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RELATIVISTIC OSCILLATOR
MODEL and DELBRÜCK SCATTERING

Relativistic Oscillator Model and Delbrück scattering

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Abstract

Elastic scattering of photons in a Lorentz-scalar potential via virtual spin-zero particle-antiparticle pairs ("Delbrück scattering") is considered. An analytic expression for the Delbrück amplitude is found exactly in case of an oscillator potential. General properties of the amplitude and its asymptotics are discussed.

1. Elastic scattering of photons in the Coulomb field of nuclei via virtual electron-positron pairs (Delbrück scattering [1]) attracted considerable interest for a long time. That is motivated by two reasons: (i) The Delbrück scattering is one of a few nonlinear quantum-electrodynamic processes which can be precisely tested by experiment [2, 3, 4, 5]. (ii) In order to extract an information on nuclear structure from differential cross sections of photon scattering on nuclei, a precise knowledge of the Delbrück amplitude may be required because of its interference with the nuclear amplitude. In some cases, the Delbrück scattering considerably modifies the differential cross section of γA-scattering [6, 7, 8, 9].

Theoretical investigations of the Delbrück scattering have a long history and many papers are devoted to this subject. Now the Delbrück amplitude is studied in detail in some approximations: (1) In the lowest-order Born approximation with respect to the parameter $Z\alpha$ (here $Z|e|$ is the charge of the nucleus, $\alpha = e^2 \simeq 1/137$ is the fine-structure constant), results were obtained for an arbitrary momentum transfer q; these results are surveyed in detail in [2]. (2) For the case of high energies ($\omega \gg m_e$, m_e being the electron mass) and small scattering angles ($q \ll \omega$), the Delbrück amplitude was obtained in [10, 11, 12, 13, 14] to all orders in $Z\alpha$. It was found that the Coulomb corrections at $Z\alpha \sim 1$ drastically change the amplitude as compared to the Born approximation. (3) At high energies and momentum transfers ($\omega, q \gg m_e$), the amplitude was also found to all orders in $Z\alpha$ [15, 16, 17]. In this case the Delbrück amplitude has a scaling behavior [18] and becomes inversely proportional to the photon energy. It has been shown that the Coulomb corrections essentially decrease the amplitude at high momentum transfer

as well. Recently some general exact expressions for the Delbrück amplitude were derived in [19], although numerical results were not presented. Some numerical results for the Delbrück amplitude obtained in the lowest-order Born approximation, in the "high-energy and small-angle" approximation, and in the "high energy and momentum transfer" approximation can be found in [20].

In the present paper we consider the Delbrück scattering in the scalar QED in the field of an oscillator Lorentz-scalar potential, or, in other words, for a relativistic oscillator. The relativistic oscillator model, being a nice theoretical laboratory, has also a few realistic applications. For instance, it can be successfully used [21] to describe interactions of collective modes of a nucleus in the region of giant resonances with the electromagnetic field. It takes the advantage of being automatically consistent with microcausality, analyticity, and dispersion relations. The Delbrück amplitude is a quantum correction to a classical part of the photon scattering by the oscillator and it is interesting to evaluate this correction explicitly to understand its significance for the physics of photon-nucleus scattering. Besides, the Delbrück amplitude for the relativistic oscillator provides an example of exact calculation of this quantity in external potentials.

2. To be specific, we consider spin-zero particles described by the Klein-Gordon equation with an oscillator Lorentz-scalar potential. We start with the Lagrangian for the charged quantum field $\hat{\phi}$ in the external electromagnetic potential A_μ:

$$L[\hat{\phi}, A](x) = \left| \partial_\mu \hat{\phi}(x) + ieA_\mu(x)\hat{\phi}(x) \right|^2 - U(r)|\hat{\phi}(x)|^2, \quad U(r) = \mu^2 + \gamma^4 r^2.$$

$$(1)$$

In the Furry representation, the Feynman rules corresponding to this Lagrangian are found from the usual Feynman rules for spin-zero particles by replacing plane waves with the normalized solutions $\langle r|n \rangle = \phi_n(r)$ of the Klein-Gordon equation in the potential $U(r)$:

$$p^2 \phi_n(r) + \gamma^4 r^2 \phi_n(r) = (E_n^2 - \mu^2)\phi_n(r), \quad p = -i\nabla \qquad (2)$$

