dispersion of the size of localized wave functions. As a physical application of the method, I find the expectation value of the persistent current in mesoscopic ring with arbitrary magnetic flux Φ. (For small Φ this problem has been solved by O. Dorokhov). The case when the random potential has finite correlation length is considered too.

1 Introduction and definition of the model.

Anderson localization is acknowledged to be a fundamental macroscopic quantum phenomena. The localization manifests itself most evidently in one dimensional case. The essence of the effect consists in all the eigenfunction of the Hamiltonian

$$\hat{\mathcal{H}} = -\frac{d^2}{dx^2} + U(x), \qquad (1)$$

to be localized wave packets providing the potential $U(x)$ is a random function of x. (See the rigorous formulations, detailed discussion and bibliography in the book [1].) This statement remains valid in the high energy limit considered in the present paper.

The only quantities that can be calculated directly are various averages over an ensemble of potentials $U(x)$. The measure of this averaging is reconstructed from the space correlation properties of a sample at our disposal. In the simplest case of the white noise statistics it takes the form:

$$\mathcal{D}U \exp\left(-\frac{1}{2D}\int_{-L}^{L} U^2(x)\,dx\right), \qquad (2)$$

$$< U(x)U(x') > = D\delta(x - x').$$

Here $(-L, L)$ is the interval, which our system takes up. P. Anderson has shown [2] that the difference from zero of the density-density correlator

can be used as a criterion of localization of the state with energy E. The correlator can be expressed as

$$p_E(x, x') = \lim_{L \to \infty} < \sum_n \delta(E - E_n)|\Psi_n(x)|^2|\Psi_n(x')|^2 > =$$
$$= \lim_{L \to \infty} \lim_{\epsilon \to +0} \frac{\epsilon}{\pi} < |G(x, x'|E + i\epsilon)|^2 >, \qquad (3)$$

where $\Psi_n(x)$ are the eigenfunctions of $\hat{\mathcal{H}}$:

$$\hat{\mathcal{H}}\Psi_n(x) = E_n\Psi_n(x)$$

and $G(x, x'|E + i\epsilon)$ is the resolvent of $\hat{\mathcal{H}}$:

$$\left(\hat{\mathcal{H}} - E\right) G(x, x'|E + i\epsilon) = \delta(x - x'), \qquad (4)$$

Indeed, the continuous spectrum wave functions are of the order of $1/L^{1/2}$ in every space point, and the sum over n gives effectively the factor L, so $p_E(x, x') \sim 1/L \to 0$. For a homogeneous in the average potential the probability to find the state localized about a given point is $\sim 1/L$, but the wave function $\Psi_n(x)$ on its own does not depend asymptotically on L here. Hence only normalizable states contribute to $p_E(x, x')$ in the thermodynamic limit. (Assume boundary conditions in the endpoints of the interval $(-L, L)$ provide the hermiticity of $\hat{\mathcal{H}}$.)

To calculate quantities like (3) two approaches have been developed. The first one is so called "phase formalism". It allows one, in principle, to derive partial differential equations of the Fokker—Plank kind for various averages over the ensemble (2) [3, 4]. (See for the review of advances [1].) However, such an approach gives explicit results only in the regime:

$$\frac{D}{E^{3/2}} \ll 1, \qquad (5)$$

corresponding to the quasiclassical kinetics. Then another method can be used [5]: extraction and summation of the infrared-singular terms of perturbation theory series (for review see [6]). The direct performance of this program requires sophisticated constructions and tedious computations.

It has been noted in [7, 8] that the sum of leading terms in the above mentioned perturbation theory corresponds to some expectation values for spin 1/2 placed in a random magnetic field with Gaussian statistics. [1]

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[1] Conceptually the same method was used in the paper [9].

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We present here derivation of this spin model somewhat modifying the line of arguments.

For the wave function of the particle we assume the following boundary conditions:

$$\frac{d}{dx}\Psi_n(x = -L) = \Psi_n(x = L) = 0. \qquad (6)$$

The Green function (4) can be expressed in terms of the solutions $u(x)$, $\tilde{u}(x)$ of the initial-value problems:

$$\left(\hat{\mathcal{H}} - E\right) u = \left(\hat{\mathcal{H}} - E\right) \tilde{u} = 0, \qquad (7)$$

$$u'(-L) = 0, \ u(-L) = 1, \ \tilde{u}(-L) = 1, \ \tilde{u}(L) = 0;$$

$$G(x, x') = \frac{1}{W} \begin{cases} u(x)\tilde{u}(x'), & x < x' \\ u(x')\tilde{u}(x), & x' < x \end{cases}$$

Here W is the Wronskian of the functions u and $\tilde{u}$:

$$W = -u'(x)\tilde{u}(x) + u(x)\tilde{u}'(x). \qquad (8)$$

All the physical quantities of interest can be defined through one solution only, say, $u(x)$ (see below). One can introduce for the function $u(x)$ the "plane-wave components" $v_1(x)$ and $v_2(x)$:

$$v_1(x) = e^{-ikx} \left(u'(x) + iku(x)\right)$$
$$v_2(x) = -e^{ikx} \left(u'(x) - iku(x)\right), E = k^2 \qquad (9)$$
$$u(x) = \frac{1}{2ik} \left(v_1(x)e^{ikx} + v_2(x)e^{-ikx}\right),$$

so that $v_1 = 0$ $(v_2 = 0)$ for the plane wave propagating from right to left (from left to right). The equation (7) is equivalent to the following first-order matrix equation:

$$\frac{d}{dx} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} = \begin{pmatrix} U(x)/2ik, & U(x)e^{-2ikx}/2ik \\ -U(x)e^{2ikx}/2ik, & -U(x)/2ik \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} \qquad (10)$$

and reduction

$$v_1(x) = v_2^*(x). \qquad (11)$$

It is seen from (10) that the derivatives v_1 and v_2 with respect to x are small along with the potential $U(x)$. That is, $v_1(x)$ and $v_2(x)$ are changed slowly compared to $\exp(\pm ikx)$. Let us rewrite (10) in more compact notations:

$$\dot{v} = \left(i\varphi(x) + \zeta^+(x)s^- + \zeta^-(x)s^+\right) \hat{v}. \qquad (12)$$

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Here
$$\hat{v} = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}$$

$$\varphi(x) = -U(x)/k, \quad \zeta^{\pm}(x) = \pm U(x)\exp(\pm 2ikx), \tag{13}$$

$s^z = \sigma^z/2, s^{\pm} = (\sigma^x \pm i\sigma^y)/2$ are the usual spin operators and the dot denotes here and below the x-derivative. The formal solution of (12) can be written in the form:

$$\hat{v}(x) = \mathcal{T}(x, -L)\hat{v}(-L)$$

$$\mathcal{T}(x, -L) = T\exp\left(\int_{-L}^{x}(i\varphi(t)s^z + \zeta^+(t)s^- + \zeta^-(t)s^+)\,dt\right), \tag{14}$$

where the sign implies the product is ordered along the interval $(-L, L)$.

