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SO(8) COLOUR AS POSSIBLE ORIGIN OF GENERATIONS

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SO(8) Colour as Possible Origin of Generations

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A B S T R A C T

A possible connection between the existence of three quark-lepton generations and the triality property of SO(8) group (the equality between 8-dimensional vectors and spinors) is investigated. SO(8) appears to be a natural one-flavour unification group. Family formation due to weak interactions results in SO(10) group and a broken triality symmetry. Further speculations about how to restore this symmetry lead finally to E_6 exceptional group and three quark-lepton families.

1. INTRODUCTION

One of the most striking features of quark-lepton spectrum is its cloning property: μ and τ families seem to be just heavy copies of electron family. Actually we have two questions to be answered: what is an origin of family formation and how many generations do exist. Recent LEP data [1] strongly suggests three quark-lepton generations. Although Calabi-Yau compactifications of the heterotic string model can lead to three generations [2], there are many such Calabi-Yau manifolds, and additional assumptions are needed to argue why the number three is preferred [3].

There is another well-known example of particle cloning (doubling of states): the existence of antiparticles. Algebraically the charge conjugation operator defines an (outer) automorphism of underlying symmetry group [4,5] and reflects the symmetry of the corresponding Dynkin diagram. We can thought that the observed triplication of states can have the same origin.

The most symmetric Dynkin diagram is associated with SO(8) group. So it is the richest in automorphisms and if SO(8) plays some dynamical role we can hope that its greatly symmetrical internal structure naturally lead to the desired multiplication of states in elementary particle spectrum. What follows is an elaboration of this idea.

Although the relevant mathematical properties of SO(8) are known for a long time [6], they have not been discussed in context of the generation problem, to my knowledge.

To make the paper as self-contained as possible, I have reproduced in it some well-known results concerning SO(10) and E_6 groups, Although I hope their treatment here doesn't merely copy the existing literature.

4. PECULIARITIES OF THE SO(8) GROUP

To explain why D_4, the Lie algebra of the SO(8) group, is the most symmetric among simple Lie algebras, let us briefly outline their classification theory. Details can be found e.g. in [7,8].

Let $\mathcal{L}$ be a complex Lie algebra and $x\in\mathcal{L}$. x is called semisimple if ad_x, linear operator on $\mathcal{L}$, defined as $ad_x(y)=[x,y]$, is diagonal in some basis. The Cartan subalgebra is the maximal Abelian subalgebra of $\mathcal{L}$ with semisimple elements. $\mathcal{L}$ can have several Cartan subalgebras, but they all have the same dimension, which is called rank of $\mathcal{L}$.

Let $\mathcal{K}$ be some Cartan subalgebra and $\mathcal{K}^*$ - its dual space (i.e. the space of the complex valued linear functions on $\mathcal{K}$). For any $\lambda\in\mathcal{K}^*$ let us define $\mathcal{L}^\lambda$ as a subspace in $\mathcal{L}$, such that for each $x\in\mathcal{L}^\lambda$ and $y\in\mathcal{K}$ the eigenvalue equation $ad_y(x)=\lambda(y)x$ is satisfied. If $\lambda\neq0$ and $\mathcal{L}^\lambda\neq0$, then λ is called a root of the algebra $\mathcal{L}$. The number of roots is finite and if R is the root system, then

$$\mathcal{L} = \mathcal{K} \oplus \sum_{\lambda\in R} \mathcal{L}^\lambda \quad \text{(direct sum of the vector spaces)} , \qquad (1)$$

where each $\mathcal{L}^\lambda$ is one-dimensional. Thus

$$\dim \mathcal{L} = \operatorname{rank} \mathcal{L} + \text{number of roots} .$$

If we select some $\{K_1,K_2,\dots,K_r\}$ basis in $\mathcal{K}$ and add nonzero vectors $E_\lambda\in\mathcal{L}^\lambda$, $\lambda\in R$, we get the Cartan-Weyl basis for $\mathcal{L}$ Lie algebra. In the case of normalization

$$Sp(ad_{K_i}\cdot ad_{K_j}) = \delta_{ij} , \quad Sp(ad_{E_\lambda}\cdot ad_{E_{-\lambda}}) = -1 , \qquad (2)$$

the commutation relations in this basis have the following form

$$[K_i,E_\lambda] = \lambda(K_i)E_\lambda , \qquad [E_\lambda,E_{-\lambda}] = -\sum_i \lambda(K_i)K_i$$
$$[E_\lambda,E_\mu] = N_{\lambda,\mu}\, E_{\lambda+\mu} . \qquad (3)$$

In the last equation $\lambda+\mu\neq0$ and $N_{\lambda,\mu}$ differs from zero only if $\lambda+\mu\in R$.

Let some basis be chosen in $\mathcal{K}^*$. A vector from $\mathcal{K}^*$ is called positive if its first nonvanishing coordinate is positive. A positive root is called simple if it is not representable as a sum of two other positive roots. The set of simple roots S spans the space $\mathcal{K}^*$ (can be served as a basis in $\mathcal{K}^*$) and any $\lambda\in R$ root has the unique representation $\lambda=\sum_{\alpha\in S} m_\alpha \alpha$, where m_α are integers all having the same signs.

For every $\lambda\in\mathcal{K}^*$ there exists uniquely defined $h_\lambda\in\mathcal{K}$ element, such that for any $x\in\mathcal{K}$

$$\lambda(x)=Sp(ad_{h_\lambda}\cdot ad_x) . \qquad (4)$$

Using this the scalar product can be defined in $\mathcal{K}^*$:

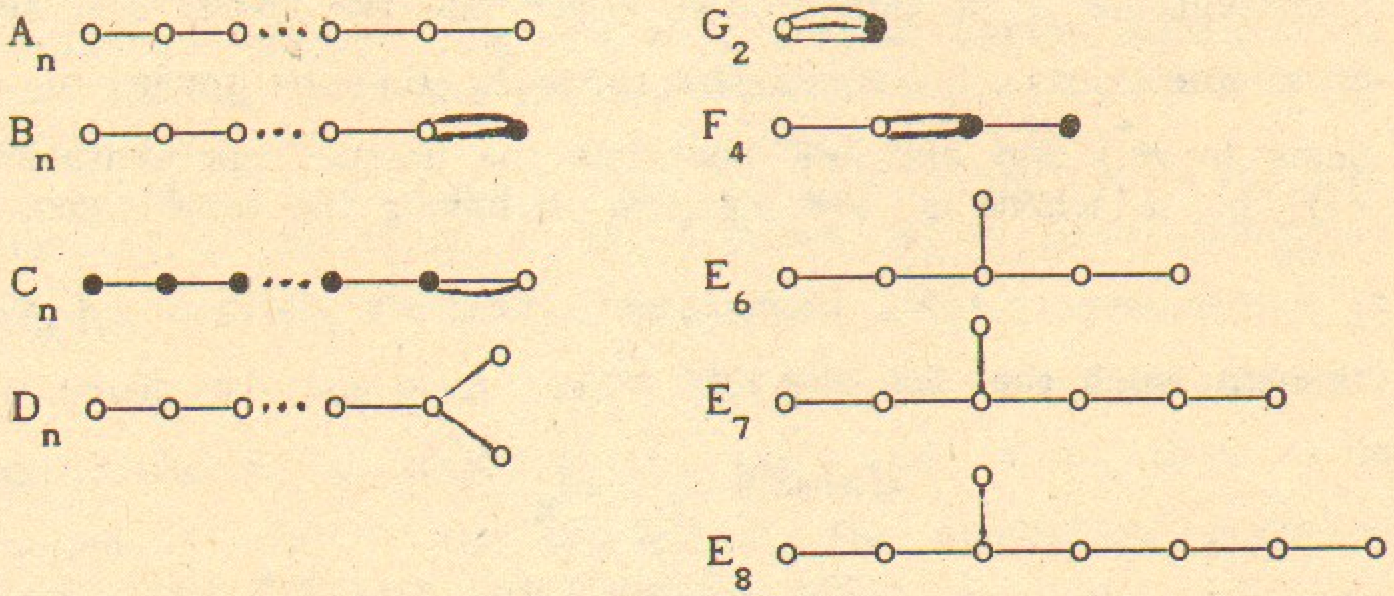
$$(\lambda,\mu) = Sp(ad_{h_\lambda}\cdot ad_{h_\mu}) . \qquad (5)$$

The angle between simple roots, which can be derived from this scalar product, turns out to be very restricted.

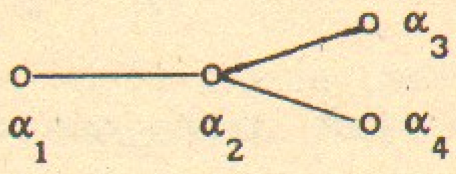
Namely, if φ is the angle between a pair of simple roots, then $4\cos^2\varphi = 0,1,2$ or 3 and $\varphi \geq 90^\circ$. Moreover, if $\varphi \neq 90^\circ$, then the ratio between their lengths turns out to be $2|\cos\varphi|$.

The crucial point is that $N_{\lambda,\mu}$ structure constants in (2) and, therefore, the structure of a simple Lie algebra, are uniquely defined by the length and angle relations among simple roots. This information is compactly represented by the Dynkin diagram. On such a diagram each simple root is depicted by a small circle, which is made black, if the root is a short one (any simple Lie algebra has simple roots with at most two different lengths and usually the longer simple roots are normalized to a length-squared of 2). Each pair of vertexes on the Dynkin diagram is connected by lines, number of which equals to $4\cos^2\varphi$, φ being the angle between corresponding simple roots.

The main classification theorem for simple Lie algebras states that there exist only four infinite series and five exceptional algebras shown below:



Now it is clear that D_4 really has the most symmetric Dynkin diagram



Actually only the symmetry with regard to the cyclic permutations of the $(\alpha_1,\alpha_3,\alpha_4)$ simple roots (which we call triality symmetry) is new, because the symmetry with regard to the interchange $\alpha_3 \longleftrightarrow \alpha_4$ (last two simple roots) is shared by other D_n Lie algebras also.

The operators of a simple Lie algebra may be represented by n×n matrices acting in n-dimensional Hilbert space. Cartan subalgebra, being abelian, is representable by simultaneously diagonal matrices. Using this, each vector in representation space can be (up to possible degeneration) labeled by some elements from $\mathcal{K}^*$, called weights. If $\lambda \in \mathcal{K}^*$ is the weight associated with the Hilbert space vector $|\lambda\rangle$, then for every $K \in \mathcal{K}$, $\alpha \in R$

$$K\,|\lambda\rangle = \lambda(K)\,|\lambda\rangle \quad \text{and} \quad E_\alpha\,|\lambda\rangle \sim |\lambda+\alpha\rangle \,. \qquad (6)$$

So E_α for positive roots act as a raising operators.

All properties of any irreducible representation are uniquely determined by its highest weight Λ for which $|\Lambda\rangle$ is annihilated by all raising operators. Other vectors in representation space can be obtained from $|\Lambda\rangle$ by means of the lowering operators $E_{-\alpha}$.

Often it is useful to label weights by their Dynkin coordinates, which for any weight λ are defined as

$$a_i = \frac{2\,(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}\,, \quad \alpha_i \in S\,. \qquad (7)$$

The convenience of the Dynkin coordinates relies upon the fact that they are always integer numbers. Moreover, for the highest weight they are non-negative integers and any such set $(a_1,a_2,\ldots,a_r)$ of non-negative integers is a set of Dynkin coordinates of the highest weight for some irreducible representation.

