



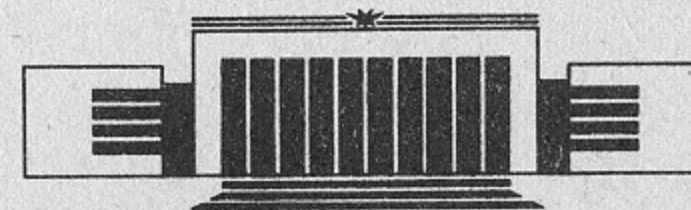
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ON POLINOMIAL FORMULATIONS OF GENERAL
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On Polynomial Formulations of General Relativity

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ABSTRACT

As basic variables in general relativity (GR) are chosen antisymmetric connection and bivectors — bilinear in tetrad area tensors subject to appropriate (bilinear) constraints. In canonical formalism we get theory with polynomial constraints some of which are II class. On partial resolving the latter we get another polynomial formulation. Separating self- and antiselfdual parts of antisymmetric tensors we come to Ashtekar constraints including those known as “reality conditions” which connect self- and antiselfdual sectors of the theory. These conditions form second class system and cannot be simply imposed on quantum states (or taken as initial conditions in classical theory). Rather these should be taken into account in operator sense by forming corresponding Dirac brackets. As a result, commutators between canonical variables are no longer polynomial, and even separate treatment of self- and antiselfdual sectors is impossible.

1. Ashtekar variables [1] attract much attention as possible tool to solve quantum constraints of GR nonperturbatively [2]. Such a possibility is connected with polynomiality of GR in the new variables. In this note two another polynomial versions of GR are suggested. Canonical formalism is developed and connection with Ashtekar variables is considered.

The issue point is Einstein-Hilbert action in the tetrad-connection variables [3]:

$$S = \frac{1}{8} \int d^4x \epsilon_{abcd} \epsilon^{\mu\nu\lambda\rho} e_\mu^a e_\nu^b [\mathcal{D}_\lambda, \mathcal{D}_\rho]^{cd} \quad (1)$$

where $\mathcal{D}_\lambda = \partial_\lambda + \omega_\lambda$ (in fundamental representation) is covariant derivative, and $\omega_\mu^{ab} = -\omega_\mu^{ba}$ is element of $so(3,1)$, Lie algebra of $SO(3,1)$ group. Raising and lowering indices is performed with the help of metric $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$, while $\epsilon^{0123} = +1$. $\alpha, \beta, \dots = 1, 2, 3$ and $\mu, \nu, \dots = 0, 1, 2, 3$ are coordinate indices and $a, b, \dots = 0, 1, 2, 3$ are local ones. Separating space and time indices we put Lagrangian density into the form

$$\mathcal{L}_0 = \pi^\alpha \circ \dot{\omega}_\alpha - h \circ \mathcal{D}_\alpha \pi^\alpha - n_\alpha \circ R^\alpha \quad (2)$$

Here

$$h = -\omega_0, \quad R_{ab}^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\gamma} [\mathcal{D}_\beta, \mathcal{D}_\gamma]_{ab}, \quad \pi_{ab}^\alpha = \frac{1}{2} \epsilon_{abcd} \epsilon^{\alpha\beta\gamma} e_\beta^c e_\gamma^d, \quad n_{\alpha ab} = -\epsilon_{abcd} e_0^c e_\alpha^d \quad (3)$$

Scalar product of two matrices (\circ) and hereafter used their dual product ($*$) and dual matrix are defined as

$$A \circ B \stackrel{\text{def}}{=} \frac{1}{2} A^{ab} B_{ab}$$

$$A * B \stackrel{\text{def}}{=} A \circ (*B) \quad (4)$$

$$*B^{ab} \stackrel{\text{def}}{=} \frac{1}{2} \epsilon^{ab}{}_{cd} B^{cd}$$

In order that n_α, π^α be of the form pointed out in (3), dual products of the type $\pi * \pi, n * n$ and traceless part of $\pi * n$ should vanish:

$$\pi^\alpha * \pi^\beta = 0 \quad (5)$$

$$\pi^\alpha * n_\beta - \frac{1}{3} (\pi^\gamma * n_\gamma) \delta_\beta^\alpha = 0 \quad (6)$$