Let us consider the expectation value of some functional of $v_1(x), v_2(x)$. Expanding the -exponential (14) and this functional in a series in the fields $\varphi(t), \zeta^{\pm}(t)$ and performing the averaging over $\mathcal{D}U(x)$ we obtain the result as a series in integrals:

$$\int dt\, dt' < \varphi(t)\varphi(t') >, \quad \int dt\, dt' < \zeta^+(t)\zeta^-(t') >,$$

$$\int dt dt' < \varphi(t)\zeta^{\pm}(t') >, \quad \int dt dt' < \zeta^+(t)\zeta^+(t') >, \quad \int dt dt' < \zeta^-(t)\zeta^-(t') > . \tag{15}$$

over some domains of the order of L.

In the last three expressions we integrate fast-oscillating functions. Therefore these integrals remain restricted in their values with increasing integration intervals and fall with an increase of energy. On their turn, integrals of the first two kinds correspond to the infrared-singular contributions and grow linearly with L. Thus to leave in the perturbation theory series the terms dominating in the large L limit the correlators: $< \zeta^+\zeta^+ >, < \zeta^-\zeta^- >$, and $< \varphi, \zeta^{\pm} >$ should be neglected. It is equivalent to the assumption that the fields φ and $\zeta^{\pm}$ are statistically independent and the weight of $\mathcal{D}\zeta^{\pm}$ - averaging is phase invariant. For the white noise statistics the corresponding integration measure has the form:

$$\mathcal{D}\varphi(x)\mathcal{D}\zeta^{\pm}(x)\exp\left\{-\frac{2}{\alpha}\int_{-L}^{L}\left(a\varphi^2(x) + \zeta^+(x)\zeta^-(x)\right)\,dx\right\}, \tag{16}$$

where

$$\alpha = \frac{D}{2k^2}, \quad a = \frac{1}{8}. \tag{17}$$

We shall consider below the parameter as an arbitrary one. (It describes the phase randomization and does not enter the results.)

The formulae (14) and (16) were first proposed for the one dimensional random potential problem in the work [7]. Our presentation of it does not refer to the existence of the Fermi level. It allows one to suppose that Abrikosov-Ryzhkin model has some universal features relevant to infrared behaviour. The model can be easily generalized to random potentials with finite correlation length. It may be usable to study spectral properties of operators which are not random in the strict sense (see Conclusion).

The terms neglected in deriving of (14), (16) are decreased by a factor $\sim 1/(kL)$ compared with the kept ones. Thus this model can be applied in the study of mesoscopic systems (see section 5), since the inequality $1/(kL) \ll 1$ for sufficiently large k is compatible with $l \geq L$, where l is the mean free path in a given potential.

The authors of the paper [7] have used the formulae (14), (16) to obtain the conductivity of one dimensional metal. Unfortunately, the calculations have being carried out there by perturbation theory method leading to cumbersome constructions, which are inadequate to the simplicity of the model. In the present paper I solve this Abrikosov—Ryzhkin model exactly with the help of functional integration method. On deriving the path integral representation I find the multipoint correlators of arbitrary density powers. With the use of these expressions I evaluate the mean-square dispersion of the size of localized wave function. As a physical application of the method I calculate the mean absolute value of the persistent current in a mesoscopic ring with an arbitrary magnetic flux Φ. (For small Φ it has been found recently in [9].) In Conclusion I analyze the localization length dependence on the correlation length of the random potential. I discuss also a quantity that could play a role of the order parameter corresponding to the localization.

2 Functional representation for averaged functionals of $\hat{v}(x)$

It is impossible to express $\hat{v}(x)$ as a functional of the fields $\varphi(x)$, $\zeta^{\pm}(x)$ explicitly. The same problem arises when one undertakes an attempt to write out a closed functional representation for the partition function of quantum

Heisenberg ferromagnet. It has been solved in the works [10]-[12] and here we take advantage of the method proposed there.

The ordered exponential $T(x, -L)$ is defined by the equation

$$\dot{T} = (i\varphi(t)s^z + \zeta^+(t)s^- + \zeta^-(t)s^-)T \qquad (18)$$

and the initial condition:

$$T(x = -L, -L) = 1. \qquad (19)$$

Let us consider the operator given as a product of usual matrix exponential:

$$\tilde{T}(x, -L) = \exp\left(s^+ \psi^-(x)\right) \exp\left(is^z \int\limits_{-L}^{x} \rho\, dt\right) \times \qquad (20)$$

$$\times \exp\left(s^- \int\limits_{-L}^{x} dt\, \psi^+(t) \exp\left(\int\limits_{-L}^{t} \rho\, d\tau\right)\right) \exp\left(-s^+ \psi^-(-L)\right).$$

Here $\psi^\pm(x), \rho(x)$ are some new fields. It obeys the equation:

$$\dot{\tilde{T}} = \left\{(i\rho + 2\psi^+\psi^-)s^z + \psi^+ s^- + (\dot{\psi}^- - i\rho\psi^- - \psi^+(\psi^-)^2)s^+\right\}\tilde{T}, \qquad (21)$$

and the last factor in (20) gives equality:

$$\tilde{T}(-L, -L) = 1. \qquad (22)$$

Thus, the change of variables in the functional integral over the measure (16):

$$i\varphi = i\rho + 2\psi^+\psi^-,$$
$$\zeta^- = \dot{\psi}^- - i\rho\psi^- - \psi^+(\psi^-)^2 \qquad (23)$$
$$\zeta^+ = \psi^+$$

brings the ordered exponential $T(x, -L)$ to the form (20):

$$T(x, -L) = \tilde{T}(x, -L), \qquad (24)$$

and allows us to obtain an explicit functional integral representation for any physical quantity to be averaged. (Parametrization of $SL(2, C)$-valued functions on two-dimensional space analogous with (23) has been used also in the paper [13]). To accomplish the change of variables in functional integral,

much like the usual integrals, we need know the map (23) in one direction only: from $(\phi, \zeta^\pm)$ to $(\rho, \psi^\pm)$. The Jacobian $\mathcal{J}[\rho, \psi^\pm]$:

$$\mathcal{D}\varphi\mathcal{D}\zeta^+\mathcal{D}\zeta^- = \mathcal{J}[\rho, \psi^\pm]\mathcal{D}\rho\mathcal{D}\psi^+\mathcal{D}\psi^- \qquad (25)$$

depends on the regularization of the map (23) and on the kind of condition imposed on the field ψ^-. The latter is necessary since there is first-order derivative of ψ^- on the right-hand side of (23). The periodic boundary condition renders the map (23) irreversible. Following papers [11], [12] we consider the field $\psi^-(x)$ as obeying an initial condition:

$$\psi^-(-L) = \psi_0, \qquad (26)$$

but, unlike [11], [12], the concrete value of ψ_0 will be picked as the situation requires.