The weight system for any irreducible representation can be easily derived from its highest weight by the following simple algorithm: if $\Lambda=(a_1,\ldots,a_r)$ is some weight in the Dynkin basis, such that $a_i>0$ and $\Lambda+\alpha_i$ is not a weight, then i-th simple root can be subtracted from Λ a_i times, each subtraction giving also a weight.

The root system can be constructed in a similar manner, because, as (2) shows, the roots are weights for the adjoint (regular) representation $x \longrightarrow ad_x$.

At last, let us cite the Weyl formula for the dimensionality of an irreducible representation:

$$N = \prod_{\alpha\in R_+} \frac{(\Lambda+\delta,\alpha)}{(\delta,\alpha)} , \tag{8}$$

where Λ is the highest weight of the irrep, R_+ is the set of the positive roots and $\delta=(1,1,\ldots,1)$ in the Dynkin basis is half the sum of the positive roots.

For D_4 the adjoint representation is (0100) and the above described algorithm gives the following root system:

$$\begin{gathered} \pm\alpha_1,\ \pm\alpha_2,\ \pm\alpha_3,\ \pm\alpha_4,\ \pm(\alpha_1+\alpha_2),\ \pm(\alpha_2+\alpha_3),\ \pm(\alpha_2+\lambda_4)\ , \\ \pm(\alpha_1+\alpha_2+\alpha_3)\ ,\ \pm(\alpha_1+\alpha_2+\alpha_4)\ ,\ \pm(\alpha_2+\alpha_3+\alpha_4)\ , \\ \pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\ ,\ \pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)\ . \end{gathered} \tag{9}$$

The simple roots, as the Dynkin diagram shows, are not orthonormal, the nonzero scalar products being:

$$\begin{gathered} (\alpha_1,\alpha_1)=(\alpha_2,\alpha_2)=(\alpha_3,\alpha_3)=(\alpha_4,\alpha_4)=2\ , \\ (\alpha_1,\alpha_2)=(\alpha_2,\alpha_3)=(\alpha_2,\alpha_4)=-1\ . \end{gathered} \tag{10}$$

Sometimes more convenient is the $\{\nu_1,\nu_2,\nu_3,\nu_4\}$ basis dual to the $\{K_1,K_2,K_3,K_4\}$ basis of the Cartan subalgebra:

$\nu_m(K_n)=\delta_{mn}$. Eq. (2) implies that ν_i-basis is orthonormal. It can be deduced from (10) that

$$\alpha_1=\nu_1-\nu_2\ ,\ \alpha_2=\nu_2-\nu_3\ ,\ \alpha_3=\nu_3-\nu_4\ ,\ \alpha_4=\nu_3+\nu_4\ , \tag{11}$$

and other roots also take the very simple form in this basis $\pm\nu_m\pm\nu_n$.

The dimensionality of any (a_1,a_2,a_3,a_4) D_4-representation can be calculated from (8) using (9), (11), $\delta=3\nu_1+2\nu_2+\nu_3$ and $\Lambda=\sum_{i=1}^{4} l_i\alpha_i$, where $l_i=\sum(A^{-1})_{ij}a_j$, $A_{ij}= \dfrac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ is so called Cartan matrix for D_4 and

$$A=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix},\quad A^{-1}=\frac{1}{2}\begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}.$$

The resulting formula (although not very convenient) has the following form:

$$\begin{gathered} N(a_1,a_2,a_3,a_4)= \\ (1+a_1)(1+a_2)(1+a_3)(1+a_4)\left(1+\frac{a_1+a_2}{2}\right)\left(1+\frac{a_2+a_3}{2}\right)\times \\ \left(1+\frac{a_2+a_4}{2}\right)\left(1+\frac{a_1+a_2+a_3}{3}\right)\left(1+\frac{a_2+a_3+a_4}{3}\right)\left(1+\frac{a_1+a_2+a_4}{3}\right)\times \\ \left(1+\frac{a_1+a_2+a_3+a_4}{4}\right)\left(1+\frac{a_1+2a_2+a_3+a_4}{5}\right)\ . \end{gathered}$$

In particular, $8_v=(1000)$, $8_c=(0010)$ and $8_s=(0001)$ basic irreps all have the same dimensionality 8 — the remarkable fact valid only for the D_4 Lie algebra. The corresponding highest weights

$$\Lambda_V=\alpha_1+\alpha_2+\frac{1}{2}(\alpha_3+\alpha_4),\quad \Lambda_C=\alpha_3+\alpha_2+\frac{1}{2}(\alpha_1+\alpha_4)$$

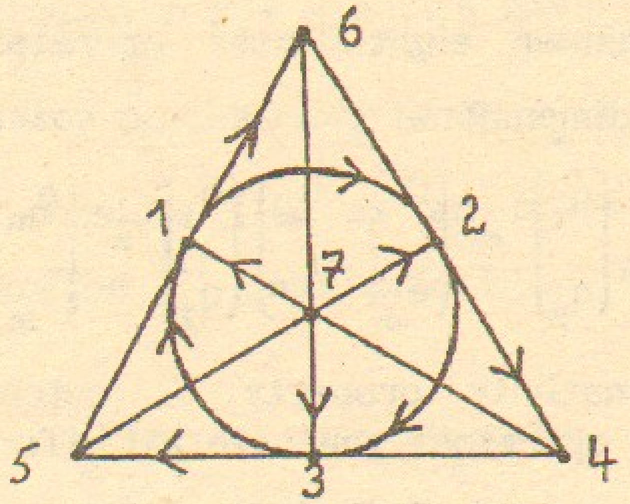
$$\text{and}\quad \Lambda_S=\alpha_4+\alpha_2+\frac{1}{2}(\alpha_1+\alpha_3)$$

are connected by the above mentioned triality symmetry. For other orthogonal groups (10...0) is a vector representation, (00...01) — a first kind spinor and (00...10) — a second kind spinor. So there is no intrinsic difference between vectors and spinors in the eight-dimensional space [9], which object is vector and which ones are spinors depends simply on how we have enumerated symmetric simple roots and so is a mere convention.

It is tempting to use this peculiarity of the SO(8) group to justify observed triplication of the quark-lepton degrees of freedom. We continue this line of thought a bit later. Now let us consider the realization of the eight-dimensional vectors and spinors through octonions [10,11], in terms of which possible connection between generations and SO(8) can be formulated most naturally.

3. OCTONIONS AND TRIALITY

Octonions can be viewed as a generalization of the complex numbers: instead of one imaginary unit we have seven imaginary units $e_A^2=-1$, $A=1\div 7$ in the octonionic algebra. The multiplication table between them is given by the following mnemonic diagram [11]



(12)

where arrows show what sign plus or minus should be taken, f.e. $e_2\cdot e_3=e_1$, $e_3\cdot e_4=-e_5$, $e_4\cdot e_6=e_2$, $e_1\cdot e_7=-e_4$ and so on.

If we consider $\{1,e_A\}$ as a basis for a real or complex vector space, then we get the algebra of real or complex octonions $q=q_0+q_Ae_A$, where q_0,q_A are scalars (real or complex numbers) and the summation over A from 1 to 7 is assumed.

The octonion algebra is not associative, but obeys a weaker condition than associativity: it is an alternative algebra. This means that the associator $(x,y,z)=x\cdot(y\cdot z)-(x\cdot y)\cdot z$ is a skew symmetric function of the x,y,z octonions.

The conjugate octonion $\bar{q}$ and the norm $N(q)$ are defined by

$$\bar{q}=q_0-q_Ae_A\ ,\qquad N(q)=q\cdot\bar{q}=\bar{q}\cdot q=N(\bar{q})=q_0^2+\sum q_A^2\ .$$

Note that the norm form satisfies $N(p\cdot q)=N(p)N(q)$ — common property of the real numbers, complex numbers, quaternions and octonions.

Using the norm form, the scalar product of octonions can be defined as

$$(p,q)=\frac{1}{2}\{N(p+q)-N(p)-N(q)\}=\frac{1}{2}(p\bar{q}+q\bar{p})=(\bar{p},\bar{q})\ . \qquad (13)$$

This scalar product reveals the following cyclic symmetry:

$$(\bar{x},y\cdot z)=(\bar{y},z\cdot x)=(\bar{z},x\cdot y)\ . \qquad (14)$$

Let us consider eight linear operators Γ_m, $m=0\div7$, acting in the 16-dimensional bioctonionic space:

$$\Gamma_m\begin{pmatrix}q_1\\q_2\end{pmatrix} = \begin{pmatrix}0 & e_m\\ \bar{e}_m & 0\end{pmatrix}\begin{pmatrix}q_1\\q_2\end{pmatrix} = \begin{pmatrix}e_m\cdot q_2\\ \bar{e}_m\cdot q_1\end{pmatrix}. \qquad (15)$$

Using the alternativity property of octonions, it can be tested that these operators generate a Clifford algebra

$$\Gamma_m\Gamma_n+\Gamma_n\Gamma_m = 2\delta_{mn}.$$

(Note, that, because of nonassociativity, the operator product is not equivalent to the product of the corresponding octonionic matrices).

The eight-dimensional vectors and spinors can be constructed in the standard way [12] from this Clifford algebra. Namely, the infinitesimal rotation in the (k,l)-plane by an angle θ is represented by the operator

$$R_{kl}=1+\frac{1}{2}\theta\,\Gamma_k\Gamma_l,$$

and the transformation law for the (bi)spinor $\Psi=\begin{pmatrix}q_1\\q_2\end{pmatrix}$ is $\Psi'=R_{kl}\Psi$, while to get a vector x_m, we should form the operator $X=x_m\Gamma_m$ and the vector transformation law is generated by

$$X'=R_{kl}X\,R_{kl}^{-1}. \qquad (16)$$

For Γ_m given by (15) the upper and lower octonionic components of the bioctonionic spinor Ψ transform independently under the 8-dimensional rotations:

$$q_1'=q_1+\frac{1}{2}\theta\, e_k\cdot(\bar{e}_l\cdot q_1) \equiv q_1+\theta\,F_{kl}(q_1)$$
$$q_2'=q_2+\frac{1}{2}\theta\, \bar{e}_k\cdot(e_l\cdot q_2) \equiv q_2+\theta\,C_{kl}(q_2) \qquad (17)$$

and for vector octonion $x=x_0+x_Ae_A$ we have

$$x'=x+\frac{1}{2}\theta\,\{e_k\cdot(\bar{e}_l\cdot x)-x\cdot(\bar{e}_k\cdot e_l)\}.$$

(To obtain this, consider (16) as an operator equation and apply both sides to the $\begin{pmatrix}1\\1\end{pmatrix}$ unit spinor).