(for brevity, we do not indicate here quantum numbers related to the angular momentum). The equation (2) coincides with that for a nonrelativistic oscillator and has the spectrum $E = \pm E_n$, where

$$E_n = \sqrt{\mu^2 + (2n+3)\gamma^2}, \quad n \geq 0. \qquad (3)$$

The parameter γ^4 determines a slope of the potential $U(r)$ in (1) and must be positive to result in a stable vacuum and bound states. It also determines

a range of the ground state, $\phi_0(r) = \gamma^{3/2}\pi^{-3/4}\exp(-\frac{1}{2}\gamma^2 r^2)$. The parameter μ^2 may be negative. However, we require $\mu^2 > -3\gamma^2$ to have a positive energy gap $2E_0$ between levels of positive and negative energies and hence a stable vacuum. In the nonrelativistic limit, $\gamma \ll \mu$, the parameter μ becomes the mass m of the particle and the oscillator parameter γ determines the oscillator frequency $\omega_0 = E_{n+1} - E_n =$const,

$$\mu \to m, \quad \gamma^2 \to m\omega_0. \qquad (4)$$

The Delbrück amplitude for the spin-zero particle is described by two Feynman diagrams shown in Fig. 1. In this figure, double lines represent the Green function for the Klein-Gordon equation in the external potential $U(r)$. The first diagram, Fig. 1a, describes the so-called seagull contribution to the photon scattering amplitude which is equal to

$$S_D = -2i\alpha(ee') \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \int d^3r\, G(r, r|\epsilon) \exp(iqr) =$$
$$= -\alpha(ee') \sum_n \frac{1}{E_n} \langle n| \exp(iqr)|n\rangle. \qquad (5)$$

Here e (k) and e' (k') are polarizations (momenta) of the incoming and outgoing photons, respectively, and $q = k - k'$; we use the radiative gauge, $ek = e'k' = 0$. $G(r, r'|\epsilon)$ is the Green function in the potential $U(r)$:

$$G(r, r'|\epsilon) = \sum_n \frac{\langle r|n\rangle\langle n|r'\rangle}{\epsilon^2 - E_n^2 + i0} =$$
$$= -i \int_0^{\infty} ds\, \langle r| \exp\left[is(\epsilon^2 - p^2 - U(r) + i0)\right] |r'\rangle, \qquad (6)$$

s being a proper time. The amplitude we use is normalized as to give, being squared, exactly the differential cross section of photon scattering.

Using a known expression for the time-dependent Green function of a nonrelativistic particle in an oscillator potential (see e.g. [22]), we obtain an explicit form for the Green function (6):

$$G(r, r'|\epsilon) = -i\gamma \int_0^{\infty} \frac{ds}{(2\pi i \sin 2s)^{3/2}} \times$$
$$\exp\left\{ \frac{is}{\gamma^2}(\epsilon^2 - \mu^2 + i0) + \frac{i\gamma^2}{2\sin 2s}\left[(r^2 + r'^2)\cos 2s - 2rr'\right] \right\}. \qquad (7)$$
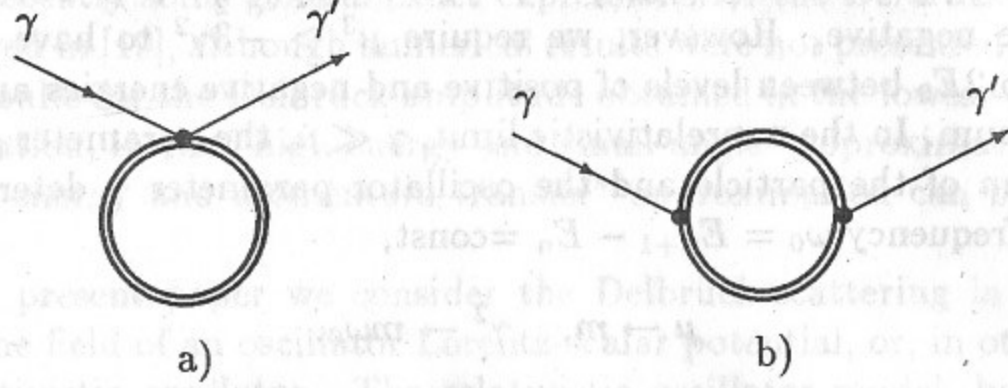
Figure 1: Diagrams of the Delbrück scattering: (a) Seagull amplitude S_D. (b) Resonance amplitude R_D.