The regularization of the map (23) is determined by the physical meaning of the model: the white-noise correlator (2) is to be considered as the limit of a smooth symmetrical correlation function. Any such a regularization of the δ-function gives for the correlators:

$$\langle\zeta^+(t) \int\limits_0^t \zeta^-(t')\, dt'\rangle = \langle\zeta^-(t) \int\limits_0^t \zeta^+(t')\, dt'\rangle \qquad (27)$$

the limiting value equal to $\frac{1}{2}\frac{\alpha}{2}$ what corresponds to the extension of a definition of the step function $\theta(x)$:

$$\theta(0) = 1/2. \qquad (28)$$

The discrete version of the change of variables (23) providing the equalities (27) has the form: ($\zeta_n^\pm = \zeta^\pm(t_n)$, $\rho_n = \rho(t_n), \ldots, n = 1, \ldots, M$, $t_n = -L + \frac{2Ln}{M}$, $h = \frac{2L}{M} \to 0$, $M \to \infty$,)

$$i\phi_n = i\rho_n + \psi_n^+(\psi_n^- + \psi_{n-1}^-),$$

$$\zeta_n^- = \frac{1}{h}(\psi_n^- - \psi_{n-1}^-) - \frac{1}{2}i\rho_n(\psi_n^- + \psi_{n-1}^-) - \frac{1}{4}\psi_n^+(\psi_n^- + \psi_{n-1}^-)^2, \qquad (29)$$

$$\zeta_n^+ = \psi_n^+.$$

All the over-diagonal elements of the differential matrix of the map (29) equal zero. Then the Jacobian $\mathcal{J}$, being the determinant of this matrix, is equal to

the product of the diagonal elements only:

$$\mathcal{J} = \text{const} \exp\left(-\frac{i}{2}\int_{-L}^{L} \rho\, dt\right). \tag{30}$$

Making the substitution (23) into the measure (16) with the use of the expressions (25), (30) we obtain the weight of averaging over the fields $(\rho, \psi^{\pm})$:

$$N\mathcal{D}\rho\mathcal{D}\psi^{+}\mathcal{D}\psi^{-} \exp\left(-S(\rho, \psi^{\pm})\right),$$

$$S(\rho, \psi^{\pm}) =$$

$$= \frac{2}{\alpha}\int_{-L}^{L} dx \left(a\rho^2 + \psi^{+}\psi^{-} - (1+4a)i\rho\psi^{+}\psi^{-} - (1+4a)(\psi^{+}\psi^{-})^2\right) +$$

$$+ \frac{i}{2}\int_{-L}^{L} dx\rho. \tag{31}$$

Here N is a normalization constant depending on αL.

In calculating of the Jacobian (25) we were considering $(\rho, \psi^{\pm})$ and $(\phi, \zeta^{\pm})$ as sets of independent complex variables or, in other words, as distinct coordinate systems in the whole space $\mathcal{C}^{3M}$ of the fields' configurations. The conditions

$$Im\varphi = 0, \zeta^{+} = (\zeta^{-})^{*}, \tag{32}$$

being from the outset embedded in the model specify the surface Σ in $\mathcal{C}^{3M}$ along which the differential form $\mathcal{D}\varphi \wedge \mathcal{D}\zeta^{+} \wedge \mathcal{D}\zeta^{-}$ or $\mathcal{D}\rho \wedge \mathcal{D}\psi^{+} \wedge \mathcal{D}\psi^{-}$ is integrated. From the point of view of the coordinates set $(\rho, \psi^{\pm})$ the equation (32) for Σ is implicit. According to the Cauchu—Poincare theorem the integration surface can be deformed in an arbitrary way in the convergence domain while an analytical function is integrated. There exists a continuous family of surfaces (homotopy) situated as a unit in "perturbative" convergence domain, which includes both the surfaces Σ and the "standard" one Σ':

$$\Sigma' = \left\{Im\rho = 0, \psi^{+} = (\psi^{-})^{*}\right\}. \tag{33}$$

The word "perturbative" means here that we check the convergence in every order of the perturbation theory expansion. (This homotopy is presented explicitly in the paper [10].) Thus, treating functional integral as the sum of

perturbation theory series [14] we can replace the surface of integration Σ by the standard one Σ'.

However, to pass from Σ to Σ' the expressions being averaged (and not just the action) should be written in the form allowing the direct analytical continuation from the surface Σ. It means constructively that the definition of any physical quantity in terms of the matrix elements of $\mathcal{T}(x, -L)$ must contain no complex conjugations.

3 The density-density correlator expression in terms of the functions $\hat{v}(x)$

The formula (3) defines the correlator $p_E(x, x')$ in terms of the singular at $\epsilon \to +0$ part of the Green function $G(x, x'|E+i\epsilon)$. When we use the representation (8) the singularity appears owing to zeros of the Wronskian $W(E)$ on the real axis. Neglecting ϵ in the numerator (8) and substituting

$$W(E \pm i\epsilon) = W(E) \pm i\epsilon W'(E) \tag{34}$$

into the denominator we obtain:

$$p_E(x, x') = \left\langle \frac{u^2(x)\tilde{u}^2(x')}{W'(E)}\delta(W)\right\rangle, x' > x. \tag{35}$$

$W(E)$ does not depend on x and, thus, we may put in (8) $x = L$:

$$W = u(L) \tag{36}$$

Being in the product with $\delta(W) = \delta(u(L))$ the solution $\tilde{u}(x)$ is proportional to $u(x)$. The proportionality coefficient is determined by the conditions (7). So, for an arbitrary functional $\mathcal{F}[\tilde{u}(x)]$ the following equality takes place:

$$\mathcal{F}[\tilde{u}(x)]\delta(u(L)) = \mathcal{F}\left[\frac{u(x)}{u'(L)}\right]\delta(u(L)). \tag{37}$$

According to (36) we can express $W'(E)$ in terms of the derivative of $u(x)$ with respect to the energy E:

$$W'(E) = \frac{\partial u(L)}{\partial E}. \tag{38}$$

The function $g(x) = \partial u(x)/\partial E$ obeys the equation

$$\left(-\frac{d^2}{dx^2} + U(x) - E\right)g(x) = u(x), \tag{39}$$

and the initial condition:

$$g(x = -L) = g'(x = -L) = 0. \tag{40}$$

The substitution $g(x) = q(x)u(x)$ leads to the first-order equation for $q'(x)$; its solution gives us:

$$g(x) = u(x) \int_{-L}^{x} \frac{dy}{u^2(y)} \int_{-L}^{y} dy_1 u^2(y_1), \tag{41}$$

and:

$$\frac{1}{W'(E)} \delta(u(L)) = \frac{1}{g(L)} \delta(u(L)) = \frac{u'(L)}{\int_{-L}^{L} u^2(y)\, dy} \delta(u(L)). \tag{42}$$