Note that

$$\frac{1}{2}[e_k\cdot(\bar{e}_l\cdot x)-x\cdot(\bar{e}_k\cdot e_l)]=$$

$$e_k(e_l,x)-e_l(e_k,x)-\frac{1}{2}[(e_k,\bar{x},e_l)+(x,\bar{e}_k,e_l)]$$

and the last term in the square brackets equals zero because of alternativity and $(\bar{x},y,z)=(2x_0-x,y,z)=-(x,y,z)$. So the vector transformation law can be represented in a more convenient form

$$x'=x+\theta\{e_k\cdot(e_l,x)-e_l\cdot(e_k,x)\}\equiv x+\theta\,G_{kl}(x). \qquad (18)$$

One more manifestation of the equality between 8-dimensional vectors and spinors is the fact [9] that each spinor transformation from (17) can be represented as a sum of four vector rotations

$$F_{0A}=\frac{1}{2}(G_{0A}+G_{A_1B_1}+G_{A_2B_2}+G_{A_3B_3}), \qquad (19)$$

where A_i,B_i are defined through the condition $e_{A_i}\cdot e_{B_i}=e_A$, and

$$F_{A_1B_1}=\frac{1}{2}(G_{A_1B_1}+G_{0A}-G_{A_2B_2}-G_{A_3B_3}). \qquad (20)$$

For example

$$F_{01}=\frac{1}{2}(G_{01}+G_{23}+G_{47}+G_{65})$$

$$F_{25}=\frac{1}{2}(G_{25}+G_{07}-G_{36}-G_{14})$$

The (anti-Hermitian) generators G_{kl} obey the following commutation relations

$$[G_{mn},G_{kl}] = \delta_{ml}G_{nk} + \delta_{nk}G_{ml} - \delta_{mk}G_{nl} - \delta_{nl}G_{mk} .$$

The same relations are valid also for the F_{kl} generators, as can be easily proved using alternativity of the octonion algebra. So the correspondence

$$\pi:\ G_{mn} \longrightarrow F_{mn} \qquad (21)$$

is an automorphism of the D_4 Lie algebra. Another D_4 automorphism is

$$\kappa:\ G_{mn} \longrightarrow KG_{mn}K = \begin{cases} -G_{mn}\ , & \text{if } m \text{ or } n = 0. \\ G_{mn}\ , & \text{if } m \text{ and } n \neq 0. \end{cases} , \qquad (22)$$

where K is the (octonionic) conjugation operator $K(q)=\bar{q}$. These two automorphisms do not commute, instead we have $\pi\cdot\kappa=(\kappa\cdot\pi)^2$.

If Ψ_1,Ψ_2 are 16-dimensional (bi)spinors, x_m — vector and $X=x_m\Gamma_m$, then $\Psi_1^T X\Psi_2$ is SO(8) invariant. This can be checked explicitly by applying spinor and vector transformation laws $\Psi'=R_{kl}\Psi$, $X'=R_{kl}XR_{kl}^{-1}$ and taking into account that (15) and the octonion multiplication table give for an operator Γ_m a symmetric 16×16 matrix.

For $\Psi_1=\begin{pmatrix}\varphi\\0\end{pmatrix}$ and $\Psi_2=\begin{pmatrix}0\\\psi\end{pmatrix}$ this invariant can be re-written as octonionic scalar product $(\bar{q},x\cdot p)$ (to take $\bar{q}$ instead of q is convenient because of cyclic symmetry (14) , which lead to the cyclic symmetric form of the triality principle described below), where $q=\varphi_0-\varphi_A e_A$, $x=x_0+x_A e_A$ and $p=\psi_0+\psi_A e_A$. From the other side, for the infinitesimal

rotation in the (k,l)-plane, $\delta(\bar{q},x\cdot p)=0$ means: (because of (17) and (18))

$$(\overline{S_{kl}(q)},x\cdot p) + (\bar{q},G_{kl}(x)\cdot p) + (\bar{q},x\cdot C_{kl}(p)) = 0\ , \qquad (23)$$

where

$$S_{kl} = KF_{kl}K\ . \qquad (24)$$

But for any SO(8) (vector) rotation $(q,p)=q_m p_m$ is obviously invariant and, as we have seen in (19) and (20), F_{kl} can be considered as such a rotation. So

$$(\overline{S_{kl}(q)},x\cdot p)=(S_{kl}(q),\overline{x\cdot p})= -(q,S_{kl}(\overline{x\cdot p}))= -(\bar{q},\overline{S_{kl}(\overline{x\cdot p})})\ ,$$

and (23) implies the following algebraic equation, valid for any two x,p octonions

$$\overline{S_{kl}(\overline{x\cdot p})} = G_{kl}(x)\cdot p + x\cdot C_{kl}(p)\ , \qquad (25)$$

(25) remains valid under any cyclic permutations of (S_{kl},G_{kl},C_{kl}). It is an algebraic expression of the equality between vectors and spinors in the-eight dimensional space [11].

Note that

$$S_{kl}=\tau(G_{kl})\ ,\ C_{kl}=\tau(S_{kl})=\tau^2(G_{kl})\ , \qquad (26)$$

where $\tau=\kappa\cdot\pi$. Indeed, only the second equation is not obvious. But $\tau^2=\pi\cdot\kappa$, and $C_{0A}=-F_{0A}$, $C_{AB}=F_{AB}$ as follows from their definitions through (17).

We can call τ the triality automorphism for the D_4 Lie algebra. It performs a cyclic interchange between vector and spinors, because G_{kl} operators realize the (1000) vector representation, S_{kl} – a first kind spinor (0001) and C_{kl} – a second kind spinor (0010), as can be established by explicit construction of the corresponding weight systems. For example, if $K_1=iC_{14}, K_2=iC_{25}, K_3=iC_{36}, K_4=iC_{07}$ represent the Cartan subalgebra and $\{\nu_i, i=1\div4\}$ is its dual basis, then the weight system and corresponding state vectors look like

$$\frac{1}{2}(-\nu_1-\nu_2-\nu_3+\nu_4) \text{ — } u_0 \qquad -\frac{1}{2}(-\nu_1-\nu_2-\nu_3+\nu_4) \text{ — } u_0^*$$

$$\frac{1}{2}(-\nu_1+\nu_2+\nu_3+\nu_4) \text{ — } u_1 \qquad -\frac{1}{2}(-\nu_1+\nu_2+\nu_3+\nu_4) \text{ — } u_1^*$$

$$\frac{1}{2}(+\nu_1-\nu_2+\nu_3+\nu_4) \text{ — } u_2 \qquad -\frac{1}{2}(+\nu_1-\nu_2+\nu_3+\nu_4) \text{ — } u_2^*$$

$$\frac{1}{2}(+\nu_1+\nu_2-\nu_3+\nu_4) \text{ — } u_3 \qquad -\frac{1}{2}(+\nu_1+\nu_2-\nu_3+\nu_4) \text{ — } u_3^*$$

So the highest weight is $\Lambda=\frac{1}{2}(\nu_1+\nu_2+\nu_3-\nu_4)=\alpha_2+\alpha_3+\frac{1}{2}(\alpha_1+\alpha_4)$ and we have (0010) irreducible representation. Here the state vectors are expressed in terms of split octonionic units [11]

$$u_0=\frac{1}{2}(e_0+ie_7) \qquad u_0^*=\frac{1}{2}(e_0-ie_7)$$
$$u_k=\frac{1}{2}(e_k+ie_{k+3}) \qquad u_k^*=\frac{1}{2}(e_k-ie_{k+3}) \quad , \qquad (27)$$

where $k=1\div3$.

In general, vector and spinors transform differently under 8-dimensional rotations, because $G_{kl}\neq S_{kl}\neq C_{kl}$. But it follows from (20) that $G_{A_1B_1}-G_{A_2B_2}$ and $G_{A_1B_1}-G_{A_3B_3}$ are invariant with regard to the triality automorphism, and so under such rotations 8-dimensional vector and both kinds of spinors transform in the same way. These transformations are automorphisms of the octonion algebra, because their generators act as derivations, as the principle of triality (25) shows. We can construct 14 linearly independent deri-

vations of the octonion algebra, because the method described above gives two independent rotations per one imaginary octonionic unit $e_A=e_{A_1}\cdot e_{B_1}$. One of possible choices is

$$L_1=i(G_{42}+G_{51}) \qquad L_8=\frac{i}{\sqrt{3}}(G_{41}+G_{52}+2G_{36})$$
$$L_2=i(G_{21}+G_{54}) \qquad L_9=\frac{i}{\sqrt{3}}(G_{51}+G_{24}+2G_{37})$$
$$L_3=i(G_{41}+G_{25}) \qquad L_{10}=\frac{i}{\sqrt{3}}(G_{54}+G_{12}+2G_{76})$$
$$L_4=i(G_{61}+G_{43}) \qquad L_{11}=\frac{i}{\sqrt{3}}(G_{62}+G_{35}+2G_{17}) \qquad (28)$$
$$L_5=i(G_{64}+G_{31}) \qquad L_{12}=\frac{i}{\sqrt{3}}(G_{32}+G_{56}+2G_{47})$$
$$L_6=i(G_{53}+G_{62}) \qquad L_{13}=\frac{i}{\sqrt{3}}(G_{43}+G_{16}+2G_{27})$$
$$L_7=i(G_{32}+G_{65}) \qquad L_{14}=\frac{i}{\sqrt{3}}(G_{31}+G_{46}+2G_{75})$$

It is well known [10] that the derivations of the octonion algebra form G_2 exceptional Lie algebra. We can rediscover this by constructing the Cartan-Weyl basis and root system from (28):

$$\frac{1}{2}(L_1\pm iL_2) \text{ — } \pm2\nu_1 \qquad \frac{1}{2}(L_9\pm iL_{10}) \text{ — } \pm(2/\sqrt{3})\nu_2$$
$$\frac{1}{2}(L_4\pm iL_5) \text{ — } \pm(\nu_1+\sqrt{3}\nu_2) \qquad \frac{1}{2}(L_{11}\pm iL_{12}) \text{ — } \pm(\nu_1+(1/\sqrt{3})\nu_2) \qquad (29)$$
$$\frac{1}{2}(L_6\pm iL_7) \text{ — } \pm(-\nu_1+\sqrt{3}\nu_2) \qquad \frac{1}{2}(L_{13}\pm iL_{14}) \text{ — } \pm(\nu_1-(1/\sqrt{3})\nu_2)$$

The role of the Cartan subalgebra basic elements play $K_1=L_3$, $K_2=L_8$ and $\{\nu_1,\nu_2\}$ is its dual basis in the root space. Note that K_1,K_2 are mutually orthogonal but not normalized to the unity (as in (2)) and so do ν_1,ν_2. But the normalizations of the K_1,K_2 are the same and we have $(\nu_1,\nu_1)=(\nu_2,\nu_2)$ and $(\nu_1,\nu_2)=0$.

The simple roots are $\alpha_1=\nu_1-\sqrt{3}\nu_2$, $\alpha_2=(2/\sqrt{3})\nu_2$ and they correspond to the Dynkin diagram  .

It follows from (28) and (29) that the first eight generators are closed under commutation and form A_2 Lie algebra — the Lie algebra of the SU(3) group. Algebraically this SU(3) can be defined as the subgroup of the octonion algebra automorphism group G_2 which leaves the seventh imaginary unit invariant. With regard to it u_k transforms as triplet, u_k^* — as antitriplet and u_0,u_0^* are singlets [11]. Therefore, if we take this SU(3) as the colour group [13,14], then all one-flavour quark-lepton degrees of freedom can be represented as one octonionic (super)field

$$q(x)=l(x)u_0+q_k(x)u_k+q_k^c(x)u_k^*+l^c(x)u_0^* , \qquad (30)$$

here $l(x)$, $q_k(x)$ are lepton and (three coloured) quark fields and $l^c(x)$, $q_k^c(x)$ — their charge conjugates.

Note that it doesn't matter what an octonion, first kind spinor, second kind spinor or vector we have in (30), because they all transform identically under SU(3).