Substituting (7) to (5) and taking elementary integrals with respect to ϵ and r, we find:

$$S_D = -\frac{i\alpha e^{i\pi/4}}{8\gamma\sqrt{\pi}}(ee') \int_0^\infty \frac{ds}{\sqrt{s}\sin^3 s} \exp\left[-is\frac{\mu^2-i0}{\gamma^2} + \frac{iq^2}{4\gamma^2}\cot s\right]. \quad (8)$$

In case of the oscillator potential $U(r)$, the Klein-Gordon operator has only a discrete spectrum (3) and therefore the Green function has only simple poles at real $\epsilon = \pm(E_n - i0)$ displaced by $i0$ in accordance with the Feynman rules. Using this analytic property, we can deform the integration path over ϵ in (5), $\epsilon \to i\epsilon$, to make it finally coincident with the imaginary axis. As a result, we get a possibility to rotate the contour of integration over s in (6)–(8), $s \to -is$, to make all the integrals well-convergent.

Note that an ultraviolet divergence at $s \to 0$ is not present in the amplitude S_D provided $q \neq 0$, so that a regularisation is not necessary. The same is valid for the second diagram, Fig. 1b, which gives the so-called resonance contribution:

$$R_D = -4i\alpha \int_{-\infty}^\infty \frac{d\epsilon}{2\pi} \int d^3r\, d^3r' \left[ie\nabla G(r,r'|\epsilon)\right] \times$$

$$\left[ie'\nabla' G(r',r|\epsilon+\omega)\right] \exp(ikr - ik'r') \quad (9)$$

$$= \alpha \sum_{n,n'} \frac{\langle n|(e'p)\exp(-ik'r)|n'\rangle\langle n'|(ep)\exp(ikr)|n\rangle}{E_n E_{n'}(E_n + E_{n'} - \omega - i0)} + (\omega \to -\omega). \quad (10)$$

It has a series of poles at $\omega = \pm(E_n + E_{n'})$ which are related with the pair photoproduction from vacuum when the pair components are captured to the

levels n, n' of the discrete spectrum. The pole at $\omega = 2E_0$ is nevertheless absent because matrix elements in (10) vanish for the s-wave states $n = n' = 0$. Using (7), we easily calculate the integrals (9) with respect to the variables ϵ, r and r' (here the substitutions $r = r_1 + r_2$ and $r' = r_1 - r_2$ are helpful). It is also convenient to replace $s_1 = s(1+x)/2$ and $s_2 = s(1-x)/2$, where s_1 and s_2 are proper times in the integral representation of the Green functions in (9). Then we get

$$R_D = -\frac{\alpha e^{i\pi/4}}{8\gamma\sqrt{\pi}} \int_0^\infty ds \cdot$$

$$\cdot \int_0^1 dx\, \frac{\sqrt{s}}{\sin^4 s}\left[\frac{\sin^2(sx)}{2\gamma^2 \sin s}(ek')(e'k) - i\cos(sx)(ee')\right] e^{iF}, \quad (11)$$

where

$$F = -s\frac{\mu^2-i0}{\gamma^2} + (1-x^2)\frac{\omega^2 s}{4\gamma^2} + \frac{q^2-2\omega^2}{4\gamma^2 \sin s}\cos(sx) + \frac{\omega^2}{2\gamma^2}\cot s. \quad (12)$$

Integrating the term in (11) proportional to (ee') by parts with respect to the variable x, we find that the term outside the integral cancels the contribution S_D of the first diagram, so that the Delbrück scattering amplitude $T_D = S_D + R_D$ finally reads:

$$T_D = -\frac{\alpha e^{i\pi/4}\omega^2}{16\gamma^3\sqrt{\pi}} \int_0^\infty ds \int_0^1 dx\, \frac{\sqrt{s}\sin(sx)}{\sin^5 s}\left[x\sin s\,(ee') - \sin(sx)(ss')\right] e^{iF}, \quad (13)$$

where $(ss') \equiv (\hat{k} \times e)(\hat{k}' \times e')$ describes a magnetic response. We see that a contribution like S_D which depends on q and is independent of ω vanishes in the total amplitude T_D, as it must be according to general consequences of gauge invariance and related low-energy theorems. Eq. (13) is our main result.

We may obtain another form of Eq. (13) by deforming the integration path over ϵ in (9) to transform the integral to $\int_{-\frac{1}{2}\omega-i\infty}^{-\frac{1}{2}\omega+i\infty} d\epsilon$. Such a deformation is always possible at low energies ω because the chains of singularities of two Green functions in (9) lying below and above the real axis, i.e. at $\epsilon = E_n - \omega - i0$ and at $\epsilon = -E_n + i0$, respectively, do not pinch the integration path provided $\omega < 2E_0$. Then we can rotate $s_1 \to -is_1$, $s_2 \to -is_2$ and get well-convergent integrals. Respectively, we may rotate the integration path in (13), $s \to -is$, and arrive at a real integral which is well suitable for calculating T_D at low energies. However, such an integral turns out to be

divergent at $\omega > E_0 + E_1$, i.e. above the nearest pole of the amplitude T_D, see Eq. (10).