Thus the correlator $p_E(x, x')$ can be written via $u(x)$ as follows:

$$p_E(x, x') = \left\langle \frac{u^2(x)u^2(x')}{u'(L)\int_{-L}^{L} u^2(y)\, dy} \delta(u(L)) \right\rangle, \quad x' > x. \tag{43}$$

In the high-energy limit (5) we can get rid of the δ-function and obtain a simple formula for $p_E(x, x')$ in terms of slowly varying amplitudes $v_{1,2}(x)$. Indeed, in neighbourhood of any given point x_0 the function $u(x)$ can be written in the form:

$$u(x) = u_{sl}(x)\sin(kx + \delta) \tag{44}$$

, where the envelope $u_{sl}(x)$ and the phase δ vary only slightly over distances of the order $\sim 1/k$. Let us average the expression (43) over the interval ΔL of the right endpoints' positions of our "space" $(-L, L)$:

$$\tilde{p}_E(x, x') = \frac{1}{\Delta L} \int_{L}^{L+\Delta L} p_E(x, x')\, dL \tag{45}$$

$$\frac{1}{k} \ll \Delta L \ll \frac{2}{\alpha} \equiv l. \tag{46}$$

(Here we introduce the standard notation l for the localization length.) In the thermodynamic limit the functions $\tilde{p}_E(x, x')$ and $p_E(x, x')$ coincide. On the other hand, the value of $u(x)$ in a given point, by the construction, does not depend on the right endpoints position. The integral in the denominator of (43) is determined by the envelope $u_{sl}(x)$ only. The variation of L from L to $L + \Delta L$ does affect it asymptotically. The averaging (45) is sufficient only for the factor $\delta(u(L))/u'(L)$. Since the conditions (46) mean that $u_{sl}(x)$ can be considered as a constant in the averaging interval, we obtain:

$$\frac{1}{\Delta L} \int_{L}^{L+\Delta L} dL \frac{1}{u'(L)} \delta(u(L)) = \frac{1}{\pi k\, u_{sl}^2(L)}, \tag{47}$$

We can derive similarly the relationship between $u_{sl}^2(x)$ and $u^2(x)$, in particular:

$$u_{sl}^2(L) \approx \frac{2}{\Delta L} \int_{L}^{L+\Delta L} dL\, u^2(L). \tag{48}$$

Substituting into (43)-(48) the expression of $u(x)$ via $\hat{v}(x)$, neglecting the contributions vanishing in the $k \to \infty$ limit and keeping in the numerator of (43) the "resonance" terms only, we obtain:

$$p_E(x, x') \approx \tilde{p}_E(x, x') \approx \frac{1}{2\pi k} \left\langle \frac{v_1(x)v_2(x)v_1(x')v_2(x')}{v_1(L)v_2(L)\int_{-L}^{L} v_1(y)v_2(y)\, dy} \right\rangle, \quad x' > x. \tag{49}$$

The "non-resonance" terms containing the oscillating factors $\exp \pm 2ik(x - x')$ will result in exponentially small in αL contributions and, thus, can be neglected.

4 Functional integration for correlators of the density -density type

The form of the expression (49) allows direct analytical continuation from the surface $v_1 = (v_2)^*$ over the functions $v_{1,2}(x)$. It is not restrictive to put $\exp(ikL) = 1$ in the thermodynamic limit. (Only if the formula (49) has been obtained already!) Initial condition for $\hat{v}(x)$ takes the form:

$$\hat{v}(-L) = ik \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{50}$$

To find $\hat{v}(x)$ we substitute into (14) the expression (20) for the evolution operator $\mathcal{T}(x, -L)$ picking the quantity ψ_0 to be equal to 1:

$$\psi_0 = \psi^-(-L) = 1. \tag{51}$$

It yields the equality:

$$\hat{v}(x) = \exp\left(-\frac{i}{2}\int\limits_{-L}^{x}\rho\,dt\right)\left(\begin{array}{c}\psi^-(x)\\1\end{array}\right), \tag{52}$$

and the expression for $p_E(x, x')$:

$$p_E(x, x') \approx \frac{1}{2\pi k}\left\langle \frac{\psi^-(x)\psi^-(x')\exp\left(-i\int\limits_{-L}^{x}\rho\,dt - i\int\limits_{-L}^{x'}\rho\,dt + i\int\limits_{-L}^{L}\rho\,dt\right)}{\psi^-(L)\int\limits_{-L}^{L}\psi^-(y)\exp\left(-i\int\limits_{-L}^{y}\rho\,dt\right)dy}\right\rangle,$$

$$x' > x, \tag{53}$$

Here the averaging over $\mathcal{D}\rho\mathcal{D}\psi^+\mathcal{D}\psi^-$ is carried out with the weight (31). To calculate this functional integral we employ a trick similar to the so-called "bosonization" in the field theory models [15]. Using the identity:

$$\exp\left(-S(\rho, \psi^\pm)\right) = \int \mathcal{D}\eta \exp\left(-\tilde{S}(\eta, \rho, \psi^\pm)\right),$$

$$\tilde{S}(\eta, \rho, \psi^\pm) =$$

$$= \frac{2}{\alpha}\int\limits_{-L}^{L}dx\left((1+4a)\eta^2 + a\rho^2 + \psi^+\dot\psi^- + (1+4a)(2\eta - i\rho)\psi^+\psi^-\right) + \frac{i}{2}\int\limits_{-L}^{L}dx\rho. \tag{54}$$

and the gauge transformation:

$$\psi^\pm(x) = \chi^\pm \exp\left(\pm(1+4a)\int\limits_{-L}^{x}dt\,(2\eta - i\rho)\right), \tag{55}$$

we get rid of the non-linear terms in the action. The Jacobian of the rotation (55) is equal to

$$\mathcal{J}_R = const\,\exp\left(-\frac{1+4a}{2}\int\limits_{-L}^{L}(2\eta - i\rho)\,dt\right), \tag{56}$$

where the regularization (29) is taken into account. The fields η and ρ enter the equation (53) via the combination

$$\int\limits_{-L}^{x}(2(1+4a)\eta - 4ia\rho)\,dt$$

only. It is natural to consider it as a new integration variable:

$$\dot\xi = 2(1+4a)\eta - 4ia\rho,$$

$$\xi(-L) = 0 \tag{57}$$

$$\mathcal{D}\rho\mathcal{D}\eta = const\,\mathcal{D}\rho\mathcal{D}\xi$$

Then the Gaussian $\mathcal{D}\rho$-integration can be done easily and we obtain the expressions for the measure:

$$const\,\mathcal{D}\xi\mathcal{D}\chi^+\mathcal{D}\chi^- \exp\left(-\frac{1}{2\alpha}\int\limits_{-L}^{L}dx\dot\xi^2 - \frac{2}{\alpha}\int\limits_{-L}^{L}dx\chi^+\dot\chi^- - \frac{\xi(L)}{2}\right) \tag{58}$$

and for the quantity to be averaged:

$$p_E(x, x') = \frac{1}{2\pi k}\left\langle \frac{\chi^-(x)\chi^-(x')\exp\left(-\xi(x) - \xi(x') + \xi(L)\right)}{\chi^-(L)\int\limits_{-L}^{L}\chi^-(y)e^{-\xi(y)}dy}\right\rangle, x' > x, \tag{59}$$