So SO(8) can be considered as a natural one-flavour quark-lepton unification group. We can call it also a generalized colour group in the Pati-Salam sense, remembering their idea about the lepton number as the fourth colour [15]. Then the triality property of the SO(8) gives a natural reason why the number of flavours should be triplicated.

4. FAMILY FORMATION AND SO(10)

Unfortunately, SO(8) is not large enough to be used as a grand unification group: there is no room for weak interactions in it. This is not surprising, because weak interactions connect two different flavours and we are considering SO(8) as a one-flavour unification group.

The following observation points out the way how SO(8) can be extended to include the weak interactions. Because $C_{AB}=F_{AB}$ and $C_{A0}=-F_{A0}$ for A,B=1÷7, the SO(8) (Hermitian) generators for the (bi)spinor transformation (17) can be represented as $M_{AB}=-iF_{AB}$ and $M_{A0}=-M_{0A}=-i\sigma_3F_{A0}$. The last equation suggests to consider $M_{A,7+k}= =-i\sigma_kF_{A0}$ generators, where k=1÷3 and summation to the modulus 10 is assumed, i.e. 7+3=0. So we have two new operators $-i\sigma_1F_{A0}$, $-i\sigma_2F_{A0}$ which mix the upper and lower (bi)spinor octonionic components. Besides, if we consider these operators as rotations, then we have to add two extra dimensions and it is expected that SO(8) will be enlarged to SO(10) in this way and two different SO(8) spinors (two different flavours) will join in one SO(10) spinor (family formation).

To complete the derivation of the SO(10) group, we need the generators of rotations in the (7+i,7+j)-planes. They must obey $[M_{A,7+i},M_{A,7+j}]=iM_{7+i,7+j}$ (no summation!). On the other hand

$$[M_{A,7+i},M_{A,7+j}]=-[\sigma_i,\sigma_j]F_{A0}F_{A0}=\frac{i}{2}\varepsilon_{ijk}\sigma_k ,$$

because $F_{A0}F_{A0}=-\frac{1}{4}$, as can be easily verified. So the final expressions for the SO(10) generators are

$$M_{AB}=-iF_{AB} \qquad M_{7+i,7+j}=\frac{1}{2}\varepsilon_{ijk}\sigma_k ,$$
$$M_{A,7+k}=-i\sigma_kF_{A0} \qquad M_{7+k,A}=-M_{A,7+k} \qquad (31)$$

where $A,B=1\div7$ and $i,j,k=1\div3$. They really satisfy the SO(10) commutation relations

$$[M_{\mu\nu},M_{\tau\rho}] = -i(\delta_{\nu\tau}M_{\mu\rho}+\delta_{\mu\rho}M_{\nu\tau}-\delta_{\mu\tau}M_{\nu\rho}-\delta_{\nu\rho}M_{\mu\tau}) \,. \qquad (32)$$

To prove this, besides the $[F_{AB},F_{CD}]$ commutator, cited earlier, the following anticommutator, which results from the alternativity of the octonion algebra, is also helpful

$$\{F_{A0},F_{B0}\} = -\frac{1}{2}\delta_{AB} \,. \qquad (33)$$

It is clear from (32), that $M_{\alpha\beta}$ $(\alpha,\beta=0,7,8,9)$ and M_{mn} $(m,n=1\div6)$ subsets of generators are closed under commutation and commute to each other. They generate the D_2 and D_3 Lie algebras of the SO(4) and SO(6) groups. Comparison of the corresponding Dynkin diagrams shows that $D_2\approx A_1\oplus A_1$ and $D_3\approx A_3$. Therefore SO(4) and SO(6) groups are locally isomorphic to the SU(2)×SU(2) and SU(4). So the SO(10) group has $SU_L(2)\times SU_R(2)\times SU(4)$ subgroup. The generators of the $SU_L(2)\times SU_R(2)$ subgroup are

$$T_L^1 = \frac{1}{2}(M_{90}-M_{78}) \qquad T_R^1 = \frac{1}{2}(M_{90}+M_{78})$$

$$T_L^2 = \frac{1}{2}(M_{08}-M_{79}) \qquad T_R^2 = \frac{1}{2}(M_{08}+M_{79}) \,.$$

$$T_L^3 = \frac{1}{2}(M_{89}-M_{70}) \qquad T_R^3 = \frac{1}{2}(M_{89}+M_{70})$$

It is interesting, that they can be rewritten as

$$T_L^1=\frac{1}{2}\sigma_1 u_0 \quad , \quad T_R^1=\frac{1}{2}\sigma_1 u_0^* \,. \qquad (34)$$

So multiplication by u_0 or u_0^* split octonion units plays the role of projection operator on the left and right weak isospin,respectively.

The SU(4) generators, which correspond to the generalized Gell-Mann matrices, are the first eight L_m operators from (28) (SU(3) generators) added by

$$L'_9=M_{35}-M_{26} \qquad L'_{13}=M_{24}-M_{15}$$

$$L'_{10}=M_{23}-M_{56} \qquad L'_{14}=M_{12}-M_{45}$$

$$L'_{11}=M_{16}-M_{34} \qquad L'_{15}=\sqrt{\frac{2}{3}}\,(M_{14}+M_{25}+M_{36}) \,.$$

$$L'_{12}=M_{46}-M_{13}$$

From them SU(4) ladder operators can be constructed

$$E_{12}=\frac{1}{2}(L_1+iL_2) \quad E_{31}=\frac{1}{2}(L_4-iL_5) \quad E_{03}=\frac{1}{2}(L'_{13}+iL'_{14})$$

$$E_{13}=\frac{1}{2}(L_4+iL_5) \quad E_{32}=\frac{1}{2}(L_6-iL_7) \quad E_{10}=\frac{1}{2}(L'_9-iL'_{10})$$

$$E_{23}=\frac{1}{2}(L_6+iL_7) \quad E_{01}=\frac{1}{2}(L'_9+iL'_{10}) \quad E_{20}=\frac{1}{2}(L'_{11}-iL'_{12})$$

$$E_{21}=\frac{1}{2}(L_1-iL_2) \quad E_{02}=\frac{1}{2}(L'_{11}+iL'_{12}) \quad E_{30}=\frac{1}{2}(L'_{13}-iL'_{14})$$

Using $e_A\cdot(e_B\cdot q)+e_B\cdot(e_A\cdot q)=-2\delta_{AB}q$, which is equivalent to (33), they also can be expressed via split octonionic units:

$$E_{ij}=-u_i\cdot(u_j^* \quad , \quad E_{0i}=-u_j\cdot(u_k \quad , \quad E_{i0}=u_j^*\cdot(u_k^* \quad . \qquad (35)$$

In the last two equations (i,j,k) is a cyclic permutation of (1,2,3). It is assumed that, for example, $E_{ij}(q)= =-u_i\cdot(u_j^*\cdot q)$. (35) together with the split octonion multiplication table

$$\begin{array}{lll} u_i \cdot u_j = \varepsilon_{ijk} u_k^* & u_i \cdot u_0^* = u_i & u_0 \cdot u_i^* = 0 \\ u_i^* \cdot u_j^* = \varepsilon_{ijk} u_k & u_i^* \cdot u_0 = u_i^* & u_0^* \cdot u_i^* = u_i^* \\ u_i \cdot u_j^* = -\delta_{ij} u_0 & u_i^* \cdot u_0^* = 0 & u_0 \cdot u_0 = u_0 \\ u_i^* \cdot u_j = -\delta_{ij} u_0^* & u_0 \cdot u_i = u_i & u_0^* \cdot u_0^* = u_0^* \\ u_i \cdot u_0 = 0 & u_0^* \cdot u_i = 0 & u_0 \cdot u_0^* = u_0^* \cdot u_0 = 0 \end{array} \qquad (36)$$

gives $E_{\alpha\beta}(u_\gamma)=\delta_{\beta\gamma}u_\alpha$ and $E_{\alpha\beta}(u_\gamma^*)=-\delta_{\alpha\gamma}u_\beta^*$, where $\alpha,\beta,\gamma=0\div3$. In other words, u_α transforms under SU(4) as a $\underline{4}$ fundamental representation and u_α^* — as its conjugate $\underline{4}^*$. So SU(4) unifies u_0 colour singlet and u_k colour triplet in one single object, and therefore plays the role of the Pati-Salam group [15].

Note that all one-family (left-handed) quark-lepton degrees of freedom are unified in one bioctonionic (super) field (16-dimensional SO(10) spinor) [16]

$$\psi_L = \begin{pmatrix} \nu(x) \\ l(x) \end{pmatrix}_L u_0 + \begin{pmatrix} q_i^u(x) \\ q_i^d(x) \end{pmatrix}_L u_i + \begin{pmatrix} l^c(x) \\ \nu^c(x) \end{pmatrix}_L u_0^* + \begin{pmatrix} q_i^{dc}(x) \\ q_i^{uc}(x) \end{pmatrix}_L u_i^* . \qquad (37)$$

The explanation why we should take the Weyl (left-handed) spinors instead of Dirac (why the weak interactions are flavour chiral) is beyond the scope of this article (I don't have any). Algebraically this indicates close interplay between space-time (space inversion) and internal symmetries [17].

Thus our construction leads to SO(10) as a natural one-family unification group. But doing so, we have broken the triality symmetry: only the spinoric octonions take part in family formation and the vectoric octonion is singled out. Can we in some way restore equivalence between vector and spinor octonions?

First of all we need to realize vector octonion in terms of the SO(10) representation and this can be done by means of 2x2 octonionic Hermitian matrices, which together with the symmetric product $X\circ Y=\frac{1}{2}(XY+YX)$ form the M_2^8 Jordan algebra [18].

Elements from M_2^8 have the form $\begin{pmatrix} \alpha & a \\ \bar{a} & \beta \end{pmatrix}$, where α,β are real numbers and $a=a_0+a_Ae_A$ is an octonion. Clearly each such element has the unique decomposition $\alpha E_1+\beta E_2+F^a$, where

$$E_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad F^a=\begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix}.$$

So M_2^8 can be represented as a direct sum $M_2^8=RE_1\oplus RE_2\oplus F$. Actually, this is the Peirce decomposition of the M_2^8 Jordan algebra relative to the orthogonal idempotents E_1,E_2 [19].

The validity of the following multiplication table can be established by calculation

$$E_i\circ E_j=\delta_{ij}E_i , \quad E_i\circ F^a=\frac{1}{2}F^a , \quad F^a\circ F^b=(a,b)\,1 , \qquad (38)$$

where $i,j=1,2$ and $1=E_1+E_2$ is the unit element of M_2^8.

Let A_2^8 to be a set of the 2x2 octonionic anti-Hermitian matrices with zero diagonal elements. If $A\in A_2^8$ and $X\in M_2^8$, then $[A,X]=AX-XA\in M_2^8$. For any $A=\begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix}$ let $a^\wedge$ denote the following linear transformation of M_2^8: $a^\wedge X=[A,X]$. Calculating, we get

$$a^\wedge E_1=-F^a , \quad a^\wedge E_2=F^a , \quad a^\wedge F^b=2(a,b)(E_1-E_2) \qquad (39)$$

(38) and (39) indicate, that $a^\wedge$ is a derivation, i.e. for any $X,Y\in M_2^8$ we have $a^\wedge(X\circ Y)=(a^\wedge X)\circ Y+X\circ(a^\wedge Y)$.