3. Eq. (13) can be further simplified in case of a low oscillator frequency and momentum transfer, $\gamma^2 \ll \mu^2$ and $q \ll \mu$. Then the contribution of the region $s \gtrsim 1$ to the integral in (13) is exponentially suppressed, as is seen after the rotation of the contour, whereas the contribution from the region $s \ll 1$ reads

$$T_D \simeq -\frac{\alpha e^{i\pi/4}\omega^2}{16\gamma^3\sqrt{\pi}}[(ee') - (ss')]\int_0^\infty \frac{ds}{s^{5/2}}\int_0^1 x^2 dx \times$$

$$\exp\left[-is\frac{\mu^2 - i0}{\gamma^2} + i\frac{q^2}{4\gamma^2 s} - i\frac{(1 - x^2)^2}{48\gamma^2}\omega^2 s^3\right] \qquad (14)$$

and is saturated by

$$s \sim s_{\text{eff}} = \begin{cases} q^2\gamma^{-2}, & \text{if} \quad \mu q \lesssim \gamma^2 \\ q\mu^{-1}, & \text{if} \quad \mu q \gtrsim \gamma^2. \end{cases} \qquad (15)$$

Respectively, at "low" energies $\omega^2 \ll \gamma^2 s_{\text{eff}}^{-3}$ which may nevertheless be high in comparison with μ, the integral (14) reduces to the modified Bessel function of the third kind $K_{3/2}$ and hence is an elementary function:

$$T_D \simeq \frac{\alpha\omega^2}{12q^3}\left(1 + \frac{\mu q}{\gamma^2}\right)\exp\left(-\frac{\mu q}{\gamma^2}\right)[(ee') - (ss')]. \qquad (16)$$

When $q \to 0$, the Delbrück amplitude becomes proportional to an effective volume of the vacuum, $V \sim q^{-3}$, probed by the photons. At high energies $\omega^2 \gg \gamma^2 s_{\text{eff}}^{-3}$, the integral (14) is saturated by x close to 1 and the answer is given by the modified Bessel function K_3:

$$T_D \simeq \pm i\omega\sqrt{3}\frac{\alpha\mu^3}{\gamma^2 q^3}K_3\left(\frac{\mu q}{\gamma^2}\right)[(ee') - (ss')], \qquad (17)$$

where the sign of plus or minus appears for $\text{Im}\,\omega > 0$ or $\text{Im}\,\omega < 0$, respectively. Since $K_3(z) \to 8z^{-3}$ when $z \to 0$, the last expression seems to have a too strong singularity $\sim q^{-6} \sim V^2$ when $q \to 0$. However, when q is very small, the assumption $\omega^2 \gg \gamma^2 s_{\text{eff}}^{-3}$ becomes violated and the Delbrück amplitude approaches to the regime of Eq. (16).

The singular behavior of the Delbrück amplitude at small q, Eq. (16), can be understood in the following way. If the external potential U is equal to

zero, the photon scattering amplitude due to the diagrams shown in Fig.1 reads [23]:

$$T = (2\pi)^3\delta^3(k - k')\frac{\alpha}{48\pi^2}e'^\mu e^\nu(k_\mu k_\nu - k^2 g_{\mu\nu})\log\frac{\Lambda^2}{m_0^2}, \qquad (18)$$

where Λ is an ultraviolet cut off and m_0 is the mass of bare scalar particles, and is proportional to $\omega^2[(ee') - (ss')]$. That means that the vacuum has an electric and magnetic polarizability per unit volume,

$$\chi_E^{\text{vac},0} = -\chi_M^{\text{vac},0} = \frac{\alpha}{48\pi^2}\log\frac{\Lambda^2}{m_0^2}. \qquad (19)$$

These universal vacuum susceptibilities are absorbed by renormalisation of electromagnetic fields and charges and are not observable. However, in the presence of an almost uniform scalar potential $U(r)$ which shifts the mass of the particles $m_0^2 \to m_{\text{eff}}^2 \equiv U(r)$, the polarizabilities get a finite meaningful piece,

$$\chi_E^{\text{vac}}(r) = -\chi_M^{\text{vac}}(r) = -\frac{\alpha}{48\pi^2}\log\frac{U(r)}{m_0^2}. \qquad (20)$$