(The asymptotic equality in the limit (5) is assumed.) The initial condition for the field $\chi^-(x)$ follows from (51):

$$\chi^-(-L) = 1. \tag{60}$$

It means that $\chi^-(x)$ contains both the fluctuating part $\chi_f^-(x)$ and the regular one:

$$\chi^-(x) = 1 + \chi_f^-(x), \chi_f^-(-L) = 0. \tag{61}$$

The component $\chi_f^-(x)$ does not contribute to $p_E(x, x')$ because the conjugated field does not appear in the broken brackets in (59). Thus, the only averaging over the field $\xi(x)$ remains. Its weight has the form

$$\exp\left(-\frac{\alpha L}{4}\right)N'\mathcal{D}\xi\exp\left(-\frac{1}{2\alpha}\int\limits_{-L}^{L}dx\dot\xi^2 - \frac{\xi(L)}{2}\right) \tag{62}$$

Here the normalization constant N' is determined by the quadratic in $\dot\xi$ term of the action:

$$N'\mathcal{D}\xi \exp\left(-\frac{1}{2\alpha}\int\limits_{-L}^{L} dx\,\dot\xi^2\right) = 1. \qquad (63)$$

The factor $\exp(-\alpha L/4)$ provides the equality $<1>=1$ for the averaging over the entire measure (62). Thus we arrive at the following path integral for the correlator $p_E(x,x')$:

$$p_E(x,x') =$$

$$= \frac{1}{2\pi k} N' \exp\left(-\frac{\alpha L}{4}\right) \int\limits_{\xi(-L)=0} \mathcal{D}\xi \exp\left(-\frac{1}{2\alpha}\int\limits_{-L}^{L} dt\,\dot\xi^2 - \frac{\xi(L)}{2}\right) \times$$

$$\times \exp\left(-\xi(x)-\xi(x')+\xi(L)\right)\left\{\int\limits_{-L}^{L}\exp\left(-\xi(t)\right)dt\right\}^{-1} =$$

$$= \frac{N'}{4\pi k\alpha}\exp\left(-\frac{\alpha L}{4}\right)\times$$

$$\times \int\limits_{0}^{\infty} d\lambda \int\limits_{\xi(-L)=0}\mathcal{D}\xi\exp\left(-\frac{1}{2\alpha}\int\limits_{-L}^{L}dt\left(\dot\xi^2+\lambda e^{-\xi}\right)+\frac{\xi(L)}{2}\right)\times$$

$$\times \exp\left(-\xi(x)-\xi(x')\right) =$$

$$= N'(4\pi k\alpha)^{-1}\int\limits_{-\infty}^{+\infty}\!\!d\sigma d\sigma'\exp\left(\frac{\sigma+\sigma'}{2}\right)\times \qquad (64)$$

$$\times \int\limits_{\xi(-L)=\sigma',\xi(L)=\sigma}\mathcal{D}\xi\exp\left(-\frac{1}{2\alpha}\int\limits_{-L}^{L}dt\left(\dot\xi^2+e^{-\xi}\right)-\frac{\alpha L}{4}\right)e^{-\xi(x)-\xi(x')}.$$

The last equality has been attained by changing of variables:

$$\lambda = e^{-\sigma'}, \xi \to \xi - \sigma' \qquad (65)$$

and by separating the integrals over the values of $\xi(t)$ in the endpoints $t = L$ and $t = -L$. The final path integral in (64) is of the Feynmann-Kac type [16] and it is equal to the following matrix element:

$$p_E(x,x') = \exp\left(-\frac{\alpha L}{4}\right)(4\pi k\alpha)^{-1}\times \qquad (66)$$

$$\times \langle e^{\xi/2}|e^{-(L-x')\hat H}e^{-\xi}e^{-(x'-x)\hat H}e^{-\xi}e^{-(x+L)\hat H}|e^{\xi/2}\rangle,$$

with the Hamiltonian:

$$\hat H = \frac{\alpha}{2}\partial_\xi^2 - \frac{1}{2\alpha}e^{-\xi}. \qquad (67)$$

The function $e^{\xi/2}$ increase when $\xi \to \infty$ and, consequently, it cannot be represented as a linear combination of the eigenfunctions of $\hat H$:

$$\hat H f_\nu(\xi) = -\frac{\alpha}{2}\nu^2 f_\nu(\xi), \quad f_\nu(\xi) = \frac{2}{\pi}\sqrt{\nu\sinh 2\pi\nu}\,K_{2i\nu}\left(\frac{2}{\alpha}e^{-\xi/2}\right), \qquad (68)$$

$$\langle f_\nu|f_{\nu'}\rangle = \delta(\nu-\nu').$$

Still, explicit solution of the corresponding evolution equation leads to the following asymptotic relation:

$$\exp(-T\hat H)e^{\xi/2} \longrightarrow \exp\left(\frac{\alpha T}{8}\right)\Upsilon_0(\xi) = \frac{2}{\alpha}\exp\left(\frac{\alpha T}{8}\right)K_1\left(\frac{2}{\alpha}e^{-\xi/2}\right). \qquad (69)$$

Here $K_\mu(z)$ is the standard notation for the modified Bessel function. The function

$$\Upsilon_0(\xi)e^{-\xi} = \frac{2}{\alpha}K_1\left(\frac{2}{\alpha}e^{-\xi/2}\right)e^{-\xi}$$

in its turn can be expanded in terms of the complete set (68). Thus, the correlator $p_E(x,x')$ is equal to:

$$p_E(x,x') =$$

$$= \frac{1}{4\pi k\alpha}\exp\left(-\frac{\alpha|x-x'|}{8}\right)\langle\Upsilon_0(\xi)e^{-\xi}|\exp\left(-|x-x'|\hat H\right)|\Upsilon_0(\xi)e^{-\xi}\rangle =$$

$$= \frac{\alpha}{\pi^3 k}\exp\left(-\frac{\alpha|x-x'|}{8}\right)\times$$

$$\times \int\limits_{0}^{\infty}d\nu\,\nu\,\mathrm{sh}2\pi\nu\exp\left(-\frac{\alpha\nu^2}{2}|x-x'|\right)\left\{\int\limits_{0}^{\infty}dy\,y\,K_1(y)K_{2i\nu}(y)\right\}^2 =$$

$$= \frac{\alpha\pi}{2k}\exp\left(-\frac{\alpha|x-x'|}{8}\right)\int\limits_{0}^{\infty}\frac{d\nu\,\nu\,\mathrm{sh}\pi\nu}{\mathrm{ch}^3\pi\nu}\left(\nu^2+\frac{1}{4}\right)^2\exp\left(-\frac{\alpha\nu^2}{2}|x-x'|\right) \quad (70)$$