$$n_\alpha * n_\beta = 0 \quad (7)$$

Conversely, having got 6 antisymmetric matrices π^α, n_α (36 components), subject to 20 conditions (5) - (7), one can check that there exists the unique, up to an overall sign, tetrad e_μ^a (16 = 36 - 20 components), in terms of which π^α, n_α are expressible according to (3). Adding (5) - (7) to (2) with the help of Lagrange multipliers gives

$$\mathcal{L} = \mathcal{L}_0 - \frac{1}{2} \mu_{\alpha\beta} \pi^\alpha * \pi^\beta - \lambda_\alpha^\beta \pi^\alpha * n_\beta - \frac{1}{2} \nu^{\alpha\beta} n_\alpha * n_\beta \quad (8)$$

$$\stackrel{\text{def}}{=} \mathcal{L}_0 - \Lambda \Phi,$$

where $\mu_{\alpha\beta}, \nu^{\alpha\beta}$ are symmetrical, $\text{tr} \lambda = \lambda_\alpha^\alpha = 0$, $\Lambda = (\mu, \nu, \lambda)$, and Φ denotes the set of constraints (5) - (7).

2. Consider the structure of constraints. In the Lagrangian formalism we first vary \mathcal{L} in Λ, h, n . This gives, together with earlier introduced Φ , also Gauss law

$$C \stackrel{\text{def}}{=} \mathcal{D}_\alpha \pi^\alpha = 0 \quad (9)$$

and

$$\lambda_\beta^\alpha * \pi^\beta + \nu^{\alpha\beta} * n_\beta + R^\alpha = 0 \quad (10)$$

Let us multiply (10) in scalar way by π^γ, n_γ . This allows, with the help of Φ , to find λ, ν ; besides, requiring $\nu^{\alpha\beta}$ be symmetrical gives constraints, having the form of momentum ones [1] - combinations of diffeomorphism generators with local rotation ones C ,

$$\mathcal{H}_\alpha \stackrel{\text{def}}{=} \epsilon_{\alpha\beta\gamma} \pi^\beta \circ R^\gamma = 0, \quad (11)$$

while requirement $\text{tr} \lambda = 0$ leads to Hamiltonian constraint¹

$$\mathcal{H}_0 \stackrel{\text{def}}{=} n_\alpha \circ R^\alpha = 0 \quad (12)$$

Constraints (11) and (12) are produced by varying action by operators

$$n_\alpha \circ \frac{\delta}{\delta n_\alpha} \quad (13)$$

and

$$\epsilon_{\alpha\beta\gamma} \pi^\beta \circ \frac{\delta}{\delta n_\gamma} \quad (14)$$

Upon varying w.r.t. π, ω we get equations of motion

$$\dot{\omega}_\alpha = -\mathcal{D}_\alpha h + * (\mu \pi + \lambda n)_\alpha \quad (15)$$

$$\dot{\pi}^\alpha = [h, \pi^\alpha] - \epsilon^{\alpha\beta\gamma} \mathcal{D}_\beta n_\gamma \quad (16)$$

Further require that constraints obtained be conserved in time. Eqs. (6) and (7) (14 components) allow to find 18 components n_α up to 4 parameters. Differentiating these will give \dot{n}_α with the same degree of indefiniteness. Namely,

$$\dot{n}_\alpha = v n_\alpha + \epsilon_{\alpha\beta\gamma} u^\beta \pi^\gamma, \quad (17)$$

where u^α, v are parameters. Knowing \dot{n}_α we can differentiate \mathcal{H}_0 and parameters λ, ν earlier obtained from (10). The rest of constraints can be differentiated with the help of (15) and (16) without problem. The Gauss law, Hamiltonian and momentum constraints are conserved identically, and the only nontrivial is condition

$$\frac{d}{dt} (\pi^\alpha * \pi^\beta) = 0, \quad (18)$$

which gives the constraint

$$G^{\gamma\delta} \stackrel{\text{def}}{=} n_\alpha * (\epsilon^{\alpha\beta\gamma} \mathcal{D}_\beta \pi^\delta + \epsilon^{\alpha\beta\delta} \mathcal{D}_\beta \pi^\gamma) = 0. \quad (19)$$

It is thus the consequence of equation of motion for connection $\delta S / \delta \omega = 0$.