Let now $\delta\in \mathrm{Der}\, M_2^8$ be some derivation and $\delta E_1=\alpha_1 E_1+\beta_1 E_2+F^{a_1}$, $\delta E_2=\alpha_2 E_1+\beta_2 E_2+F^{a_2}$. Then from $\delta E_i=\delta(E_i\circ E_i)=2E_i\circ\delta E_i$ follows that $\alpha_1=\beta_1=\alpha_2=\beta_2=0$ and from $0=\delta(E_1\circ E_2)=\delta E_1\circ E_2+E_1\circ\delta E_2 \longrightarrow a_1=-a_2$. So $\delta E_1=-F^{a_2}$, $\delta E_2=F^{a_2}$ and $\delta-a_2^\wedge$ will annihilate E_1,E_2 idempotents. Thus any $\delta\in\mathrm{Der}\, M_2^8$ element can be represented as

$$\delta=\Delta+a^\wedge , \qquad (40)$$

where Δ is the derivation which annihilates E_i idempotents. This decomposition is unique: if $\delta=0$, then $a^\wedge=-\Delta$ annihilates E_i and $a^\wedge E_1=-F^a$ shows that $a=0$.

Let $\Delta\in\mathrm{Der}\, M_2^8$ and $\Delta E_i=0$. Then applying Δ to $E_i\circ F^a=\frac{1}{2}F^a$ we get $E_i\circ\Delta F^a=\frac{1}{2}\Delta F^a$ and therefore $\Delta F^a\in F$. So Δ induces linear transformation D in the octonionic space:

$$\Delta F^a=F^{D(a)} . \qquad (41)$$

Applying Δ to $F^a\circ F^b=(a,b)1$ we get $(Da,b)+(a,Db)=0$, i.e. (a,b) scalar product is invariant under D and $D\in D_4$.

Vice versa, if $D\in D_4$, then Δ: $M_2^8\longrightarrow M_2^8$ linear mapping, defined as $\Delta E_i=0$, $\Delta F^a=F^{D(a)}$ is a derivation. If we denote $\Delta=D^\wedge$, then the general element from M_2^8 will have the form $\delta=a^\wedge+D^\wedge$. This indicates that $\dim(\mathrm{Der}\, M_2^8)=8+28=36$. It can be tested, by explicit construction of the root system, that $\mathrm{Der}\, M_2^8=B_4$ — the Lie algebra of the SO(9) group.

Acting by both sides of equation on the E_i,F^a basis, the following commutation relations can be verified:

$$[D_1^\wedge,D_2^\wedge]=[D_1,D_2]^\wedge ,\quad [a^\wedge,b^\wedge]=D^\wedge_{ab} ,\quad [D^\wedge,a^\wedge]=D(a)^\wedge , \qquad (42)$$

where $D_{ab}\in D_4$ is defined through $D_{ab}(x)=4(a,x)b-4(b,x)a$. Let us check, for example, the third equation:

$$[D^\wedge,a^\wedge]E_1=D^\wedge a^\wedge E_1=-F^{D(a)}=D(a)^\wedge E_1$$

$$[D^\wedge,a^\wedge]E_2=D^\wedge a^\wedge E_2=F^{D(a)}=D(a)^\wedge E_2$$

$$[D^\wedge,a^\wedge]F^x=-a^\wedge D^\wedge F^x=-2(a,Dx)(E_1-E_2)=2(Da,x)(E_1-E_2)=D(a)^\wedge F^x .$$

Let SM_2^8 be a set of elements from M_2^8 with zero trace and for any $T\in SM_2^8$ let $T^\wedge$ stand for the following linear transformation of M_2^8: $T^\wedge X=T\circ X\equiv X\cdot T$. If $\delta\in\mathrm{Der}\, M_2^8$ and $\delta+T^\wedge=0$, then $0=(\delta+T^\wedge)1=T^\wedge 1=T$ shows that $\delta=T^\wedge=0$. Therefore we have a direct sum of vector spaces $L=\mathrm{Der}(M_2^8)\oplus SM_2^{8\wedge}$, where $SM_2^{8\wedge}=\{T^\wedge, T\in SM_2^8\}$.

In L we have $\delta\in\mathrm{Der}\, M_2^8$, $F^{a\wedge}$, $E^\wedge$ (where $E=E_1-E_2$) elements and their linear combinations. By direct calculations the following commutation relations between them can be derived

$$[F^{a\wedge},F^{b\wedge}]=-\frac{1}{4}D^\wedge_{ab} ,\quad [E^\wedge,F^{a\wedge}]=\frac{1}{2}a^\wedge ,\quad [\delta,T^\wedge]=\delta(T)^\wedge . \qquad (43)$$

So L is closed under commutation and therefore is a Lie algebra. To find out what a Lie algebra it is, we need its root system, which can be obtained from the following Cartan-Weyl basis for L [20]:

$$K_1=iG^\wedge_{41} \quad K_2=iG^\wedge_{52} \quad K_3=iG^\wedge_{63} \quad K_4=iG^\wedge_{70} \quad K_5=E^\wedge$$

$$g_{1j}=T^\wedge_{1j} \quad g_{10}=T^\wedge_{10} \quad g_{01}=T^\wedge_{01} \quad X_{11}=\frac{1}{2}u_1^\wedge+\left(F^{u_1}\right)^\wedge$$

$$X^*_{11}=-\frac{1}{2}(u^*_1)^\wedge+\left(F^{u^*_1}\right)^\wedge \quad X_{21}=R^\wedge_1 \quad X^*_{21}=(R^*_1)^\wedge \quad Y_{11}=S^\wedge_1$$

$$Y^*_{1i}=(S^*_i)^\wedge \quad Y_{2i}=-\frac{1}{2}u_i^\wedge+\left(F^{u_i}\right)^\wedge \quad Y^*_{2i}=\frac{1}{2}(u^*_i)^\wedge+\left(F^{u^*_i}\right)^\wedge$$

$$W^+_L=\frac{1}{2}u_0^\wedge+\left(F^{u_0}\right)^\wedge \qquad W^-_L=-\frac{1}{2}(u^*_0)^\wedge+\left(F^{u^*_0}\right)^\wedge$$

$$W^+_R=\frac{1}{2}(u^*_0)^\wedge+\left(F^{u^*_0}\right)^\wedge \qquad W^-_R=-\frac{1}{2}u_0^\wedge+\left(F^{u_0}\right)^\wedge \tag{44}$$

where $i,j=1\div3$; u_α, u^*_α are the split octonionic units (27) and the D_4 transformations are defined through

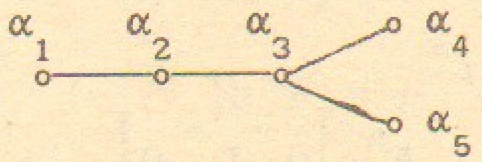
$$\begin{array}{ll}
2T_{12}=i(G_{42}+G_{51})-(G_{54}+G_{21}) & 2T_{21}=i(G_{42}+G_{51})+(G_{54}+G_{21}) \\
2T_{13}=i(G_{61}+G_{43})-(G_{31}+G_{64}) & 2T_{31}=i(G_{61}+G_{43})+(G_{31}+G_{64}) \\
2T_{23}=i(G_{53}+G_{62})-(G_{65}+G_{32}) & 2T_{32}=i(G_{53}+G_{62})+(G_{65}+G_{32}) \\
2T_{10}=i(G_{53}-G_{62})-(G_{65}-G_{32}) & 2T_{01}=i(G_{53}-G_{62})+(G_{65}-G_{32}) \\
2T_{20}=i(G_{61}-G_{43})-(G_{31}-G_{64}) & 2T_{02}=i(G_{61}-G_{43})+(G_{31}-G_{64}) \\
2T_{30}=i(G_{42}-G_{51})-(G_{54}-G_{21}) & 2T_{03}=i(G_{42}-G_{51})+(G_{54}-G_{21}) \\
2S_1=i(G_{40}+G_{71})+(G_{10}-G_{74}) & 2S^*_1=i(G_{40}+G_{71})-(G_{10}-G_{74}) \\
2S_2=i(G_{50}+G_{72})+(G_{20}-G_{75}) & 2S^*_2=i(G_{50}+G_{72})-(G_{20}-G_{75}) \\
2S_3=i(G_{60}+G_{73})+(G_{30}-G_{76}) & 2S^*_3=i(G_{60}+G_{73})-(G_{30}-G_{76}) \\
2R_1=i(G_{40}-G_{71})+(G_{10}+G_{74}) & 2R^*_1=i(G_{40}-G_{71})-(G_{10}+G_{74}) \\
2R_2=i(G_{50}-G_{72})+(G_{20}+G_{75}) & 2R^*_2=i(G_{50}-G_{72})-(G_{20}+G_{75}) \\
2R_3=i(G_{60}-G_{73})+(G_{30}+G_{76}) & 2R^*_3=i(G_{60}-G_{73})-(G_{30}+G_{76})
\end{array} \tag{45}$$

and G_{mn} operators were defined earlier in (18).

A computation of commutators shows that (44) is indeed a Cartan-Weyl basis and the roots have a simple form $\pm\nu_m\pm\nu_n$ ($n,m=1\div5$), where $\{\nu_m\}$ basis in root space is dual to the $\{K_m\}$ basis of the Cartan subalgebra. Note the correspondence between the Cartan-Weyl ladder operators and roots:

$$\begin{array}{lll}
g_{ij} — \nu_i-\nu_j & g_{i0} — -\nu_j-\nu_k & g_{0i} — \nu_j+\nu_k \\
X_{1i} — \nu_5+\nu_i & X^*_{1i} — -\nu_5-\nu_i & X_{2i} — \nu_4+\nu_i \\
X^*_{2i} — -\nu_4-\nu_i & Y_{1i} — -\nu_4+\nu_i & Y^*_{1i} — \nu_4-\nu_i \\
Y_{2i} — -\nu_5+\nu_i & Y^*_{2i} — \nu_5-\nu_i & W^+_L — \nu_4+\nu_5 \\
W^-_L — -\nu_4-\nu_5 & W^+_R — \nu_5-\nu_4 & W^-_R — -\nu_5+\nu_4
\end{array} \tag{46}$$

The simple roots are $\alpha_1=\nu_1-\nu_2$, $\alpha_2=\nu_2-\nu_3$, $\alpha_3=\nu_3-\nu_4$, $\alpha_4=\nu_4-\nu_5$, $\alpha_5=\nu_4+\nu_5$ and they correspond to the Dynkin diagram



So L is D_5 Lie algebra and the (44) operators, acting in the 10-dimensional complex vector space generated by the M^8_2 basic elements (the complexification of M^8_2), give its (10000) irreducible representation.

Thus, now we have at hand the realization of spinoric octonions as a 16-dimensional SO(10) spinor $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ and vectoric octonion as a 10-dimensional SO(10) vector $\begin{pmatrix} \alpha & q \\ \bar{q} & \beta \end{pmatrix}$. How to unify them? The familiar unitary symmetry example how to unify an isodublet and an isotriplet in the 3x3 complex Hermitian matrix can give us a hint and so let us consider 3x3 octonion Hermitian matrices.

5. E_6, TRIALITY AND FAMILY TRIPLICATION

Together with the symmetric product, 3×3 octonion Hermitian matrices form the M_3^8 exceptional Jordan algebra [10]. For it we can repeat the considerations of the preceding section.