The formula (70) up to the redefinition $\alpha/2 = l^{-1}$ coincides with the well known result [1],[6]. This method allows us to compute the high-order correlators as well. For example: $(x_1 < x_2 < \ldots < x_{2m}, m > 1)$

$$2\pi k\,p_E^{(q,m)}(x_1,x_2,\ldots x_{2m-1},x_{2m}) =$$

$$= 2\pi k \lim_{L \to \infty} \langle \sum_n \delta(E - E_n) |\Psi_n(x_1)|^{2q} |\Psi_n(x_2)|^{2q} \times$$

$$\times \ldots \times |\Psi_n(x_{2m-1})|^{2q} |\Psi_n(x_{2m})|^{2q} \rangle =$$

$$= \frac{2^{2qm}(qm-1)! \, k}{(2qm-2)!} \times$$

$$\times \lim_{L \to \infty} \lim_{\epsilon \to +0} \epsilon^{2qm-1} \langle |G(x_1,x_2|E+i\epsilon)\ldots G(x_{2m-1},x_{2m}|E+i\epsilon)|^{2q} \rangle =$$

$$= \left\langle \left(\prod_{j=1}^{m} u^{2q}(x_{2j-1}) \tilde{u}^{2q}(x_{2j}) \right) \frac{2\pi k}{(W'(E))^{2qm-1}} \delta(W(E)) \right\rangle = \quad (71)$$

$$= \left\langle \left(\prod_{j=1}^{m} u^{2q}(x_{2j-1}) u^{2q}(x_{2j}) \right) \frac{2\pi k}{\left(\int_{-L}^{L} u^2(y)\,dy \right)^{2qm-1} u'(L)} \delta(u(L)) \right\rangle \approx$$

$$\approx \left\langle \prod_{j=1}^{2m} (v_1(x_j)v_2(x_j))^q \left(\int_{-L}^{L} v_1(y)v_2(y)\,dy \right)^{-2qm+1} (v_1(L)v_2(L))^{-1} \right\rangle =$$

$$= \frac{N'}{(2\alpha)^{2qm-1}(2qm-2)!} \exp\left(-\frac{\alpha L}{4} \right) \times$$

$$\times \int_{-\infty}^{+\infty} d\sigma d\sigma' e^{(\sigma+\sigma')/2} \int_{\xi(-L)=\sigma', \xi(L)=\sigma} \mathcal{D}\xi \exp\left(-\frac{1}{2\alpha} \int_{-L}^{L} dt \left(\dot{\xi}^2 + e^{-\xi} \right) \right) \times$$

$$\times \exp\left(-\sum_{j=1}^{2m} q\xi(x_j) \right) =$$

$$= \frac{1}{(2\alpha)^{2qm-1}(2qm-2)!} \exp\left(-\frac{\alpha(x_{2m}-x_1)}{8} \right) \times$$

$$\times \langle \Upsilon_0(\xi)| \prod_{j=1}^{2m-1} e^{-q\xi} e^{-(x_{j+1}-x_j)\hat{H}} |\Upsilon_0(\xi)e^{-q\xi} \rangle =$$

$$= \left(\frac{\alpha}{8}\right)^{2mq-1} 2^{4(q+m)-6} \frac{\pi^{m+1}}{\Gamma(2mq-1)(\Gamma(2q))^2} \left(\frac{\Gamma(q)}{\Gamma(q+1/2)} \right)^{2m-2} \times$$

$$\times \exp\left(-\frac{\alpha(x_{2m}-x_1)}{8} \right) \times$$

$$\times \int_0^\infty \int_0^\infty \prod_{j=1}^{2m-1} d\nu_j \exp\left(-\frac{\nu_j^2 \alpha}{2} \Delta x_j \right) \nu_j \, \mathrm{sh} 2\pi\nu_j \, P^{(q)}(\nu_1) P^{(q)}(\nu_{2m-1}) \times$$

$$\times \prod_{s=1}^{2m-2} Q^{(q)}(\nu_s, \nu_{s+1}),$$

where $\Delta x_j = x_{j+1} - x_j$ and the functions $P^{(q)}(\nu)$ and $Q^{(q)}(\nu, \nu')$ are defined as follows:

$$P^{(q)}(\nu) = \frac{1}{\cosh^2 \pi\nu} \left[(q-1/2)^2 + \nu^2 \right] \prod_{j=1}^{q-1} \left((j-1/2)^2 + \nu^2 \right)^2, \quad (72)$$

$$Q^{(q)}(\nu, \nu') = \frac{\nu^2 - \nu'^2}{\cosh 2\pi\nu - \cosh 2\pi\nu'} \prod_{j=1}^{q-1} \left[1 + \frac{2}{j^2}(\nu^2 + \nu'^2) + \frac{1}{j^4}(\nu^2 - \nu'^2) \right]. \quad (73)$$

The expression for $p_E^{(q,1)}(x_1, x_2)$ can be obtained from (71) by the formal substitution 1 for the product from $s = 1$ to $s = 2m-2$ and by putting $m = 1$ in the remaining integral.

As one of possible applications of the formulae (71)-(72) let us consider the dispersion of the sizes of localized wave functions. It is seen from (71) that the distant exponential asymptotics of the probability distributions does not fluctuate. On the other hand, it would be natural to define the wave packet size R_E as some integral property. For example, let us define R_E as:

$$R_E^{-1} = \frac{4}{3} \int_{-L}^{L} dx \, |\psi(x)|^4. \quad (74)$$

The coefficient 4/3 cancels the mean value of the fast oscillating factor

$$\sin^4(kx + \delta)$$

(see (44)). Then we have:

$$\langle R_E^{-1} \rangle = \frac{4}{3\rho(E)} \int_{-L}^{L} dx \, p_E(x,x) = \frac{2\alpha}{9} = \frac{4}{9l}. \quad (75)$$

Here $\rho(E)$ is the density of states at large E:

$$\rho(E) = \frac{L}{\pi k}. \quad (76)$$

The expectation value of the square of R_E^{-1} can be found from the correlation function $\langle |\psi(x)\psi(x')|^4 \rangle$:

$$\langle R_E^{-2} \rangle = \frac{32}{9\rho(E)} \int_{-L}^{L} dx \int_{-L}^{x} dx' \, p_E^{(2,1)}(x',x). \quad (77)$$

Using the explicit expression for $p_E^{(2,1)}(x', x)$:

$$p_E^{(2,1)}(x', x) = \frac{\pi \alpha^3}{576} \times$$

$$\times e^{-\alpha(x-x')/8} \int_0^\infty d\nu \, \text{cxp} \left(-\frac{\alpha\nu^2}{2}(x - x') \right) \frac{\text{sh}\,\pi\nu}{\text{ch}^3\pi\nu} \nu \left(\frac{9}{4} + \nu^2 \right)^2 \left(\frac{1}{4} + \nu^2 \right)^4.$$