Finally, differentiating (19) allows us to find $\mu_{\alpha\beta}$. Indeed, dependence on $\mu_{\alpha\beta}$ arises due to terms with $\dot{\omega}$, see (15), and has the form

$$G^{\alpha\beta} = 2 \det \|g_{\alpha\beta}\| \epsilon_{abcd} e_0^a e_1^b e_2^c e_3^d (g^{\gamma\delta} g^{\epsilon\zeta} - g^{\gamma\epsilon} g^{\delta\zeta}) \mu_{\epsilon\zeta} + \dots, \quad (20)$$

$$g_{\alpha\beta} \stackrel{\text{def}}{=} e_\alpha^a e_{\beta a}$$

¹ More accurately, our constraint \mathcal{H}_0 is combination of Hamiltonian constraint [1], whose effect in combination with C are shifts in time, and of \mathcal{H}_α ; see below

Due to nondegeneracy of metric the equation $\dot{G} = 0$ is uniquely solvable for μ .

In Hamiltonian formalism denote by \tilde{q} the momentum, conjugate to coordinate q . In particular, (π, ω) already form canonical pair (\tilde{q}, q) . Hamiltonian density is

$$\mathcal{H} = \sum_q \tilde{q}\dot{q} - \mathcal{L}. \quad (21)$$

First, the primary constraints can be found:

$$\tilde{\Lambda} = 0, \quad \tilde{h} = 0, \quad \tilde{n} = 0. \quad (22)$$

Their conservation leads to secondary constraints

$$0 = \frac{d}{dt}\tilde{q} = \{\tilde{q}, \int \mathcal{H} d^3x\} = -\frac{\partial \mathcal{H}}{\partial q}, \quad (q \neq \pi, \omega), \quad (23)$$

which are easily recognised to be the earlier obtained in Lagrangian formalism constraints Φ , C and (10). The Poisson bracket is defined as usual, in particular

$$\{\omega_{\alpha}^{ab}(x), \pi_{cd}^{\beta}(x')\} = (\delta_c^a \delta_d^b - \delta_d^a \delta_c^b) \delta_{\alpha}^{\beta} \delta^3(x - x') \quad (24)$$

The further Dirac procedure of extracting the constraints completely repeats the above consideration in the Lagrangian formalism. As a result, the following nondynamical (i.e. different from π, ω) variables remain undefined: h, \dot{h} , being Lagrange multipliers at constraints C, \tilde{h} ; 4 parameters in n and the same number of those in \dot{n} (see (17)) being Lagrange multipliers at constraints \mathcal{H}_{μ} and at four combinations of \tilde{n} , respectively. This means that corresponding constraints are I class.

In particular, to describe evolution of physical observables (which are natural to thought of as functions of π, n, ω) it is sufficient to use the set of pairs $(\pi, \omega), (\tilde{n}, n)$ as phase space. Then phase manifold of GR is defined by I class constraints,

$$C, \quad \mathcal{H}_{\mu}, \quad 4 \text{ combinations } \tilde{n}, \quad (25)$$

and by the others, II class ones:

$$\Phi, \quad G^{\alpha\beta}, \quad 14 \text{ combinations } \tilde{n}. \quad (26)$$

The number of the degrees of freedom is equal to the number of canonical pairs minus the number of I class constraints and half of the number of II class ones. Let $[A]$ be the number of components of a value A . Then the number of I class combinations of \tilde{n} (4 in (25)) arises as $[n] - [\Lambda] + [\mu]$. The

same is the number of constraints \mathcal{H}_{μ} . As a result, the number of the degrees of freedom turns out to be expressible as

$$[\omega] - [n] - [h] + [\Lambda] - 2[\mu] = 2, \quad (27)$$

as it would expected.