A general element from M_3^8 has the form

$$X=\begin{pmatrix} \alpha & x_3 & \bar{x}_2 \\ \bar{x}_3 & \beta & x_1 \\ x_2 & \bar{x}_1 & \gamma \end{pmatrix}$$

and can be uniquely presented as $X=\alpha E_1+\beta E_2+\gamma E_3+F_1^{x_1}+F_2^{x_2}+F_3^{x_3}$ where

$$E_1=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad E_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad E_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F_1^a=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & \bar{a} & 0 \end{pmatrix} \qquad F_2^a=\begin{pmatrix} 0 & 0 & \bar{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \qquad F_3^a=\begin{pmatrix} 0 & a & 0 \\ \bar{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is the Peirce decomposition of M_3^8 relative to the mutually orthogonal idempotents E_i and $RE_i=\{\alpha E_i:\ \alpha\in R\}$, $F_i=\{F_i^a:\ a$ is an octonion$\}$ are its Peirce components. In particular, if (i,j,k) is some permutation of $(1,2,3)$ and $X\in M_3^8$ obeys $E_j\circ X=E_k\circ X=\frac{1}{2}X$, then $X\in F_i$.

The multiplication table looks like (no summation!)

$$E_i\circ E_j=\delta_{ij}E_i \qquad E_i\circ F_j^a=\frac{1}{2}(1-\delta_{ij})F_j^a$$

$$F_i^a\circ F_i^b=(a,b)(E_j+E_k) \qquad F_i^a\circ F_j^b=\frac{1}{2}F_k^{\overline{ab}} \quad , \tag{47}$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$.

Let A_3^8 designate a set of the 3×3 octonion anti-Hermitian matrices with zero diagonal elements. If $A\in A_3^8$ and $X\in M_3^8$ then $[A,X]=AX-XA\in M_3^8$. Therefore we have $A^\wedge:\ M_3^8\longrightarrow M_3^8$ linear transformation defined as $A^\wedge X=[A,X]$.

For any $A=\begin{pmatrix} 0 & a_3 & \bar{a}_2 \\ -\bar{a}_3 & 0 & a_1 \\ -a_2 & -\bar{a}_1 & 0 \end{pmatrix}$ matrix let us designate $\{a_1,a_2,a_3\}=A^\wedge$. By direct calculations its effect on the M_3^8 basic elements can be got:

$$\{a_1,a_2,a_3\}E_1=-F_2^{a_2}-F_3^{a_3} \quad \{a_1,a_2,a_3\}E_2=F_3^{a_3}-F_1^{a_1}$$

$$\{a_1,a_2,a_3\}E_3=F_2^{a_2}+F_1^{a_1} \quad \{a_1,a_2,a_3\}F_1^a=F_2^{\overline{a_3a}}+F_3^{\overline{aa_2}}+2(a,a_1)(E_2-E_3)$$

$$\{a_1,a_2,a_3\}F_2^a=F_3^{\overline{a_1a}}-F_1^{\overline{aa_3}}+2(a,a_2)(E_1-E_3) \tag{48}$$

$$\{a_1,a_2,a_3\}F_3^a=-F_1^{\overline{a_2a}}-F_2^{\overline{aa_1}}+2(a,a_3)(E_1-E_2)$$

Using them, it can be shown that if $A\in A_3^8$, then $A^\wedge$ is a derivation.

Let $\delta\in\mathrm{Der}\ M_3^8$ be some derivation. From $\delta E_i=\delta(E_i\circ E_i)=$ $=2E_i\circ\delta E_i$ it follows

$$\delta E_i \circ (1-2E_i)=0\ ,\quad i=1\div 3\ . \qquad (49)$$

Let $\delta E_i=\alpha_i E_1+\beta_i E_2+\gamma_i E_3+F_1^{x_1^i}+F_2^{x_2^i}+F_3^{x_3^i}$. Then from (47) we get

$$\delta E_1 \circ (1-2E_1)= -\alpha_1 E_1+\beta_1 E_2+\gamma_1 E_3+F_1^{x_1^1}$$

$$\delta E_2 \circ (1-2E_2)= \alpha_2 E_1-\beta_2 E_2+\gamma_2 E_3+F_2^{x_2^2}\ ,$$

$$\delta E_3 \circ (1-2E_3)= \alpha_3 E_1+\beta_3 E_2-\gamma_3 E_3+F_3^{x_3^3}$$

hence (49) is fulfilled if $\alpha_i=\beta_i=\gamma_i=0$ and $x_1^1=x_2^2=x_3^3=0$. Furthermore, $E_i \circ E_j=0$ $(i\neq j)$ indicates $\delta E_i \circ E_j+E_i \circ \delta E_j=0$. Substituting here $\delta E_i=F_j^{x_j^i}+F_k^{x_k^i}$ we get $x_1^2=-x_1^3$, $x_2^1=-x_2^3$, $x_3^1=-x_3^2$ and therefore

$$\delta E_1=-F_2^{x_2^3}-F_3^{x_3^2} \qquad \delta E_2=-F_1^{x_1^3}+F_3^{x_3^2} \qquad \delta E_3= F_1^{x_1^3}+F_2^{x_2^3}\ . \qquad (50)$$

Comparing (48) and (50) we conclude that $\delta-\{x_1^3,x_2^3,x_3^2\}$ derivation annihilates all three idempotents.

Thus any $\delta \in \mathrm{Der}\, M_3^8$ derivation decomposes as $\delta=\Delta+A^\wedge$, where $A\in A_3^8$ and Δ derivation annihilates E_i idempotents. This decomposition is unique: if $\Delta+A^\wedge=0$, then $A^\wedge E_i=0$, $i=1\div 3$ and (48) shows that $A=0$.

Let $\Delta E_i=0$. Acting by Δ on the equation $E_i \circ F_j^a=\frac{1}{2}F_j^a$ $(i\neq j)$ we get $E_i \circ \Delta F_j^a=\frac{1}{2}\Delta F_j^a$. This shows that the Peirce

components F_i are invariant under Δ, and Δ induces Δ_i linear transformations in the octonion space:

$$\Delta F_i^a = F_i^{\Delta_i(a)}\ . \qquad (51)$$

Applying Δ to $F_i^a \circ F_i^b=(a,b)(E_j+E_k)$ we get

$$F_i^{\Delta_i a} \circ F_i^b+F_i^a \circ F_i^{\Delta_i b}=0\ ,$$

i.e. $(\Delta_i a,b)+(a,\Delta_i b)=0$. So (a,b) scalar product is invariant under Δ_i, hence $\Delta_i \in D_4$.

Similarly, from $F_1^a \circ F_2^b=\frac{1}{2}F_3^{\overline{ab}}$ it follows

$$F_1^{\Delta_1 a} \circ F_2^b + F_1^a \circ F_2^{\Delta_2 b} = \frac{1}{2}F_3^{\Delta_3(\overline{ab})}\ ,$$

or $\overline{\Delta_3(\overline{ab})} = (\Delta_1 a)\cdot b + a\cdot(\Delta_2 b)$. This shows that $\Delta_1,\Delta_2,\Delta_3$ form a triality triplet and $\Delta_2=\tau(\Delta_1)$, $\Delta_3=\tau^2(\Delta_1)$, τ being the triality automorphism (26).

Let us denote Δ derivation, which annihilates E_i idempotents and in the Peirce components acts according to (51), by $\{\Delta_1,\Delta_2,\Delta_3\}$. Thusa general element from $\mathrm{Der}\, M_3^8$ is $\delta=\{a_1,a_2,a_3\}+\{\Delta_1,\Delta_2,\Delta_3\}$. So, because a triality triplet is uniquely defined by its first element, $\dim(\mathrm{Der}\, M_3^8)=$ $=3\times 8+28=52$. It can be tested, by constructing the root system, that $\mathrm{Der}\, M_3^8$ is F_4 exceptional Lie algebra.

The commutation relations in the $\mathrm{Der}\, M_3^8$ are given by

$$[\{\Delta_1,\Delta_2,\Delta_3\},\{\Delta_1',\Delta_2',\Delta_3'\}]=\{[\Delta_1,\Delta_1'],[\Delta_2,\Delta_2'],[\Delta_3,\Delta_3']\}$$

$$[\{\Delta_1,\Delta_2,\Delta_3\},\{a_1,a_2,a_3\}]=\{\Delta_1 a_1,\Delta_2 a_2,\Delta_3 a_3\} \qquad (52)$$

$$[\{a,0,0\},\{b,0,0\}]=\{\Delta^1_{ab},\Delta^2_{ab},\Delta^3_{ab}\} \quad [\{a,0,0\},\{0,b,0\}]=\{0,0,\overline{ab}\}$$

$$[\{0,a,0\},\{0,b,0\}]=\{\Delta^3_{ab},\Delta^1_{ab},\Delta^2_{ab}\} \quad [\{0,0,a\},\{b,0,0\}]=\{0,\overline{ab},0\}$$

$$[\{0,0,a\},\{0,0,b\}]=\{\Delta^2_{ab},\Delta^3_{ab},\Delta^1_{ab}\} \quad [\{0,a,0\},\{0,0,b\}]=\{\overline{ab},0,0\},$$

where $\Delta^1_{ab},\Delta^2_{ab},\Delta^3_{ab}$ triality triplet is defined through

$$\Delta^1_{ab}(x)=4(a,x)b-4(b,x)a$$

$$\Delta^2_{ab}(x)=\bar{b}\cdot(a\cdot x)-\bar{a}\cdot(b\cdot x)\ .$$

$$\Delta^3_{ab}(x)=(x\cdot a)\cdot\bar{b}-(x\cdot b)\cdot\bar{a}$$

As an illustration, let us prove that

$$[\{a,0,0\},\{0,b,0\}]F_1^x=\{0,0,\overline{ab}\}F_1^x\ .$$

Using (48), we get

$$[\{a,0,0\},\{0,b,0\}]F_1^x=F_2^{2(a,x)b-\bar{a}\cdot(x\cdot b)}\ .$$

Furthermore, because of octonion alternativity

$$2(a,x)b-\bar{a}\cdot(x\cdot b)=(\bar{x}\cdot a)\cdot b-(\bar{a},x,b)=(\bar{x}\cdot a)\cdot b+(\bar{x},a,b)=\bar{x}\cdot(a\cdot b)\ ,$$

and

$$[\{a,0,0\},\{0,b,0\}]F_1^x=F_2^{\bar{x}\cdot(a\cdot b)}=\{0,0,\overline{ab}\}F_1^x\ .$$

Let SM_3^8 be a set of elements from M_3^8 with zero trace. For any $T\in SM_3^8$ let $T^\wedge$ denote the following linear trans-

formation of M_3^8: $T^\wedge X==T\circ X$. As in the case of M_2^8, we have a direct sum of the vector spaces $L=Der(M_2^8)\oplus SM_2^{8\wedge}$.