The last formula is valid when the potential $U(r)$ is constant at distances $\sim m_{\text{eff}}^{-1}$ which are characteristic for creating those vacuum polarizabilities:

$$m_{\text{eff}}^{-1} \ll \Delta r \equiv U(r)\left[\frac{dU(r)}{dr}\right]^{-1}. \qquad (21)$$

For the potential $U(r) = \mu^2 + \gamma^4 r^2$, it means

$$\begin{cases} r \gg \gamma^{-1}, & \text{if} \quad \gamma^2 r \gtrsim \mu \\ r \ll \mu^3\gamma^{-4}, & \text{if} \quad \gamma^2 r \lesssim \mu \end{cases} \qquad (22)$$

In the case $\gamma \ll \mu$, both the above regions overlap and the formula (20) is valid everywhere. Then the scattering amplitude of low-energy photons is equal to

$$T_D = \omega^2\alpha_E(q)(ee') + \omega^2\alpha_M(q)(ss') \qquad (23)$$

with

$$\alpha_E(q) = -\alpha_M(q) = -\frac{\alpha}{48\pi^2}\int\exp(iqr)\log\frac{U(r)}{m_0^2}d^3r. \qquad (24)$$

Taking this elementary integral, we find

$$\alpha_E(q) = -\alpha_M(q) = C\delta^3(q) + \frac{\alpha}{12q^3}(1 + a)e^{-a}, \quad a = \frac{\mu q}{\gamma^2}. \qquad (25)$$

in complete accordance with (16). Here C is an infinite constant due to the polarizability of the vacuum in the whole space; it is infinite because the potential $U(r) \to \infty$ when $r \to \infty$.

The approximation of a uniform potential (20) turns out to be inapplicable at high energies when the virtual particles of the mass m_{eff} produced by the photon propagate to a distance $\sim \hbar c/\Delta E \sim \omega/m_{\text{eff}}^2$ which is large in comparison with the scale Δr of variation of the potential $U(r)$. This just happens when $\omega^2 \gg \gamma^2 s_{\text{eff}}^{-3}$ and the amplitude T_D becomes predominantly imaginary, see Eq. (17).

Keeping in (13) terms of the next order in s^2, we can find a correction to the expansion of T_D in powers of the momentum transfer. At "low" energies it reads

$$T_D \simeq \text{Eq. (16)} + \frac{\alpha\omega^2 e^{-a}}{1440\mu\gamma^2}[(19+2a)(ss') - (17+2a)(ee')] . \qquad (26)$$

It determines an asymptotics of the helicity-non-flip amplitude T_D^{++} because the piece (14) contributes to only the helicity-flip amplitude T_D^{+-}, as easily seen from the relation $(ee') = \pm(ss') = (1 \pm \cos\theta)/2$ for the helicity-non-flip and helicity-flip case, respectively, θ being the scattering angle.

The singular behavior of T_D at $q \to 0$ disappears when the potential $U(r)$ has a finite range. For example, in case of a cut oscillator potential,

$$\log\frac{U(r)}{m_0^2} = -A\exp\left(-\frac{\omega_0^2 r^2}{A}\right), \qquad (27)$$

where $A = \log\frac{U(\infty)}{U(0)} = O\left(\frac{\omega_0}{m_0}\right) \ll 1$ and ω_0 is the frequency of small oscillations, the polarizabilities (24) are finite:

$$\alpha_E(q) = -\alpha_M(q) = \frac{\alpha A^{5/2}}{48\sqrt{\pi}\omega_0^3}\exp\left(-\frac{Aq^2}{4\omega_0^2}\right). \qquad (28)$$

At energies $\omega = O(\omega_0)$ the corresponding Delbrück amplitude (23) is less by the factor of $O((\omega_0/m_0)^{3/2})$ than the ordinary nonrelativistic photon scattering amplitude $T_0 \simeq \frac{\alpha\omega^2}{m_0(\omega_0^2 - \omega^2)}(ee')$ by a particle bound at the ground state.

4. In conclusion, in the present paper we calculated for the first time the amplitude of the Delbrück scattering in the scalar QED in case of a scalar oscillator potential and, at low oscillator frequency, investigated its asymptotic behavior at low and high energies and low momentum transfer. A close relation of the Delbrück scattering with the vacuum polarisation was demonstrated.

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Relativistic Oscillator Model and Delbrück Scattering

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