3. Some disadvantage of formulation (25), (26) is that \tilde{n} are not purely I or II class constraints but rather their nontrivial combinations. We can pass to another polynomial version of GR by noting that Φ can be solved for n as

$$n = \epsilon_{\alpha\beta\gamma} w^{\beta} \pi^{\gamma} + v \epsilon_{\alpha\beta\gamma} \pi^{\beta} \pi^{\gamma}, \quad (28)$$

where w^{α}, v are parameters. Then constraints $G^{\alpha\beta}, \mathcal{H}_0$ at $v \neq 0$ are equivalent to the following ones:

$$\tilde{G}^{\alpha\beta} \stackrel{\text{def}}{=} \pi^{\gamma} * ([\pi^{\alpha}, \mathcal{D}_{\gamma} \pi^{\beta}] + ([\pi^{\beta}, \mathcal{D}_{\gamma} \pi^{\alpha}]) \quad (29)$$

$$\tilde{\mathcal{H}}_0 \stackrel{\text{def}}{=} \epsilon_{\alpha\beta\gamma} \pi^{\alpha} \pi^{\beta} \circ R^{\gamma} \quad (30)$$

Phase manifold in terms of (π, ω) is given by constraints

$$C, \quad \mathcal{H}_{\alpha}, \quad \tilde{\mathcal{H}}_0 \quad (31)$$

and

$$\pi^{\alpha} * \pi^{\beta}, \quad \tilde{G}^{\alpha\beta} \quad (32)$$

of I and II class, respectively.

In quantum theory II class constraints cannot be simply imposed on states since due to their noncommutativity this will lead to vanishing the wavefunction itself. Instead, these should be taken into account in the operator sense by assigning to quantum commutators the values of the Dirac rather than Poisson brackets. Dirac brackets arise from Poisson ones when projecting orthogonally to the II class constraint surface in the phase space:

$$\{f, g\}_D \stackrel{\text{def}}{=} \{f, g\} - \{f, \Theta_A\} (\Delta^{-1})^{AB} \{\Theta_B, g\}, \quad (33)$$

where $\{\Theta_A\}$ is the full set of II class constraints, and Δ^{-1} is matrix inversed to that of their Poisson brackets:

$$(\Delta^{-1})^{AB} \{\Theta_B, \Theta_C\} = \delta_C^A \quad (34)$$

Now when Θ_A are constraints (32), the matrix Δ^{-1} is easy to find. Poisson brackets on constraint surface take the form

$$\left\{ \int \left(\frac{1}{2} \mu_{\alpha\beta} \pi^\alpha * \pi^\beta + \frac{1}{2} m_{\alpha\beta} \tilde{G}^{\alpha\beta} \right) d^3x, \int \left(\frac{1}{2} \mu'_{\alpha\beta} \pi^\alpha * \pi^\beta + \frac{1}{2} m'_{\alpha\beta} \tilde{G}^{\alpha\beta} \right) d^3x \right\} =$$

$$\int (\det \|g_{\alpha\beta}\|)^2 [\text{tr}(m\mu' - \mu m') + \text{tr}\mu \text{tr}m' - \text{tr}m \text{tr}\mu' + \text{tr}(m\chi) \text{tr}m' - \text{tr}m \text{tr}(m'\chi)] d^3x, \quad (35)$$

$$\chi_{\alpha\beta} \stackrel{\text{def}}{=} \epsilon_{\alpha\gamma\delta} \pi^\gamma * \mathcal{D}_\beta \pi^\delta.$$

Here m, m', μ, μ' are test functions (symmetric matrices), while raising and lowering indices is made with the help of metric $g_{\alpha\beta}$. Inverting the bilinear form (36) which leads to Δ^{-1} offers no difficulties, and Dirac bracket of any quantities f, g turns out to be local:

$$\{f, g\}_D = \{f, g\} - \frac{1}{2} \int (\det \|g_{\alpha\beta}\|)^{-2} [\text{tr}(\{f, \phi\} \{ \tilde{G}, g \} - \{f, \tilde{G}\} \{ \phi, g \}) - \frac{1}{2} \text{tr}\{f, \phi\} \text{tr}\{ \tilde{G}, g \} + \frac{1}{2} \text{tr}\{f, \tilde{G}\} \text{tr}\{ \phi, g \} + \frac{1}{2} \text{tr}\{f, \phi\} \text{tr}(\chi \{ \phi, g \}) - \frac{1}{2} \text{tr}(\{f, \phi\} \chi) \text{tr}\{ \phi, g \}] d^3x, \quad (36)$$