Commutation relations between $\delta\in Der\ M_3^8$, $(F_1^a)^\wedge$, $(F_2^a)^\wedge$, $(F_3^a)^\wedge$, $E_{23}^\wedge$, $E_{13}^\wedge$ (where $E_{ij}=E_i-E_j$) elements of L can be obtained using octonion properties and the result is

$$[(F_1^a)^\wedge,(F_1^b)^\wedge]=-\tfrac{1}{4}\{\Delta^1_{ab},\Delta^2_{ab},\Delta^3_{ab}\} \quad [(F_2^a)^\wedge,(F_2^b)^\wedge]=-\tfrac{1}{4}\{\Delta^3_{ab},\Delta^1_{ab},\Delta^2_{ab}\}$$

$$[(F_3^a)^\wedge,(F_3^b)^\wedge]=-\tfrac{1}{4}\{\Delta^2_{ab},\Delta^3_{ab},\Delta^1_{ab}\} \quad [(F_1^a)^\wedge,(F_2^b)^\wedge]=-\tfrac{1}{4}\{0,0,\overline{ab}\}$$

$$[(F_3^a)^\wedge,(F_1^b)^\wedge]=\tfrac{1}{4}\{0,\overline{ab},0\} \quad [(F_2^a)^\wedge,(F_3^b)^\wedge]=-\tfrac{1}{4}\{\overline{ab},0,0\}$$

$$[E_{13}^\wedge,(F_1^a)^\wedge]=\frac{1}{4}\{a,0,0\} \quad [E_{13}^\wedge,(F_2^a)^\wedge]=\frac{1}{2}\{0,a,0\}\ . \qquad (53)$$

$$[E_{13}^\wedge,(F_3^a)^\wedge]=\frac{1}{4}\{0,0,a\} \quad [E_{23}^\wedge,(F_1^a)^\wedge]=\frac{1}{2}\{a,0,0\}$$

$$[E_{23}^\wedge,(F_2^a)^\wedge]=\frac{1}{4}\{0,a,0\} \quad [E_{23}^\wedge,(F_3^a)^\wedge]=-\frac{1}{4}\{0,0,a\}$$

$$[E_{23}^\wedge,E_{13}^\wedge]=0 \quad [\delta,T^\wedge]=\delta(T)^\wedge$$

As we see, L is closed under commutation. To rediscover the known fact [10,21] that L is E_6 exceptional Lie algebra, let us consider the following basis ($G^\tau\equiv\tau(G)$, $G^{\tau^2}\equiv\tau^2(G)$):

$$K_1=i\{G_{41},G^\tau_{41},G^{\tau^2}_{41}\} \quad K_2=i\{G_{52},G^\tau_{52},G^{\tau^2}_{52}\} \quad K_3=i\{G_{63},G^\tau_{63},G^{\tau^2}_{63}\}$$

$$K_4=i\{G_{70},G^\tau_{70},G^{\tau^2}_{70}\} \quad K_5=E_{23}^\wedge \quad K_6=\frac{1}{\sqrt{3}}(2E_{13}^\wedge-E_{23}^\wedge)$$

$$g_{ij}=i\{T_{ij},T^\tau_{ij},T^{\tau^2}_{ij}\} \quad g_{i0}=i\{T_{i0},T^\tau_{i0},T^{\tau^2}_{i0}\} \quad g_{0i}=i\{T_{0i},T^\tau_{0i},T^{\tau^2}_{0i}\}$$

$$X_{1i}=\frac{1}{2}\{u_i,0,0\}+\left(F_1^{u_i}\right)^\wedge \quad X^*_{1i}=-\frac{1}{2}\{u^*_i,0,0\}+\left(F_1^{u^*_i}\right)^\wedge \quad X_{2i}=\{R_i,R_i^\tau,R_i^{\tau^2}\}$$

$$X^*_{2i}=\{R^*_i,R_i^{*\tau},R_i^{*\tau^2}\} \qquad Y_{1i}=\{S_i,S_i^\tau,S_i^{\tau^2}\} \qquad Y^*_{1i}=\{S^*_i,S_i^{*\tau},S_i^{*\tau^2}\}$$

$$Y_{2i}=-\frac{1}{2}\{u_i,0,0\}+\left(F_1^{u_i}\right)^\wedge \qquad Y^*_{2i}=\frac{1}{2}\{u^*_i,0,0\}+\left(F_1^{u^*_i}\right)^\wedge$$

$$W^+_L=\frac{1}{2}\{u_0,0,0\}+\left(F_1^{u_0}\right)^\wedge \qquad W^-_L=-\frac{1}{2}\{u^*_0,0,0\}+\left(F_1^{u^*_0}\right)^\wedge$$

$$W^+_R=\frac{1}{2}\{u^*_0,0,0\}+\left(F_1^{u^*_0}\right)^\wedge \qquad W^-_R=-\frac{1}{2}\{u_0,0,0\}+\left(F_1^{u_0}\right)^\wedge \qquad (54)$$

$$X_{Li}=-\frac{1}{2}\{0,0,u_i\}+\left(F_3^{u_i}\right)^\wedge \qquad X^*_{Li}=\frac{1}{2}\{0,0,u^*_i\}+\left(F_3^{u^*_i}\right)^\wedge$$

$$Y_{Li}=-\frac{1}{2}\{0,u_i,0\}+\left(F_2^{u_i}\right)^\wedge \qquad Y^*_{Li}=\frac{1}{2}\{0,u^*_i,0\}+\left(F_2^{u^*_i}\right)^\wedge$$

$$X_{Ri}=\frac{1}{2}\{0,0,u_i\}+\left(F_3^{u_i}\right)^\wedge \qquad X^*_{Ri}=-\frac{1}{2}\{0,0,u^*_i\}+\left(F_3^{u^*_i}\right)^\wedge$$

$$Y_{Ri}=\frac{1}{2}\{0,u_i,0\}+\left(F_2^{u_i}\right)^\wedge \qquad Y^*_{Ri}=-\frac{1}{2}\{0,u^*_i,0\}+\left(F_2^{u^*_i}\right)^\wedge$$

$$V^+_L=\frac{1}{2}\{0,u^*_0,0\}+\left(F_2^{u^*_0}\right)^\wedge \qquad V^-_L=-\frac{1}{2}\{0,u_0,0\}+\left(F_2^{u_0}\right)^\wedge$$

$$V^+_R=-\frac{1}{2}\{0,0,u_0\}+\left(F_3^{u_0}\right)^\wedge \qquad V^-_R=\frac{1}{2}\{0,0,u^*_0\}+\left(F_3^{u^*_0}\right)^\wedge$$

$$V_L=\frac{1}{2}\{0,0,u_0\}+\left(F_3^{u_0}\right)^\wedge \qquad V^*_L=-\frac{1}{2}\{0,0,u^*_0\}+\left(F_3^{u^*_0}\right)^\wedge$$

$$V_R=-\frac{1}{2}\{0,u^*_0,0\}+\left(F_2^{u^*_0}\right)^\wedge \qquad V^*_R=\frac{1}{2}\{0,u_0,0\}+\left(F_2^{u_0}\right)^\wedge$$

To find the roots, we need the results of action of the ad_{K_μ}, $\mu=1\div 6$ operators on this basis (their eigenvalues). The calculations by means of (52) and (53) show that (54) is the Cartan-Weyl basis and the roots are

$$\pm\nu_m\pm\nu_n \ , \quad \pm\Lambda_\alpha\pm\frac{1}{2}(\nu_5+\sqrt{3}\nu_6) \ , \quad \pm M_\alpha\pm\frac{1}{2}(\nu_5-\sqrt{3}\nu_6) \ , \qquad (55)$$

where $m,n=1\div 5$, $\alpha=1\div 4$ and

$$M_i=M_4-\nu_j-\nu_k \qquad M_4=\frac{1}{2}(\nu_1+\nu_2+\nu_3+\nu_4)$$

$$\Lambda_i=\Lambda_4-\nu_j-\nu_k \qquad \Lambda_4=\frac{1}{2}(\nu_1+\nu_2+\nu_3-\nu_4)$$

The correspondence (46) remains valid. The other Cartan-Weyl basic elements correspond to the roots

$$X_{Li} \ — \ M_i+\frac{1}{2}(\nu_5-\sqrt{3}\nu_6) \qquad X^*_{Li} \ — \ -M_i-\frac{1}{2}(\nu_5-\sqrt{3}\nu_6)$$

$$Y_{Li} \ — \ \Lambda_i-\frac{1}{2}(\nu_5+\sqrt{3}\nu_6) \qquad Y^*_{Li} \ — \ -\Lambda_i+\frac{1}{2}(\nu_5+\sqrt{3}\nu_6)$$

$$X_{Ri} \ — \ M_i-\frac{1}{2}(\nu_5-\sqrt{3}\nu_6) \qquad X^*_{Ri} \ — \ -M_i+\frac{1}{2}(\nu_5-\sqrt{3}\nu_6)$$

$$Y_{Ri} \ — \ \Lambda_i+\frac{1}{2}(\nu_5+\sqrt{3}\nu_6) \qquad Y^*_{Ri} \ — \ -\Lambda_i-\frac{1}{2}(\nu_5+\sqrt{3}\nu_6) \qquad (56)$$

$$V^+_L \ — \ -\Lambda_4+\frac{1}{2}(\nu_5+\sqrt{3}\nu_6) \qquad V^-_L \ — \ \Lambda_4-\frac{1}{2}(\nu_5+\sqrt{3}\nu_6)$$

$$V_L \ — \ -M_4-\frac{1}{2}(\nu_5-\sqrt{3}\nu_6) \qquad V^*_L \ — \ M_4+\frac{1}{2}(\nu_5-\sqrt{3}\nu_6)$$

$$V^+_R \ — \ -M_4+\frac{1}{2}(\nu_5-\sqrt{3}\nu_6) \qquad V^-_R \ — \ M_4-\frac{1}{2}(\nu_5-\sqrt{3}\nu_6)$$

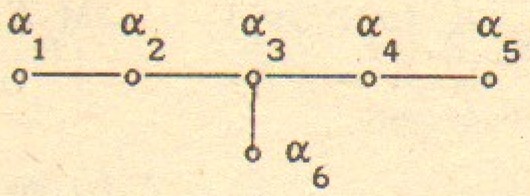
$$V_R \ — \ -\Lambda_4-\frac{1}{2}(\nu_5+\sqrt{3}\nu_6) \qquad V^*_R \ — \ \Lambda_4+\frac{1}{2}(\nu_5+\sqrt{3}\nu_6)$$

The simple roots from (55) are

$$\alpha_1=\frac{1}{2}(\nu_1-\nu_2-\nu_3-\nu_4-\nu_5-\sqrt{3}\nu_6)\qquad \alpha_2=\nu_4+\nu_5\qquad \alpha_3=\nu_3-\nu_4$$

$$\alpha_4=\nu_4-\nu_5\qquad \alpha_5=\frac{1}{2}(\nu_1-\nu_2-\nu_3-\nu_4-\nu_5+\sqrt{3}\nu_6)\qquad \alpha_6=\nu_2-\nu_3\ ,$$

and they correspond to the Dynkin diagram



(it can be shown that $(\nu_m,\nu_n)=\frac{1}{24}\delta_{mn}$).

Hence L is E_6 exceptional Lie algebra. The way how it was constructed shows the close relationship between D_5 and E_6: the latter is connected to the exceptional Jordan algebra M_3^8 and the former — to the Jordan algebra M_2^8 [22].