(78)

and evaluating the integrals over dx' and $d\nu$ we obtain:

$$\langle R_E^{-2} \rangle \approx 0.23 \frac{1}{l^2}.$$

(79)

With (75) it gives us the mean square relative dispersion of R_E^{-1}:

$$\frac{\langle R_E^{-2} \rangle - \langle R_E^{-1} \rangle^2}{\langle R_E^{-2} \rangle} \approx 0.13.$$

(80)

5 Mean current in 1D mesoscopic ring with the magnetic flux Φ

Let us consider one dimensional metal ring in transverse magnetic field. The expectation value of the current operator for one electron stationary state becomes non-zero and the energy receives -odd term. Then the Fermi levels for left and right directions of the mean velocity turn out to be shifted one about another. As a result, a persistent current flows along this ring in the ground state [17]. There are arguments ([19], [9]) that the total current is about the mean current I corresponding to one-electron state on the Fermi level. We will assume the ring size $2L$ to be comparable the mean free path. Then the localization effects do not lead to the total suppression of I, but I is rendered to be a nontrivial function of the magnetic field (see below). (The case of ordered inhomogeneous conductor has been considered in the paper [20]).

There exists the gauge by which the wave function of an electron in the ring with the magnetic flux Φ obeys the boundary condition:

$$\psi(L) = \exp(2\pi i\Phi)\,\psi(-L),$$

(81)

and the Hamiltonian has the previous form (1). The mean absolute value of current corresponding to a state with energy E can be represented in the limit (5) as follows [18], [9]:

$$I = \left\langle \frac{\pi k}{L} \sum_n \delta(E - E_n)|j_n| \right\rangle,$$

$$j_n = -\frac{1}{2\pi} \frac{\partial E_n}{\partial \Phi}.$$

(82)

($h = c = e = 1$, the magnetic flux quantum is equal to 1.) The condition (81) is nonlocal, therefore, the formula (82) for I cannot be rewritten in terms of functions like $u(x)$, $\tilde{u}(x)$ of Sections 1,2. It has been shown in [9], however, that I can be expressed directly via the elements of the $\mathcal{T}$-matrix (14). It is worth noting that we have functional representation just for them.

Indeed, by the construction, the matrix $\mathcal{T} \equiv \mathcal{T}(-L, L)$ satisfies the "unitarity" conditions:

$$\sigma^z \mathcal{T}^\dagger \sigma^z = \mathcal{T}^{-1}, \quad \det \mathcal{T} = 1.$$

(83)

Therefore we can parametrize in the following way:

$$\mathcal{T} = \begin{pmatrix} \cosh \Gamma e^{i\alpha_s}, & \sinh \Gamma e^{i\beta_s} \\ \sinh \Gamma e^{-i\beta_s}, & \cosh \Gamma e^{-i\alpha_s} \end{pmatrix},$$

(84)

where Γ, α_s and β_s are slowly varying real functions of L. The mapping of the initial data space at the point $x = -L$ into the space of solutions of the equation (7) at the point $x = L$ is realized in the basis $(u' \pm iku)$ by the transfer-matrix :

$$T = \exp(ikL\sigma^z)\mathcal{T}\exp(ikL\sigma^z) =$$

$$= \begin{pmatrix} \cosh \Gamma e^{i(\alpha_s + kL)}, & \sinh \Gamma e^{i\beta_s} \\ \sinh \Gamma e^{-i\beta_s}, & \cosh \Gamma e^{-i(\alpha_s + kL)} \end{pmatrix}.$$

(85)

The condition (81) is equivalent for the matrix to have the eigenvalue $e^{i\theta}$, $\theta = 2\pi\Phi$:

$$\det(T - e^{i\theta}) = 1,$$

or

$$\tau(E) \equiv \cosh \Gamma \cos(\alpha_s + kL) = \cos\theta.$$

(86)

The last equation defines the set of the energy $E = k^2$ values, and, using the formula (82) for j_n we obtain:

$$\sum_n \delta(E - E_n)|j_n| = \sum_n \delta(E - E_n) \left| \frac{\sin\theta}{\tau'(E)} \right| = \delta(\tau(E) - \cos\theta)|\sin\theta|.$$

(87)

We can get rid of the δ - function in much the same manner as in the previous section. Averaging over the interval ΔL $(1/k \ll \Delta L \ll l)$ of the (half-) circumferences L we find:

$$I = \langle \frac{\pi k}{L} \delta \left(\tau(E) - \cos\theta \right) \rangle |\sin\theta| \approx \left\langle \frac{1}{\sqrt{\sinh^2\Gamma + \sin^2\theta}} \right\rangle \left| \frac{k}{L} \sin\theta \right| =$$

$$= \left| \frac{k}{L} \sin\theta \right| \frac{1}{\sqrt{\pi}} \left\langle \int\limits_{-\infty}^{+\infty} d\mu \exp\left(-\mu^2(\sinh^2\Gamma + \sin^2\theta)\right) \right\rangle. \tag{88}$$

It is important that $\sinh^2\Gamma$ can be rewritten so that the direct analytical continuation from the surface Σ becomes possible:

$$\sinh^2\Gamma = (1,0)\mathcal{T}^t s^- \mathcal{T} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{89}$$

Here the sign t denotes, as usually, transposition, and complex conjugation does not enter this formula. For $\sinh^2\Gamma$ to have the most simple form in terms of the fields $\rho, \psi^\pm$ the field ψ^- should obey zero-valued initial condition:

$$\psi^-(-L) = 0. \tag{90}$$

Then substituting (20) into (89) we obtain:

$$\sinh^2\Gamma = \psi^-(L) \int\limits_{-L}^{L} dt\, \psi^+(t) \exp\left(-i\int\limits_{t}^{L} \rho\, d\tau\right). \tag{91}$$

The right-hand side of (91) is bilinear in the fields $\psi^\pm$ but it is non-local. It is convenient to start from the transformation of the Hubbard-Stratonovich kind:

$$I = \left| \frac{k\sin\theta}{\sqrt{\pi} L} \right| \int\limits_{-\infty}^{+\infty} d\mu \exp\left(-\mu^2\sin^2\theta\right) \times$$

$$\times \left\langle \exp\left\{-\mu^2\psi^-(L) \int\limits_{-L}^{L} dt\, \psi^+(t) \exp\left(-i\int\limits_{t}^{L} \rho\, d\tau\right)\right\} \right\rangle =$$

$$= \left| \frac{k\sin\theta}{\sqrt{\pi} L} \right| \int\limits_{-\infty}^{+\infty} d\mu \int dz\, dz^* \exp\left(-\mu^2\sin^2\theta - |z|^2\right) \times$$

$$\times \left\langle \exp\left\{-i\mu z\psi^-(L) - i\mu z^* \int\limits_{-L}^{L} dt\, \psi^+(t) \exp\left(-i\int\limits_{t}^{L} \rho\, d\tau\right)\right\} \right\rangle. \tag{92}$$