$$\phi^{\alpha\beta} \stackrel{\text{def}}{=} \pi^\alpha * \pi^\beta$$

Procedure of performing trace refers to indices of functions \tilde{G}, ϕ, χ , while integration variable x is their argument. Dirac bracket turns out to be non-polynomial (due to occurrence of $(\det \|g_{\alpha\beta}\|)^{-2}$). Also note that different components of ω do not commute.

4. Finally, let us pass to Ashtekar variables and decompose for that antisymmetric tensors A^{ab} into selfdual $+A$ and antiselfdual $-A$ parts,

$$A = +A + -A, \quad \pm A = \frac{1}{2} (A \pm i^* A), \quad i^*(\pm A) = \pm(\pm A), \quad (37)$$

each of which embed into complex 3D vector space by expanding over basis of (anti-)selfdual matrices

$$\pm \Sigma_{ab}^k = \pm i (\delta_a^k \delta_b^0 - \delta_b^k \delta_a^0) + \epsilon_{kab}, \quad (38)$$

so that

$$\pm A^{ab} = \pm A^k \pm \Sigma_k^{ab} / 2 \stackrel{\text{def}}{=} \pm \vec{A} \cdot \pm \vec{\Sigma}^{ab} / 2 \quad (39)$$

(matrices $-i \pm \Sigma_{ab}^k$ are chosen to obey algebra of Pauli matrices σ^k). Then for real tensor quantity $+A = \overline{-A}$ (overlining means usual complex conjugation).

At such embedding the constraints become sums or differences between monoms of only selfdual and of only antiselfdual fields. It is convenient to group these as follows:

$$2i\phi^{\alpha\beta} = +\vec{\pi}^\alpha \cdot +\vec{\pi}^\beta - -\vec{\pi}^\alpha \cdot -\vec{\pi}^\beta = 0 \quad (40)$$

$$-2i\tilde{G}^{\alpha\beta} = +\vec{\pi}^\gamma \cdot +\vec{\pi}^{(\alpha} \times +\mathcal{D}_\gamma +\vec{\pi}^{\beta)} - -\vec{\pi}^\gamma \cdot -\vec{\pi}^{(\alpha} \times -\mathcal{D}_\gamma -\vec{\pi}^{\beta)} = 0 \quad (41)$$

$$C \circ (+\vec{\Sigma} - -\vec{\Sigma}) / 2 = +\mathcal{D}_\alpha +\vec{\pi}^\alpha - -\mathcal{D}_\alpha -\vec{\pi}^\alpha = 0 \quad (42)$$

$$C \circ (+\vec{\Sigma} + -\vec{\Sigma}) / 2 = +\mathcal{D}_\alpha +\vec{\pi}^\alpha + -\mathcal{D}_\alpha -\vec{\pi}^\alpha = 0 \quad (43)$$

$$2\mathcal{H}_\alpha = \epsilon_{\alpha\beta\gamma} (+\vec{\pi}^\beta \cdot +\vec{R}^\gamma + -\vec{\pi}^\beta \cdot -\vec{R}^\gamma) = 0 \quad (44)$$

$$-4\mathcal{H}_0 = \epsilon_{\alpha\beta\gamma} (+\vec{\pi}^\alpha \times +\vec{\pi}^\beta \cdot +\vec{R}^\gamma + -\vec{\pi}^\alpha \times -\vec{\pi}^\beta \cdot -\vec{R}^\gamma) = 0. \quad (45)$$