(47) multiplication table shows that M_3^8 has three M_2^8 subalgebras, consisting correspondingly from elements

$$\begin{pmatrix}\alpha & a & 0\\ \overline{a} & \beta & 0\\ 0 & 0 & 0\end{pmatrix},\quad \begin{pmatrix}\alpha & 0 & \overline{a}\\ 0 & 0 & 0\\ a & 0 & \beta\end{pmatrix}\quad \text{and}\quad \begin{pmatrix}0 & 0 & 0\\ 0 & \alpha & a\\ 0 & \overline{a} & \beta\end{pmatrix}.$$

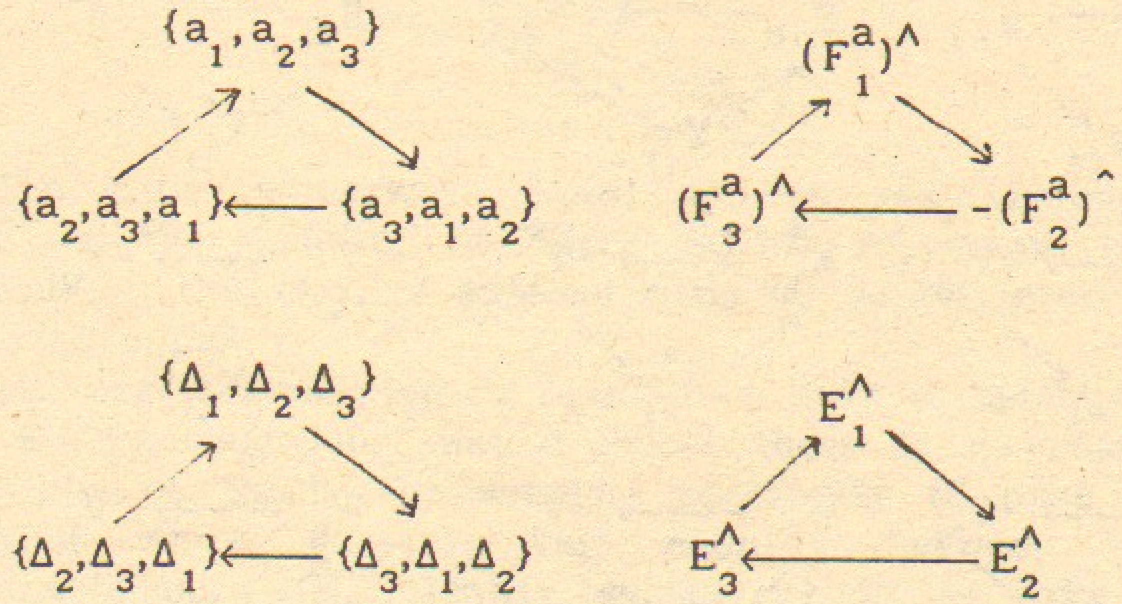
Therefore E_6 has three equivalent D_5 subalgebras. It is interesting to note that the corresponding SO(10) subgroups in E_6 are the analogies of the U-,V- and T-spin SU(2) subgroups in SU(3). This becomes clear if note, that when octonions are changed to real numbers the above described construction for D_5 gives A_1 Lie algebra and for E_6 — A_2 Lie algebra.

Let D_5^i be that D_5 subalgebra of E_6 which acts in the M_2^8 Jordan algebra, formed from the F_i^a, E_j, E_k elements. D_5^i

consists from $\{\Delta_1,\Delta_2,\Delta_3\}$, $(F_i^a)^\wedge$, $E_{jk}^\wedge$ $(E_{jk}=E_j-E_k)$, $\{\delta_{i1}a_1,\delta_{i2}a_2,\delta_{i3}a_3\}$ operators and their (complex) linear combinations. Therefore the intersection of these D_5^i subalgebras is D_4 formed from the $\{\Delta_1,\Delta_2,\Delta_3\}$ triality triplets, and their unification gives the whole E_6 algebra.

The triality automorphism for D_4 can be continued on E_6:

τ :



(57)

It can be verified that (57) actually is an E_6 automorphism, i.e. that (52),(53) commutation relations are invariant under it. For example, let us consider

$$[\{a_1,a_2,a_3\},(F_1^a)^\wedge]=\left(F_2^{\overline{a_3a}}\right)^\wedge+\left(F_3^{\overline{aa_2}}\right)^\wedge+2(a,a_1)E_{23}^\wedge\ .$$

We have

$$\tau(\{a_1,a_2,a_3\})=\{a_3,a_1,a_2\}\quad,\quad \tau(F_1^{a\wedge})=-F_2^{a\wedge}$$

and

$$\tau\left(\left(F_2^{\overline{a_3a}}\right)^\wedge+\left(F_3^{\overline{aa_2}}\right)^\wedge+2(a,a_1)E_{23}^\wedge\right)=-\left(F_3^{\overline{a_3a}}\right)^\wedge+\left(F_1^{\overline{aa_2}}\right)^\wedge-2(a,a_1)E_{13}^\wedge.$$

But with the help of (48),(53) we find that really

$$[\{a_3,a_1,a_2\},-(F_2^a)^\wedge]=-\left(F_3^{\overline{a_3a}}\right)^\wedge+\left(F_1^{\overline{aa_2}}\right)^\wedge-2(a,a_1)E_{13}^\wedge\ .$$

This τ automorphism causes a cyclic permutation of the D^i subalgebras

$$\tau: \quad D^1_5 \longrightarrow D^2_5 \longrightarrow D^3_5 \longrightarrow D^1_5 \qquad (58)$$

and the following cyclic permutations in the (54) basis

$$g_{ij} \longrightarrow g_{ij} \longrightarrow g_{ij} \longrightarrow g_{ij} \qquad g_{i0} \longrightarrow -X_{2i} \longrightarrow Y_{1i} \longrightarrow g_{i0} \qquad g_{0i} \longrightarrow -X^*_{2i} \longrightarrow Y^*_{1i} \longrightarrow g_{0i}$$

$$X_{1i} \longrightarrow -Y_{Li} \longrightarrow X_{Ri} \longrightarrow X_{1i} \qquad X^*_{1i} \longrightarrow -Y^*_{Li} \longrightarrow X^*_{Ri} \longrightarrow X^*_{1i} \qquad Y_{2i} \longrightarrow -Y_{Ri} \longrightarrow X_{Li} \longrightarrow Y_{2i} \qquad (59)$$

$$Y^*_{2i} \longrightarrow -Y^*_{Ri} \longrightarrow X^*_{Li} \longrightarrow Y^*_{2i} \qquad W^+_L \longrightarrow -V^-_L \longrightarrow V_L \longrightarrow W^+_L \qquad W^-_L \longrightarrow -V^+_L \longrightarrow V^*_L \longrightarrow W^-_L$$

$$W^-_R \longrightarrow -V^*_R \longrightarrow V^+_R \longrightarrow W^-_R \qquad W^+_R \longrightarrow -V_R \longrightarrow V^-_R \longrightarrow W^+_R$$

On the Cartan subalgebra its effect is given by

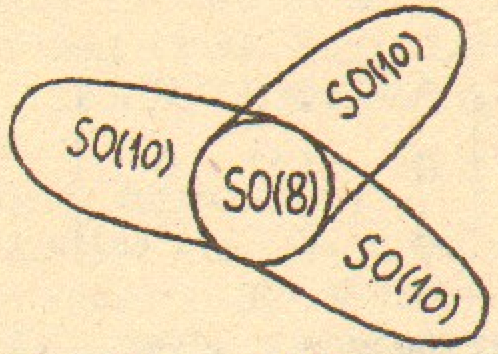
$$\tau(K_1)=\frac{1}{2}(K_1-K_2-K_3-K_4) \qquad \tau(K_2)=\frac{1}{2}(-K_1+K_2-K_3-K_4)$$

$$\tau(K_3)=\frac{1}{2}(-K_1-K_2+K_3-K_4) \qquad \tau(K_4)=\frac{1}{2}(K_1+K_2+K_3-K_4) \qquad (60)$$

$$\tau(K_5)=-\frac{1}{2}(K_5+\sqrt{3}\;K_6) \qquad \tau(K_5)=\frac{1}{2}(\sqrt{3}\;K_5-K_6)$$

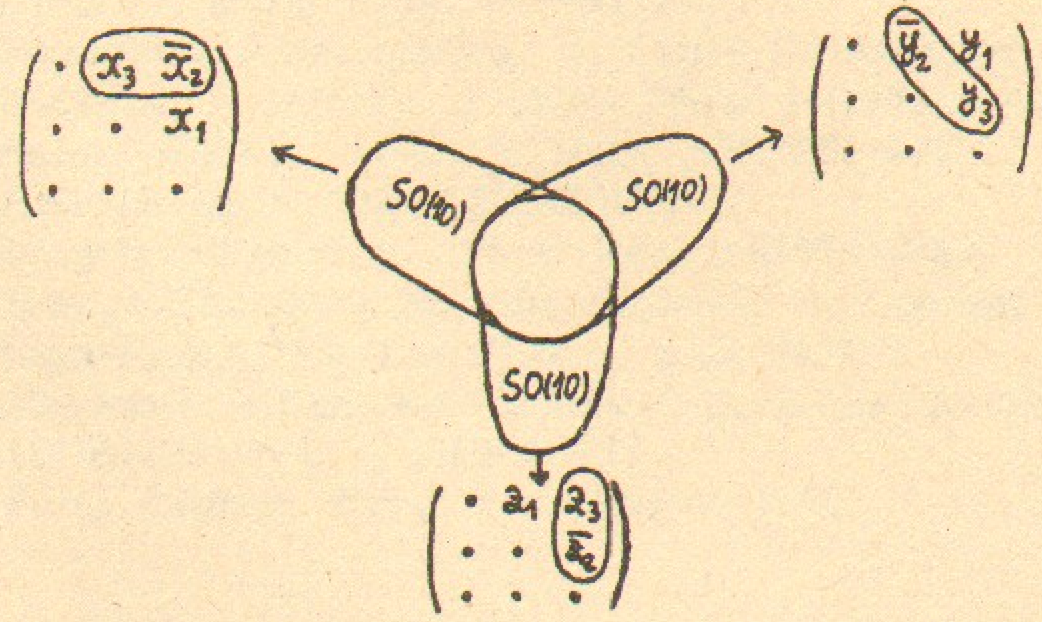
So E_6 exceptional Lie group is very closely related to triality. Firstly, it unifies the spinoric and vectoric

octonions in one 27-dimensional irreducible representation (algebraically they unify in the M^8_3 exceptional Jordan algebra). Secondly, its internal structure also reveals a very interesting triality picture:

$E_6 =$ 

The equality between SO(8) spinors and vectors now results in the equality of three SO(10) subgroups (in the existence of the triality automorphism τ, which interchanges these subgroups).

To form a quark-lepton family, we have to select one of these SO(10) subgroups. But a priori there is no reason to prefer any of them. The simplest possibility to have family formation which respects this equality between various SO(10) subgroups (E_6 triality symmetry) is to take three copies of M^8_3 and arrange matters in such a way that in the first M^8_3 the first SO(10) subgroup acts as a family formating group, in the second M^8_3 — the second SO(10) and in the third one — the third SO(10):



More formally, we have 27+27+27 reducible representation of E_6, such that when we go from one irreducible subspace to another, representation matrices are rotated by the triality automorphism τ : for any $a \in E_6$ element a representation matrix looks like

$$\begin{pmatrix} A & 0 & 0 \\ 0 & \tau(A) & 0 \\ 0 & 0 & \tau^2(A) \end{pmatrix},$$

where $a \longrightarrow A$ correspondence gives a 27 irreducible representation.

6. CONCLUSION

If we take seriously that octonions play some underlying dynamical role in particle physics and SO(8) appears as a one-flavour unification group, then the triality property of SO(8) gives a natural reason for the existence of three quark-lepton generations. Family formation from two flavours due to weak interactions can be connected naturally enough to SO(10) group, but with the triality symmetry violated. An attempt to restore this symmetry leads to the exceptional group E_6 and three quark-lepton families.

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Z.K. Silagadze

SO(8) Colour as Possible Origin of Generations

З.К. Силагадзе

SO(8) Цвет как возможный источник поколений

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