turning the exponent to the linear in $\psi^\pm$ combination. Retracing the same path (54)-(56) as in the previous section, and introducing the variable $\xi(t)$ in a manner like (57):

$$\dot{\xi} = -2(1+4a)\eta + 4ia\rho, \cdot$$
$$\xi(L) = 0 \tag{93}$$
$$\mathcal{D}\rho\mathcal{D}\eta = \text{const}\, \mathcal{D}\rho\mathcal{D}\xi$$

we bring the $\mathcal{D}\rho\mathcal{D}\chi$-integration to the Gaussian form. Performing it we find:

$$I = \exp(-\frac{\alpha L}{4}) \left| \frac{k}{L}\sin\theta \right| \frac{1}{\pi^{3/2}} N' \int\limits_{-\infty}^{+\infty} d\mu \int dz\, dz^* \exp\left(-\mu^2\sin^2\theta - |z|^2\right) \times$$

$$\times \int\limits_{\xi(L)=0} \mathcal{D}\xi \exp\left(-\frac{1}{2\alpha}\int\limits_{-L}^{L} dx \left(\dot{\xi}^2 + \alpha^2\mu^2|z|^2 e^{-\xi}\right) - \frac{\xi(-L)}{2}\right). \tag{94}$$

Let us change μ for the integration variable σ:

$$\alpha^2\mu^2|z|^2 = e^{-\sigma}, \tag{95}$$

and shift the trajectory $\xi(x)$ by $-\sigma$:

$$\xi(x) \to \xi(x) - \sigma.$$

Then the integral over $(z\, z^*)$-plane is calculated exactly and we obtain the representation of I as the matrix element:

$$I = \exp(-\frac{\alpha L}{4}) \left| \frac{k}{\sqrt{\pi}\alpha L}\sin\theta \right| \langle \Upsilon_2(\xi)| \exp(-2L\hat{H})|\Upsilon_1(\xi)\rangle, \tag{96}$$

Here $\hat{H}$ is defined in (67), and the functions $\Upsilon_{1,2}(\xi)$ have the following forms:

$$\Upsilon_1(\xi) = \exp\left(-\frac{\xi}{2}\right) \tag{97}$$

and

$$\Upsilon_2(\xi) = \frac{\sqrt{\pi}}{2} \exp\left\{-\frac{2}{\alpha}\exp\left(-\frac{\xi}{2}\right)|\sin\theta|\right\}. \tag{98}$$

Using the expansion in terms of the complete set (68) and the integral representation for $K_{2i\nu}(y)$ we find the explicit formula for the current I: $(l^{-1} = \alpha/2)$

$$I = \frac{k}{L}\left(\frac{l}{L}\right)^{1/2} \exp\left(-\frac{L}{4l}\right) \sin^2\theta \int\limits_{-\infty}^{+\infty} \frac{dt\, \cosh t}{\sin^2\theta + \sinh^2 t} \exp\left(-\frac{l}{L}t^2\right) \tag{99}$$

When $\sin^2\theta$ runs from 0 to 1, the value of I increase monotonically from 0 to I_{max}. The first order of the Taylor series in θ agrees with the result of [9]. It is worth noting that the parameter l/L determines not only the absolute value of I, but its dependence on the magnetic field as well:

$$\frac{I}{I_{max}} = \frac{\sin^2\theta}{\int\limits_{-\infty}^{+\infty} \frac{dy}{\cosh y} \exp\left(-\frac{l}{L}y^2\right)} \int\limits_{-\infty}^{+\infty} \frac{dt \cosh t}{\sin^2\theta + \sinh^2 t} \exp\left(-\frac{l}{L}t^2\right) \quad (100)$$

This formula allows us, in principle, to find l/L from the run of the experimental curve $I(\theta)$. Finishing this section I would like to emphasize that all its formulae are exact in the limit (5) and no analogue of the "non-resonance" terms have appeared.

6 Conclusion

Let us suppose for the random potential $U(x)$ to have the finite correlation length κ^{-1}:

$$< U(x)U(x') > = \frac{1}{2}D\kappa\exp\left(-\kappa|x-x'|\right). \quad (101)$$

We can take it into account in the limit $1 \ll \kappa l$ by the renormalization of the parameter α:

$$\alpha_\kappa = \frac{\alpha_0}{1 + 4k^2/\kappa^2}. \quad (102)$$

It should be noted that this renormalization may be sufficient since the inequalities $1 \ll \kappa l$ and $k/\kappa \sim 1$ can take place simultaneously.

Indeed, the correlator (101) corresponds to the following measure of the integration over the fields $\zeta^\pm$:

$$\mathcal{D}\zeta^\pm \exp\left\{-\frac{2}{\alpha}\int\limits_{-L}^{L}\left(\frac{1}{\kappa^2}|\dot\zeta|^2 + 2i\frac{k}{\kappa^2}(\dot\zeta^+\zeta^- - \zeta^+\dot\zeta^-) + \left(1 + \frac{4k^2}{\kappa^2}\right)|\zeta|^2\right)dx\right\}. \quad (103)$$

On performing the "bosonisation" and passing to the variable ξ we obtain some effective action $S_{eff}(\xi)$. The terms with derivatives $\dot\zeta^\pm$ in the exponent of (103) would produce the terms of $S_{eff}(\xi)$ containing derivatives as well. The localization length l is the only parameter of the length dimension occurring in the unperturbed problem. Therefore the contributions of

those "non-markovian" terms would be suppressed by powers of the quantity $(\kappa l)^{-1}$. Neglecting them we come to the formula (102).

The variable σ' appeared in (64), (65) could be considered as the global order parameter corresponding to the localization. In fact, the non-zero value of the correlator $p_E(x, x')$ in the thermodynamic limit is the consequence of the following relationship:

$$\lim_{L\to\infty}\frac{\sigma'}{\alpha L} = \frac{1}{2} > 0. \quad (104)$$

On the other hand, the quantity $\lambda = \exp(-\sigma')$ is conjugated to the wave function norm. Then the inequality (104) corresponds to the exponential in the average increase of the functions $u, \tilde u$ (7).

It is worth noting that the large-scale behaviour of wave functions is governed by some averaged characteristics of the potential $U(x)$. In the perturbation theory framework they appear as some infrared singular integrals. The integrands are multipoint products of potential with fast-oscillating exponentials. Thus, if the quantity

$$b_\Delta(x, d) = \int\limits_{x-\Delta}^{x+\Delta} dx' \exp\left(ik(x-x')\right)U(x)U(x') \quad (105)$$

starting from some Δ

$$k^{-1} \ll \Delta \ll l$$

becomes independent on x, the Abrikosov-Ryzhkin model with some effective a and α can be used to investigate the properties of the wave functions.

The method similar to that presented here allows one to derive path integral representation for any averaged combination of Green functions at arbitrary energy E [21]. Unfortunately, I succeeded in computing such path integrals in some simplest cases only.

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