Here $\pm \mathcal{D}_\alpha(\cdot) = \partial_\alpha(\cdot) - \pm \vec{\omega}_\alpha \times (\cdot)$, $\pm \vec{R}^\alpha = -\epsilon^{\alpha\beta\gamma} [\pm \mathcal{D}_\beta, \pm \mathcal{D}_\gamma] / 2$, while $(\alpha \dots \beta)$ means the sum of objects with indices $\alpha \dots \beta$ and $\beta \dots \alpha$. Equations (40), (41) and (42) at $+\vec{\pi} = \overline{-\vec{\pi}}$ present 6 + 6 + 3 real conditions. If the condition $+\vec{\pi} = \overline{-\vec{\pi}}$ is not assumed beforehand, we deal with 6+6+3 complex equations on $+\vec{\pi}, +\vec{\omega}$. It follows from (40) that some U exists, an element of $SO(3, C)$, such that $+\vec{\pi}^\alpha = U -\vec{\pi}^\alpha$. Then (41) and (42) allow us to find 9 components of connection: $+\mathcal{D}_\alpha = U -\mathcal{D}_\alpha U^\dagger$. As a result, (43) - (45) are fulfilled separately for (+) and (-)-components. Thus we arrive at the Ashtekar constraints:

$$\mathcal{D}_\alpha \vec{\pi}^\alpha, \quad \epsilon_{\alpha\beta\gamma} \vec{\pi}^\beta \cdot \vec{R}^\gamma, \quad \epsilon_{\alpha\beta\gamma} \vec{\pi}^\alpha \times \vec{\pi}^\beta \cdot \vec{R}^\gamma, \quad (46)$$

where $\vec{\pi}, \vec{R}$ are (+) or (-)-components.

Equations (40), (41) in the theory with pseudoEuclidean signature are known as reality conditions [4]. These equations, however, survive in the real theory with Euclidean signature. In both cases these connect self- and antiselfdual sectors of theory and their being the II class constraints leads to commutators (Dirac brackets) of π, ω different from canonical ones and nonpolynomial (see (36)). In particular, different components of ω do not commute and there is no such representation of commutation relations that ω be c -number. On the other hand, the components of π do not commute with each other, and one might try to use representation in which π is c -number. However, when imposing the I class constraints on states we shall

not get analytical functionals of π as solutions (physical case of pseudoEuclidean signature is considered). Nonanalyticity occurs in the dependence of commutators on the (real) metric $g_{\alpha\beta}$. Indeed, in selfdual components $g^{\alpha\beta} \det \|g_{\gamma\delta}\| = \pi^\alpha \circ \pi^\beta$ becomes real part of $2\bar{\pi}^\alpha \cdot \bar{\pi}^\beta$.

Besides, self- and antiselfdual components do not commute. For example, it is easy to find from general formula (36) that

$$\{ {}^+\omega_\alpha^i(x), {}^-\pi^{\beta k}(x') \}_D = -(\det \|g_{\alpha\beta}\|)^{-1} ({}^+\pi^{\gamma i} {}^-\pi_{\gamma k} \delta_\alpha^\beta + {}^+\pi^{\beta i} {}^-\pi_{\alpha k}) \delta^3(x - x'). \quad (47)$$

We thus have considered 2 formulations of GR with polynomial Lagrangian. These are defined by the sets of constraints (25), (26) and (31), (32) on phase spaces of pairs (π, ω) , (\bar{n}, n) and of (π, ω) , respectively. The first version turns out to be appropriate for generalisation to discrete Regge gravity where it allows one to put theory into quasipolynomial form [5]. Second version leads, on separating self- and antiselfdual components of tensors, to Ashtekar variables. It turns out that due to commutators Ashtekar variables are not free from nonpolynomiality and nonanalyticity.

References

- [1] Ashtekar, A.(1986) *Phys.Rev.Lett.*, **57**, 2224.
Ashtekar, A.(1987) *Phys.Rev.*, **D36**, 1787.
- [2] Jacobson, T., and Smolin, L.(1988) *Nucl.Phys.*, **B299**, 295.
- [3] Schwinger, J.(1963) *Phys.Rev.*, **130**, 1253.
- [4] Ashtekar, A., Romano, J.D., and Tate, R.S.(1988) "New variables for gravity: inclusion of matter" Syracuse University preprint.
- [5] Khatsymovsky, V.(1993) "Regge calculus in the canonical form" Novosibirsk preprint BINP 93-42, submitted to *Gen.Rel.Grav.*

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